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SETS OF UNIQUENESS FOR THE GROUP OF INTEGERS OF A p-SERIES FIELD

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§ 1. Introduction.

Let G denote the group of integers of a p-series field, where p is a prime $\geqq 2$. Thus, any element $\bar{x} \in G$ can be represented as a sequence $\{x_i\}_{i=0}^{\infty}$ with $0 \leqq x_i < p$ for each $i \geqq 0$. Moreover, the dual group $\{\Psi_m\}_{m=0}^{\infty}$ of G can be described by the following process. If m is a non-negative integer with $m = \sum_{k=0}^{\infty} \alpha_k p^k$, $0 \leqq \alpha_k < p$ for each k, and if $\bar{x} \in G$ then

(1) $\Psi_m(\bar{x}) = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}(\bar{x}),$

where for each integer $k \geqq 0$ and for each $x = \{x_i\} \in G$, the functions ϕ_k are defined by

(2) $\phi_k(\bar{x}) = \exp(2\pi i x_k/p).$

In the case that $p = 2$, the group G is the dyadic group introduced by Fine [**1**] and the functions $\{\Psi_m\}_{m=0}^{\infty}$ are the Walsh-Paley functions. A detailed account of these groups and basic properties can be found in [**4**].

One of these basic properties is that the group G can be identified with the unit interval $[0, 1)$. This is accomplished by associating with each element $\bar{x} = \{x_i\} \in G, 0 \leqq x_i < p$, the point $x = \sum_{i=0}^{\infty} x_i/p^{i+1}$. It is well-known that the map $\bar{x} \rightarrow x$ takes Haar measure on G to Lebesgue measure on $[0, 1)$. Moreover, if we neglect the set D, of p-rationals, this map is one-to-one and onto. It becomes a group homomorphism if we define the p-sum of two real numbers $x, y \in [0, 1)$ by

$$x \dotplus y = \sum_{i=0}^{\infty} (x_i \oplus y_i)/p^{i+1}$$

where

$$x = \sum_{i=0}^{\infty} x_i/p^{i+1}, \; y = \sum_{i=0}^{\infty} y_i/p^{i+1},$$

and $x_i \oplus y_i$ represents the sum of x_i and y_i, modulo p. Abusing the notation slightly, we shall set $\Psi_m(x) = \Psi_m(\bar{x})$ for $x \in [0, 1)$ and $m = 0, 1, \ldots$. Since each Ψ_m is a character on G, we have that

(3) $\Psi_m(x \dotplus y) = \Psi_m(x)\Psi_m(y),$

for $x, y \in [0, 1)$ and $m = 0, 1, \ldots$.

Define the *p-sum* of two non-negative integers n and l as follows. If $m =$

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$\sum_{i=0}^{\infty} \alpha_i p^i$ and if $l = \sum_{i=0}^{\infty} \beta_i p^i$, with $0 \leqq \alpha_i, \beta_i < p$, then

$$m \dot{+} l = \sum_{i=0}^{\infty} (\alpha_i \oplus \beta_i) p^i.$$

It is clear from equation (1) that

$$(4) \qquad \Psi_{m \dot{+} l}(x) = \Psi_m(x)\Psi_l(x),$$

for $x \in [0, 1)$ and $m, l = 0, 1, \ldots$. We shall denote the p-sum of an integer l with itself $(p - 1)$ times by $\dot{-}l$. Since addition of coordinates is modulo p, we observe that $l \dot{-} l = 0$.

Define the *p-product* of a non-negative integer $m = \sum_{k=0}^{\infty} \alpha_k p^k$ with a real number x (which either belongs to $[0, 1)$ or to the set $\{1, 2, \ldots\}$) by

$$m \circ x = (\alpha_0 \circ x) \dot{+} (\alpha_1 p \circ x) \dot{+} (\alpha_2 p^2 \circ x) \dot{+} \ldots,$$

where the numbers $\alpha p^l \circ x$ are defined as follows. If $x = \sum_{i=0}^{\infty} x_i/p^{i+1}$ belongs to the interval $[0, 1)$, then

$$\alpha p^l \circ x = \sum_{i=0}^{\infty} \alpha \otimes x_{i+l}/p^{i+1},$$

where $\alpha \otimes x_{i+l}$ represents the product of α with x_{i+l}, modulo p. If $x = \sum_{i=0}^{\infty} \beta_i p^i$ is a non-negative integer, then

$$\alpha p^l \circ x = \sum_{i=0}^{\infty} \alpha \otimes \beta_i p^{i+l}$$

where $\alpha \otimes \beta_i$ represents the product of α with β_i, modulo p.

