GENERATING FUNCTIONS FOR BESSEL FUNCTIONS

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1. Introduction. On replacing the parameter n in Bessel's differential equation

$$(1.1) \qquad x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (x^2 - n^2)v = 0$$

by the operator $y(\partial/\partial y)$, the partial differential equation $Lu = 0$ is constructed, where

$$(1.2) \quad L = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y} + x^2 = \left(x\frac{\partial}{\partial x}\right)^2 - \left(y\frac{\partial}{\partial y}\right)^2 + x^2.$$

This operator annuls $u(x, y) = v(x)y^n$ if, and only if, $v(x)$ satisfies (1.1) and hence is a cylindrical function of order n. Thus every generating function of a set of cylindrical functions is a solution of $Lu = 0$.

It is shown in § 2 that the partial differential equation $Lu = 0$ is invariant under a three-parameter Lie group. This group is then applied to the systematic determination of generating functions for Bessel functions, following the methods employed in two previous papers **(4; 5)**.

2. Group of operators. The operators

$$A = y\frac{\partial}{\partial y}, B = y^{-1}\frac{\partial}{\partial x} + x^{-1}\frac{\partial}{\partial y}, C = -y\frac{\partial}{\partial x} + x^{-1}y^2\frac{\partial}{\partial y}$$

satisfy the commutator relations $[A, B] = -B$, $[A, C] = C$, $[B, C] = 0$, and therefore generate a three-parameter Lie group. From these relations and the operator identity

$$(2.1) \qquad -x^{-2}L = BC - 1,$$

where L is the operator (1.2), it follows that A, B, C are commutative with $x^{-2}L$ and therefore convert every solution of $Lu = 0$ into a solution. In particular

$$(2.2) \qquad \begin{cases} AJ_n(x)y^n = nJ_n(x)y^n, AJ_{-n}(x)y^n = nJ_{-n}(x)y^n, \\ BJ_n(x)y^n = J_{n-1}(x)y^{n-1}, BJ_{-n}(x)y^n = -J_{-n+1}(x)y^{n-1}, \\ CJ_n(x)y^n = J_{n+1}(x)y^{n+1}, CJ_{-n}(x)y^n = -J_{-n-1}(x)y^{n+1}, \end{cases}$$

where n is an arbitrary complex number.

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The operator A generates the trivial group $x' = x$, $y' = ty$, $(t \neq 0)$, which is used for purposes of normalization. The extended form of the group generated by the commutative operators B, C is described by

(2.3) $$e^{bB+cC}f(x, y) = f([(x - 2cy)(x + 2b/y)]^{\frac{1}{2}}, [y(xy + 2b)/(x - 2cy)]^{\frac{1}{2}}),$$

where b and c are arbitrary constants and $f(x, y)$ an arbitrary function, the signs of the radicals being chosen so that the right member reduces to $f(x, y)$ when $b = c = 0$. If $f(x, y)$ is annulled by L, so is the right member of (2.3).

3. Generating functions annulled by operators of the first order. Since $J_\nu(x)y^\nu$ is annulled by L and $A - \nu$, it follows from the operator identity

$$e^{bB+cC}Ae^{-bB-cC} = A + bB - cC$$

(**4**, p. 1035) and (2.3) that

(3.1) $$\begin{aligned} G(x, y) &= e^{bB+cC}J_\nu(x)y^\nu \\ &= (xy + 2b)^{\frac{1}{2}\nu}(xy^{-1} - 2c)^{-\frac{1}{2}\nu}J_\nu([x - 2cy)(x + 2b/y)]^{\frac{1}{2}}) \end{aligned}$$

is annulled by L and $A + bB + cC - \nu$. While any cylindrical function of order ν may be employed in place of $J_\nu(x)$, it is sufficient to confine attention to the Bessel functions of the first kind.