Let n be a fixed positive integer, and denote the set of n-dimensional vectors whose coordinates are non-negative integers by I^n. If $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ belong to I^n, then define the *p-dot product of A and B* by

$$A \circ B = (a_1 \circ b_1) \dot{+} (a_2 \circ b_2) \dot{+} \ldots \dot{+} (a_n \circ b_n);$$

for $x \in [0, 1)$ define the *p-scalar product of x and A* by

$$x \circ A = (a_1 \circ x, a_2 \circ x, \ldots, a_n \circ x).$$

A sequence $\{V_j\}_{j=1}^{\infty} \subseteq I^n$ is said to be *p-normal* if given any non-zero vector $A \in I^n$, we have $A \circ V_j \to +\infty$, as $j \to \infty$.

Finally, let E be a subset of the interval $[0, 1)$ and for any character series $S = \sum_{k=0}^{\infty} a_k \Psi_k$ set

$$S_N(x) = \sum_{k=0}^{N-1} a_k \Psi_k(x), \ x \in [0, 1), \ N = 1, 2, \ldots.$$

The set E is said to be a *p-set of uniqueness* if the only character series S which satisfies $S_N(x) \to 0$ as $N \to \infty$, for $x \in [0, 1) \sim E$, is the zero series. The set E is said to be a *$pH^{(n)}$-set* if there exists an open, connected set $\Delta \subseteq R^n$ and a p-normal sequence $\{V_j\}$ of vectors in I^n such that for all $x \in E$ and for all integers $j \geqq 1$, the point $x \circ V_j$ never belongs to Δ. For the trigonometric analogues of these concepts, see [**6**, p. 346].

In Section 2, we shall sketch proofs of the following two theorems.

THEOREM 1. *Suppose that f is integrable on $[0, 1)$, that Z is a countable subset of $[0, 1)$, and that $S = \sum a_k\Psi_k$ is a character series which satisfies $p^{-m}S_{p^m}(x) \to 0$, as $m \to \infty$, for each $x \in [0, 1)$. If $S_pm(x)$ converges to $f(x)$, as $m \to \infty$, for $x \in [0, 1) \sim Z$, then S is the G-Fourier series of f, i.e.,*

$$a_k = \int_0^1 f(x)\Psi_k(x)dx, \quad \text{for } k = 0, 1, \ldots.$$

THEOREM 2. *Let E be a subset of $[0, 1)$. A sufficient condition that E be a p-set of uniqueness is the existence of a sequence of polynomials on G, say*

$$\lambda_j(x) = \sum_{k=0}^{n_j} c_k^{(j)}\Psi_k(x) \quad j = 1, 2, \ldots.$$

which vanish for $x \in E \sim Z_j$, where Z_j is a countable set $(j = 1, 2, \ldots)$, and whose coefficients satisfy three properties:

(3) $\sum_{k=0}^{n_j} |c_k^{(j)}| \leqq C < \infty \qquad j = 1, 2, \ldots.$

(4) $|c_0^{(j)}| \geqq A > 0 \qquad j = 1, 2, \ldots.$

(5) $\lim_{j\to\infty} c_k^{(j)} = 0 \qquad k = 1, 2, \ldots.$

In both cases, the proofs we outline follow closely those given earlier in the Walsh-Paley case. For Theorem 1, see [**2**]; for Theorem 2, see [**3**].

In Section 3, we shall apply these results to prove the following theorem.

THEOREM 3. *Let E be a subset of $[0, 1)$. If E is countable or if E is a $pH^{(n)}$-set, then E is a p-set of uniqueness.*

In Section 4 we shall discuss specific examples of $2H^{(1)}$-sets, thereby providing the first new perfect sets of uniqueness for Walsh-Paley series since 1949 (see [**3**] and [**5**].)

§ 2. Uniqueness and Localization. For each $x \in [0, 1)$ and each non-negative integer m, we define $\alpha_m(x) = q/p^m$ by insisting that $q \leqq p^m x < q + 1$. We also set $\beta_m(x) = \alpha_m(x) + p^{-m}$ and $\alpha_m'(x) = \alpha_m(x) - p^{-m}$.

Recall that D represents the set of p-rationals in the interval $[0, 1)$. The following lemma is the key to the proof of Theorem 1. It was proved in the special case $p = 2$ in [**2**]. By replacing each occurrence of 2^m by p^m, and by subdividing each interval into p even subintervals instead of halves, the proof in [**2**] can also be used to establish this result:

LEMMA 1. *Let G be a function defined on D which satisfies the following three properties:*

$$\limsup_{m\to\infty} G(\alpha_m'(x)) \geqq G(x) \qquad x \in D;$$
$$\liminf_{m\to\infty} [G(\beta_m(x)) - G(\alpha_m(x))] \leqq 0 \qquad x \in [0, 1);$$
$$\liminf_{m\to\infty} p^m[G(\beta_m(x)) - G(\alpha_m(x))] \leqq 0 \qquad x \in [0, 1) \sim Z,$$

for some countable set Z. Then G is monotone decreasing on D.

The proof of Theorem 1 proceeds as follows: Set

$$F(x) = \int_0^x f(t)dt$$

and, when it exists,

$$L(x) = \sum_{k=0}^{\infty} a_k \int_0^x \Psi_k(t)dt,$$

for $x \in [0, 1)$. Observe that $L(x)$ is defined for each $x \in D$. In fact, since each character Ψ_k is constant on any interval of the form $J = [q/p^m, (q+1)/p^m)$ when $k < p^m$, and satisfies $\int_J \Psi_k(t)dt = 0$ when $k \geqq p^m$, it is the case that

$$(6) \quad L(\beta_m(x)) - L(\alpha_m(x)) = (\beta_m(x) - \alpha_m(x))S_{p^m}(x)$$

for $m = 1, 2, \ldots$ and $x \in [0, 1)$.

Apply the Vitali-Carathéodory Theorem to F, to choose an absolutely continuous function ϕ which uniformly approximates F, and whose derivative is dominated by f. Verify, using (6) and the hypotheses of Theorem 1, that $\phi - L$ satisfies the three conditions in Lemma 1. Hence, $\phi - L$ is monotone decreasing on D. Since ϕ approximates F, it follows that $F - L$ is monotone decreasing on D. By symmetry, $L - F$ is also monotone decreasing on D.

Consequently, $L(x) = \int_0^x f(t)\,dt$ for all $x \in D$. Now, instead of showing that L is essentially absolutely continuous, [**2**], verify directly that S is the G-Fourier series of f. Indeed, fix an integer k and choose p-rationals α_m and β_m such that $\Psi_k(x) = \Psi_k(\alpha_m)$ for $x \in [\alpha_m, \beta_m)$, and so that $[0, 1) = \bigcup_{m=1}^{M} [\alpha_m, \beta_m)$. Then by what we just showed,

$$\int_0^1 f(x)\Psi_k(x)dx = \sum_{m=1}^{M} \int_{\alpha_m}^{\beta_m} f(x)\Psi_k(x)dx = \sum_{m=1}^{M} \Psi_k(\alpha_m) \times [L(\beta_m) - L(\alpha_m)].$$

However, we can choose n_0 so large (see (6)) that

$$L(\beta_m) - L(\alpha_m) = \int_{\alpha_m}^{\beta_m} S_{n_0}(t)dt.$$

Consequently,

$$\int_0^1 f(x)\Psi_k(x)dx = \int_0^1 \Psi_k(t)S_{n_0}(t)dt.$$

Since the functions $\{\Psi_k\}$ are orthonormal, the right hand side reduces to a_k, as required.

The proof of Theorem 2 in the Walsh-Paley case relies heavily on a formal product of polynomials with series.

LEMMA 2. *Let* $\lambda(x) = \sum_{k=0}^{N_0} c_k\Psi_k(x)$ *be a polynomial on* G, *and let* $S(x) = \sum_{k=0}^{\infty} a_k\Psi_k(x)$ *be a character series on* G. *Define a series* λS *by*

$$\lambda S(x) = \sum_{k=0}^{\infty} \tilde{a}_k\Psi_k(x),\ x \in [0, 1),$$

where $\tilde{a}_k = \sum_{l=0}^{N_0} c_l a_{k\dot{-}l}$ *for* $k = 0, 1, \ldots$. *If* $a_k \to 0$, *as* $k \to \infty$, *then* $\tilde{a}_k \to 0$ *and*

(7) $\lim_{m\to\infty}[\lambda S_m(x) - \lambda(x)S_m(x)] = 0$

uniformly on $[0, 1)$.