If $b = 0$, we choose $c = 1$, so that

$$G(x, y) = (xy)^\nu(x^2 - 2xy)^{-\frac{1}{2}\nu}J_\nu([x^2 - 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_n J_{\nu+n}(x)y^{\nu+n}.$$

The indicated expansion is justified by the observation that $(xy)^{-\nu}G(x, y)$ is an entire function of x and y. Since G is annulled by $A - C - \nu$, we find, with the aid of (2.2), that $g_{n-1} = ng_n$ $(n = 1, 2, \ldots)$. Multiplying G by $(xy)^{-\nu}$ and then setting $x = 0$, noting that

(3.2) $$x^{-\nu}J_\nu(x)]_{x=0} = \frac{1}{2^\nu\Gamma(\nu + 1)},$$

we have $g_0 = 1$; hence $g_n = 1/n!$. Thus

(3.3) $$x^\nu(x^2 - 2xy)^{-\frac{1}{2}\nu}J_\nu([x^2 - 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} J_{\nu+n}(x)y^n/n!,$$

which may be identified with Lommel's first formula (**3**, p. 140).

If $c = 0$, we choose $b = 1$, whence

(3.4) $$\begin{aligned} G(x, y) &= (y^2 + 2y/x)^{\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}) \\ &= (2 + xy)^\nu(x^2 + 2x/y)^{-\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}). \end{aligned}$$

From the last expression it is evident that G has a Laurent expansion about $y = 0$:

$$G(x, y) = \sum_{n=-\infty}^{\infty} g_n J_n(x)y^n, \qquad |xy| < 2.$$

Since this function is annulled by $A + B - \nu$, we find, with the aid of (2.2), that $g_{n+1} = (\nu - n)g_n$, $(n = 0, \pm 1, \pm 2, \ldots)$. Setting $x = 0$, we have $g_0 = 1/\Gamma(\nu + 1)$; hence $g_n = 1/\Gamma(\nu - n + 1)$. Replacing y by y^{-1}, we obtain

$$(3.5)\quad (xy)^{-\nu}(x^2 + 2xy)^{\frac{1}{2}\nu}J_\nu([x^2 + 2xy]^{\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} J_n(x)(-y)^n/\Gamma(\nu + n + 1), \qquad |2y| > |x|.$$

Writing (3.4) in the form

$$G(x, y) = (xy)^\nu(1 + 2/xy)^\nu(x^2 + 2x/y)^{-\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}),$$

it is evident that $(xy)^{-\nu}G$ is expressible as a power series in y^{-1}, convergent for $|xy| > 2$. We obtain, after simplification,

$$(3.6)\qquad (1 + 2y/x)^{\frac{1}{2}\nu}J_\nu([x^2 + 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} J_{\nu-n}(x)y^n/n!, \qquad |2y| < |x|,$$

which may be identified with Lommel's second formula (**3**, p. 140).

If $bc \neq 0$, it proves convenient to choose $b = \frac{1}{2}w$, $c = -\frac{1}{2}w$, whence

$$(3.7)\quad G(x, y) = (w + xy)^{\frac{1}{2}\nu}(w + x/y)^{-\frac{1}{2}\nu}J_\nu([w^2 + x^2 + wx(y + y^{-1})]^{\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} g_n J_n(x)y^n, \qquad |xy| < |w|.$$

Replacing y by $2y/x$ and then setting $x = 0$, we obtain, with the aid of (3.2),

$$(1 + 2y/w)^{\frac{1}{2}\nu}J_\nu([w^2 + 2wy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_n y^n/n!, \qquad |2y| < |w|.$$

Comparing with (3.6), we infer that $g_n = J_{\nu-n}(w)$, $(n = 0, 1, 2, \ldots)$. Similarly, replacing y by $x/2y$ and then setting $x = 0$, we obtain

$$w^\nu(w^2 + 2wy)^{\frac{1}{2}\nu}J_\nu([w^2 + 2wy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_{-n}(-y)^n/n!.$$

Comparing with (3.3) we conclude that $g_{-n} = J_{\nu+n}(w)$, $(n = 0, 1, 2, \ldots)$. Hence

$$(3.8)\quad (w + xy)^{\frac{1}{2}\nu}(w + x/y)^{-\frac{1}{2}\nu}J_\nu([w^2 + x^2 + wx(y + y^{-1})]^{\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} J_{\nu-n}(w)J_n(x)y^n, \qquad |xy| < |w|,$$

which may be identified with Graf's addition theorem (**3**, p. 359) by substituting $y = -e^{-i\phi}$. Another expansion of (3.7), valid for $|xy| > |w|$, may be obtained from (3.8) by replacing y by y^{-1}, interchanging x and w, and multiplying by y^ν.

We have now obtained, in normalized form, functions annulled by L and differential operators of the first order of the form $r_1A + r_2B + r_3C + r_4$, where the r's are constants and $r_1 \neq 0$. Generating functions annulled by $r_2B + r_3C + r_4$ are not included in (3.1) but may be derived independently.

Since $[B, C] = 0$, we seek a solution of the simultaneous equations $(B - 1)u = 0$, $(C - 1)u = 0$. This solution is annulled by $r_2 B + r_3 C + r_4$, normalized so that $r_2 + r_3 + r_4 = 0$. By (2.1) it is also annulled by L. We find the solution to be the familiar generating function

$$e^{\frac{1}{2}x(y-y^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) y^n \tag{3.9}$$

of the Bessel functions of integral order.

4. Generating functions annulled by $A(B - C) + \frac{1}{2}(B + C) + 4\alpha - 1$**.** By a suitable choice of new variables the equation $Lu = 0$ may be transformed into one solvable by separation of variables. A solution so obtained, if possessed of suitable analytic properties, provides a generating function for Bessel functions. We shall present several examples.

Choosing new variables

$$\xi = \tfrac{1}{2}x(y^{-1} - y + 2i), \eta = \tfrac{1}{2}x(y^{-1} - y - 2i),$$

the equation $Lu = 0$ is transformed into

$$4\xi \frac{\partial^2 u}{\partial \xi^2} - 4\eta \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \xi} - 2\frac{\partial u}{\partial \eta} - (\xi - \eta)u = 0.$$

Four linearly independent solutions are obtained by separation of variables:

$$\begin{aligned}
u_1 &= e^{-\frac{1}{2}(\xi+\eta)} {}_1F_1(\alpha; \tfrac{1}{2}; \xi) {}_1F_1(\alpha; \tfrac{1}{2}; \eta),\\
u_2 &= \xi^{\frac{1}{2}} e^{-\frac{1}{2}(\xi+\eta)} {}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \xi) {}_1F_1(\alpha; \tfrac{1}{2}; \eta),\\
u_3 &= \eta^{\frac{1}{2}} e^{-\frac{1}{2}(\xi+\eta)} {}_1F_1(\alpha; \tfrac{1}{2}; \xi) {}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \eta),\\
u_4 &= (\xi\eta)^{\frac{1}{2}} e^{-\frac{1}{2}(\xi+\eta)} {}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \xi) {}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \eta),
\end{aligned}$$

where α is an arbitrary constant. These functions are also annulled by

$$\begin{aligned}
4\xi \frac{\partial^2}{\partial \xi^2} &+ 2\frac{\partial}{\partial \xi} - \xi + 1 - 4\alpha\\
&= 4\xi(\xi - \eta)^{-2} L - A(B - C) - \tfrac{1}{2}(B + C) + 1 - 4\alpha,
\end{aligned}$$

where A, B, C are the operators of § 2, and hence by

$$A(B - C) + \tfrac{1}{2}(B + C) + 4\alpha - 1.$$

This operator provides recurrence relations for the coefficients of the expansions of the generating functions; but these relations will not be used.

When expressed in terms of x and y, the function u_1 is seen to have a Laurent expansion about $y = 0$:

$$\begin{aligned}
e^{\frac{1}{2}x(y-y^{-1})} &{}_1F_1(\alpha; \tfrac{1}{2}; \tfrac{1}{2}x/y - \tfrac{1}{2}xy + ix) {}_1F_1(\alpha; \tfrac{1}{2}; \tfrac{1}{2}x/y - \tfrac{1}{2}xy - ix)\\
&= \sum_{n=-\infty}^{\infty} g_n J_n(x) y^n.
\end{aligned}$$

Replacing y by $2y/x$ and then setting $x = 0$, we have by (3.2)

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$$e^{y}[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2=\sum_{n=0}^{\infty}g_n y^n/n!.$$

By Kummer's formula,

$$e^{y}[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2={}_1F_1(\alpha;\tfrac{1}{2};-y)\,{}_1F_1(\tfrac{1}{2}-\alpha;\tfrac{1}{2};y).$$

The expansion of the right member in powers of y may be obtained with the aid of Chaundy's formula

$${}_1F_1(a;c;-y)\,{}_1F_1(a';c';y)=\sum_{n=0}^{\infty}\frac{(a)_n(-y)^n}{n!(c)_n}\,{}_3F_2\left[\begin{matrix}a',\,1-c-n;\,1\\ c',\,1-a-n\end{matrix}\right]$$