To prove this lemma we begin with a simple observation. If $k = \sum_{j=0}^{\infty} \beta_j p^j$ is a non-negative integer, with $0 \leqq \beta_j < p$, and if q and N are fixed natural numbers, then a necessary and sufficient condition that $(q - 1)p^N \leqq k < qp^N$ is that

$$\sum_{j=n}^{\infty} \beta_j p^j = (q - 1)p^N.$$

It follows that if k and l are non-negative integers which satisfy $l < p^N$ and $(q - 1)\, p^N \leqq k < qp^N$, then

(8) $(q - 1)p^N \leqq k \dot{+} l < qp^N$.

In particular, since $\dot{-}\, l = l \dot{+} l \dot{+} \ldots \dot{+} l$ ($(p - 1)$ − terms), we see that $k \dot{-} l \to \infty$ as $k \to \infty$, for each integer $l \geqq 0$. Thus $\tilde{a}_k \to 0$ as $k \to \infty$ because $a_k \to 0$ as $k \to \infty$.

To show that (7) holds, fix N so large that $\dot{-}\, l < p^N$ for all $l < N_0$, and fix $x \in [0, 1)$. By (8), if $l < N_0$ then

$$\sum_{k=0}^{qp^N-1} a_{k\dot{-}l}\Psi_{k\dot{-}l}(x) = \sum_{k=0}^{qp^N-1} a_k\Psi_k(x).$$

Since $\Psi_{k\dot{-}l}(x)\Psi_l(x) = \Psi_k(x)$ for all integers $k, l \geqq 0$, we therefore obtain the following identity:

$$\lambda S_{qp^N}(x) = \lambda(x)S_{qp^N}(x)$$

for $q = 1, 2, \ldots$.

Let m be a positive integer. Choose a non-negative integer q which satisfies $qp^N \leqq m < (q + 1)p^N$. By the identity derived in the preceeding paragraph, we have

$$\lambda S_m(x) - \lambda(x)S_m(x) \equiv \sum_{k=qp^N}^{m-1} \tilde{a}_k\Psi_k(x) - \lambda(x) \sum_{qp^N}^{m-1} a_k\Psi_k(x).$$

In particular,

$$|\lambda S_m(x) - \lambda(x)S_m(x)| \leqq p^N\{\sup_{k\geqq qp^N}|\tilde{a}_k| + \|\lambda\|_\infty \sup_{k\geqq qp^N}|a_k|\}.$$

Since both a_k and $\tilde{a}_k$ tend to zero as $k \to \infty$, we have verified (7), and thus have completed the proof of Lemma 2.

To prove Theorem 2, let $S = \sum a_k \Psi_k$ be a character series which converges to zero off E. Fix an integer j, and consider the product $\lambda_j S$. By Lemma 2, the assumption concerning S, and the hypothesis concerning the vanishing of λ_j, the Walsh series $\lambda_j S$ converges to zero off the countable set Z_j. Hence by Theorem 1, the coefficients of $\lambda_j S$ must vanish. By writing down the explicit formula for those coefficients, as given by Lemma 2, we are therefore lead to the equation

$$a_k = (-1/c_0^{(j)}) \sum_{l=0}^{n_j} c_l^{(j)} a_{k \dot{-} l}$$

for $k = 0, 1, \ldots$. By using (3), (4) and (5) to estimate this sum, for large j, one can easily show that $a_k = 0$ for $k = 0, 1, \ldots$. In particular, S is the zero series, as required.

§ 3. A proof of theorem 3. Suppose first that E is countable. Observe, since every p-rational x has a p-adic expansion which terminates in zeros, that

$$S(x) = \sum_{k=0}^{\infty} a_k \Psi_k(x)$$

which converges at a p-rational, necessarily satisfies $a_k \to 0$ as $k \to \infty$. It follows that such a series also satisfies $p^{-m} S_{p^m}(x) \to 0$ as $m \to \infty$, for each $x \in [0, 1)$. Consequently, Theorem 1 proves that E is a p-set of uniqueness.