(**2**, p. 70). However, this expansion may be expressed in a more suitable form by means of the transformation formula

$${}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\,1\\ \beta_1,\beta_2\end{matrix}\right]=\frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\times{}_3F_2\left[\begin{matrix}\beta_1-\alpha_1,\,\beta_1-\alpha_2,\,\alpha_3;\,1\\ \beta_1,\,\beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right]$$

(**1**, p. 98), whence

$$(4.1)\quad {}_1F_1(a;c;-y)\,{}_1F_1(a';c';y)=\sum_{n=0}^{\infty}\frac{(c+c'-a-a')_n}{(c)_n}y^n\,{}_3F_2\left[\begin{matrix}c'-a',\,c+c'+n-1,\,-n;\,1\\ c',\,c+c'-a-a'\end{matrix}\right].$$

Thus

$$e^{y}[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2=\sum_{n=0}^{\infty}{}_3F_2(\alpha,n,-n;\tfrac{1}{2};\tfrac{1}{2};1)y^n/n!,$$

and g_n is determined for $n=0,1,2,\ldots$. Since the generating function is unaltered when y is replaced by $-y^{-1}$, $g_{-n}=g_n$. Hence

$$(4.2)\quad e^{\frac{1}{2}x(y-y^{-1})}\,{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy+ix)\,{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy-ix)=\sum_{n=-\infty}^{\infty}{}_3F_2(\alpha,n,-n;\tfrac{1}{2};\tfrac{1}{2};1)J_n(x)y^n.$$

Since $\xi^{\frac{1}{2}}=(\tfrac{1}{2}x)^{\frac{1}{2}}(y^{-\frac{1}{2}}+iy^{\frac{1}{2}})$, u_2 has an expansion of the form

$$\sum_{n=-\infty}^{\infty}[a_nJ_{n+\frac{1}{2}}(x)+b_nJ_{-n-\frac{1}{2}}(x)]y^{n+\frac{1}{2}}.$$

Applying the methods described above, we obtain, after multiplying by $(2y/x)^{\frac{1}{2}}$

$$(4.3)\quad (1+iy)e^{\frac{1}{2}x(y-y^{-1})}\,{}_1F_1(\alpha+\tfrac{1}{2};3/2;\tfrac{1}{2}x/y-\tfrac{1}{2}xy+ix)\times{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy-ix)$$
$$=(\pi/2x)^{\frac{1}{2}}\sum_{n=0}^{\infty}{}_3F_2(\alpha,n+1,-n;1,\tfrac{1}{2};1)J_{n+\frac{1}{2}}(x)[iy^{n+1}+(-y)^{-n}].$$

Replacing i by $-i$, we obtain the expansion which arises similarly from u_3.

Since $(\xi\eta)^{\frac{1}{2}} = \frac{1}{2}x(y + y^{-1})$, u_4 has a Laurent expansion about $y = 0$. We obtain, after replacing α by $\alpha - \frac{1}{2}$,

$$(4.4)\quad \tfrac{1}{2}x(y + y^{-1})e^{\frac{1}{2}x(y-y^{-1})}{}_1F_1(\alpha; 3/2; \tfrac{1}{2}x/y - \tfrac{1}{2}xy + ix) \times {}_1F_1(\alpha; 3/2; \tfrac{1}{2}x/y - \tfrac{1}{2}xy - ix) = \sum_{n=-\infty}^{\infty} n\, {}_3F_2(\alpha, n+1, 1-n; 3/2; 3/2; 1) J_n(x) y^n.$$

With the aid of these results the elementary solutions of the three-dimensional wave equation in parabolic cylindrical co-ordinates may be expressed in terms of cylindrical wave functions.

5. Generating functions annulled by $B^2 + 8CA + 4C$. When we choose new variables $\xi = xy - (x/y)^{\frac{1}{2}}$, $\eta = xy + (x/y)^{\frac{1}{2}}$, the equation $Lu = 0$ becomes

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{4}(\xi - \eta)u = 0.$$

The following solutions are obtained by separation of variables:

$$u_1 = {}_0F_1(2/3; -[\xi + z]^3/36)\,{}_0F_1(2/3; -[\eta + z]^3/36),$$
$$u_2 = (\xi + z)\,{}_0F_1(4/3; -[\xi + z]^3/36)\,{}_0F_1(2/3; -[\eta + z)^3/36),$$
$$u_3 = (\eta + z)\,{}_0F_1(2/3; -[\xi + z]^3/36)\,{}_0F_1(4/3; -[\eta + z]^3/36),$$
$$u_4 = (\xi + z)(\eta + z)\,{}_0F_1(4/3; -[\xi + z]^3/36)\,{}_0F_1(4/3; -[\eta + z]^3/36$$

where z is an arbitrary constant. These functions are also annulled by

$$\frac{\partial^2}{\partial \xi^2} + \frac{1}{4}(\xi + z) = 2\xi[(\xi - \eta)(\xi^2 - \eta^2)]^{-1}L + \frac{B^2}{16} + \tfrac{1}{2}CA + \frac{C}{4} + \frac{z}{4}$$

and hence by $R = B^2 + 8CA + 4C + 4z$.

The functions u_1 and u_4 have expansions of the form

$$\sum_{n=-\infty}^{\infty} g_n J_n(x) y^n.$$

Applying R, we obtain the recurrence relation

$$g_{n+2} + 4zg_n + 4(2n - 1)g_{n-1} = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

by means of (2.2). No explicit solution is available for arbitrary z. A solution is readily obtained for $z = 0$. We find that

$$(5.1)\quad {}_0F_1(2/3; -[xy - (x/y)^{\frac{1}{2}}]^3/36)\,{}_0F_1(2/3; -[xy + (x/y)^{\frac{1}{2}}]^3/36) = \sum_{m=-\infty}^{\infty} (-24)^m \Gamma(m + 1/6) J_{3m}(x) y^{3m}/\Gamma(1/6),$$

$$(5.2)\quad \frac{1}{8}(x^2y^2 - x/y)\,{}_0F_1(4/3; -[xy - (x/y)^{\frac{1}{2}}]^3/36) \times {}_0F_1(4/3; -[xy + (x/y)^{\frac{1}{2}}]^3/36) = \sum_{m=-\infty}^{\infty} (-24)^m \Gamma(m + 5/6) J_{3m+2}(x) y^{3m+2}/\Gamma\left(\frac{5}{6}\right)$$

For u_2 we obtain similarly

$$(5.3)\quad [xy - (x/y)^{\frac{1}{2}}]{}_0F_1(4/3; -[xy - (x/y)^{\frac{1}{2}}]^3/36)\; {}_0F_1(2/3; -[xy + (x/y)^{\frac{1}{2}}]^3/36)$$
$$= -(\pi/2)^{\frac{1}{2}} \sum_{m=0}^{\infty} (-24)^{-m} J_{3m+1/2}(x) y^{-3m-1/2}/m!$$
$$+ 2 \sum_{m=-\infty}^{\infty} (-24)^m \Gamma(m + \tfrac{1}{2}) J_{3m+1}(x) y^{3m+1}/\Gamma(\tfrac{1}{2}).$$

6. Generating functions annulled by $A^2 + \alpha(2CA + C) + \beta C^2 - \nu^2$. If we choose new variables

$$\xi = \tfrac{1}{2}[(x^2 + 2a^2xy)^{\frac{1}{2}} - (x^2 + 2b^2xy)^{\frac{1}{2}}],$$
$$\eta = \tfrac{1}{2}[(x^2 + 2a^2xy)^{\frac{1}{2}} + (x^2 + 2b^2xy)^{\frac{1}{2}}], \qquad (a^2 \neq b^2),$$

where a and b are constants and the signs of the radicals are chosen so that $\xi = 0$, $\eta = x$ when $y = 0$, the equation $Lu = 0$ becomes

$$\xi^2 \frac{\partial^2 u}{\partial \xi^2} - \eta^2 \frac{\partial^2 u}{\partial \eta^2} + \xi \frac{\partial u}{\partial \xi} - \eta \frac{\partial u}{\partial \eta} + (\xi^2 - \eta^2) u = 0.$$

Comparing with (1.1), it follows that L annuls the four functions $J_{\pm\nu}(\xi)J_{\pm\nu}(\eta)$, where ν is arbitrary. These functions are also annulled by

$$\xi^2 \frac{\partial^2}{\partial \xi^2} + \xi \frac{\partial}{\partial \xi} + \xi^2 - \nu^2 = \xi^2(\xi^2 + 2c\xi\eta + \eta^2)^{-1} L + A^2 + \tfrac{1}{2}(a^2 + b^2)(2CA + C) + a^2b^2C^2 - \nu^2,$$

where $c = (a^2 + b^2)/(a^2 - b^2)$, and hence by

$$R = A^2 + \tfrac{1}{2}(a^2 + b^2)\,(2CA + C) + a^2b^2C^2 - \nu^2.$$

Employing the methods described previously, and applying the well-known formulae

$$J_\mu(\alpha z) J_\nu(\beta z) = \frac{(\frac{1}{2}\alpha z)^\mu (\frac{1}{2}\beta z)^\nu}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\alpha z)^{2n}}{n!\,\Gamma(\mu + n + 1)} \times F(-n, -\mu - n; \nu + 1; \beta^2/\alpha^2),$$

$$F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z - 1}\right),$$

the following results are obtained:

$$(6.1)\quad 2^{2\nu}\Gamma(\nu + 1)(a^2 - b^2)^{-\xi} J_\nu(\xi) J_\nu(\eta)$$
$$= \sum_{n=0}^{\infty} \frac{(ab)^n}{n!} F\left(-n, n + 2\nu + 1; \nu + 1; \frac{(a + b)^2}{4ab}\right) J_{\nu+n}(x) y^{\nu+n},$$
$$(a^2 \neq b^2, ab \neq 0, \nu \neq -1, -2, \ldots).$$

$$(6.2)\quad 2^{2\nu}\Gamma(\nu+1)(a^2-b^2)^{-\nu}J_\nu(\xi)J_{-\nu}(\eta)$$
$$= \sum_{n=0}^{\infty} \frac{(-ab)^n}{n!} F\left(-n, n+2\nu+1; \nu+1; \frac{(a+b)^2}{4ab}\right) J_{-\nu-n}(x)y^{\nu+n},$$
$$|y| < \text{Min}(|x/2a^2|, |x/2b^2|),\ (a^2 \neq b^2, ab \neq 0, \nu \neq -1, -2, \ldots).$$

$$(6.3)\quad \Gamma(\nu+1)(a+b)^\nu(a-b)^{-\nu}J_\nu(\xi)J_{-\nu}(\eta)$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-ab)^n}{\Gamma(n-\nu+1)} F\left(-n, n+1; \nu+1; -\frac{(a-b)^2}{4ab}\right) J_n(x)y^n,$$
$$|y| > \text{Max}(|x/2a^2|, |x/2b^2|),\ (a^2 \neq b^2, ab \neq 0, \nu \neq -1, -2, \ldots),$$

and the left member has the value $(\sin \nu\pi)/\nu\pi$ when $x = 0$.

The excluded case $ab = 0$ may be treated similarly. Setting $a = 0$, $b^2 = -2$, the following generating functions, annulled by $A^2 - 2CA - C - \nu^2$, are obtained:

$$(6.4)\quad J_\nu(\tfrac{1}{2}[x - (x^2-4xy)^{\frac{1}{2}}])J_\nu(\tfrac{1}{2}[x + (x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu+n+1)} \binom{2\nu+2n}{n} J_{\nu+n}(x)(y/2)^{\nu+n},$$

$$(6.5)\quad J_\nu(\tfrac{1}{2}[x - (x^2-4xy)^{\frac{1}{2}}])J_{-\nu}(\tfrac{1}{2}[x + (x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)} \binom{2\nu+2n}{n} J_{-\nu-n}(x)(y/2)^{\nu+n},\ |4y| < |x|.$$

$$(6.6)\quad e^{\nu\pi i}J_\nu(\tfrac{1}{2}[x - (x^2-4xy)^{\frac{1}{2}}])J_{-\nu}(\tfrac{1}{2}[x + (x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1-\nu)\Gamma(n+1+\nu)} J_n(x)(2y)^n$$
$$+ i\pi^{-\frac{1}{2}} \sin \nu\pi \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-\nu)_n(\frac{1}{2}+\nu)_n}{n!} J_{n+\frac{1}{2}}(x)(2y)^{-n-\frac{1}{2}},\ |4y| > |x|,$$

where the left member has the value $(\sin \nu\pi)/\nu\pi$ when $x = 0$. Formulae (6.4) and (6.5) are limiting cases of formulae (6.1) and (6.2) respectively.

References

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