Suppose that E is a $pH^{(n)}$-set. That is to say, suppose that there is an open, connected set $\Delta \subseteq R^n$ and a p-normal sequence $\{V_j\}_{j=1}^{\infty} \subseteq I^n$ such that for all $x \in E$ and for all integers $j \geqq 1$, the point $x \circ V_j$ never belongs to Δ. For simplicity, we suppose that $n = 2$, and set $V_j = (a_j, b_j)$ for $j = 1, 2, \ldots$. We may suppose that $\Delta = J_1 \times J_2$, where each J_i is a subinterval of $[0, 1)$ with p-rational endpoints, say $J_i = [\alpha_i, \beta_i)$.

Denote, for $i = 1$ and 2, the characteristic function of the interval J_i by μ_i, and observe that μ_i is a polynomial on G, say

$$\mu_1(x) = \sum_{m=0}^{M} \gamma_m \Psi_m(x)$$

and

$$\mu_2(x) = \sum_{l=0}^{L} \delta_l \Psi_l(x).$$

We intend to show that the functions $\lambda_j(x) = \mu_1(a_j \circ x)\mu_2(b_j \circ x)$, $j = 1, 2, \ldots$, satisfy the hypotheses of Theorem 2 with respect to E, thereby showing that E is a p-set of uniqueness. Above all, we need to be sure that each λ_j is a polynomial.

LEMMA 3. *Suppose that m and k are non-negative integers. Then* $\Psi_k(m \circ x) = \Psi_{m \circ k}(x)$ *for* $x \in [0, 1)$.

To verify this lemma, we begin by observing that by (2), and the definition of $\alpha p^l \circ k$, the following formula subsists for $x = \sum_{i=0}^{\infty} x_i/p^{i+1}$ and for non-negative integers N, l, and α, with $0 \leqq \alpha < p$:

$$\phi_N(\alpha p^l \circ x) = \exp\left(2\pi i \alpha \otimes x_{N+l}/p\right).$$

But $\exp(2\pi i) = 1$, so we can replace the product modulo p by αx_{N+l}. Hence by (1), and the definition of $\alpha p^l \circ p^N$, we obtain

$$\phi_N(\alpha p^l \circ x) = \Psi_{\alpha p^l \circ p^N}(x).$$

Hence the lemma holds in the special case when $k = p^N$ and $m = \alpha p^l$. In the case when $k = \sum_{i=0}^{\infty} \beta_i p^i$ but $m = \alpha p^l$, we have by (1) that

$$\Psi_k(\alpha p^l \circ x) = \prod_{i=0}^{\infty} \phi_i^{\beta_i}(\alpha p^l \circ x).$$

By the previous case, then,

$$(9) \qquad \Psi_k(\alpha p^l \circ x) = \prod_{i=0}^{\infty} \phi_{i+l}^{\alpha\beta_i}(x).$$

According to (1) and the definition of $\alpha p^l \circ k$, the right hand side of (9) is identical to $\Psi_{\alpha p^l \circ k}(x)$, as required. Finally, if $m = \sum_{i=0}^{\infty} \alpha_i p^i$ then by definition of $m \circ x$ and (3), we have

$$\Psi_k(m \circ x) = \Psi_k(\alpha_0 \circ x)\Psi_k(\alpha_1 p \circ x) \dots$$

By the preceeding case, and equation (4), this leads directly to $\Psi_k(m \circ x) = \Psi_{m \circ k}(x)$, and thus establishes the lemma.

We are now prepared to verify that the functions λ_j satisfy the hypotheses of Theorem 2.

For the time being, let j be fixed. Since each μ_i is the characteristic function of J_i $(i = 1, 2)$ and since $x \in E$ implies that $(a_j \circ x, b_j \circ x) \notin J_1 \times J_2$, it is clear that $\lambda_j(x) = 0$ for $x \in E$.

Next, by Lemma 3, we know that

$$\mu_1(a_j \circ x) = \sum_{m=0}^{M} \gamma_m \Psi_{a_j \circ m}(x)$$

and

$$\mu_2(b_j \circ x) = \sum_{l=0}^{L} \delta_l \Psi_{b_j \circ l}(x),$$

for $x \in [0, 1)$. Hence

$$\lambda_j(x) \equiv \sum_{m=0}^{M} \sum_{l=0}^{L} \gamma_m \delta_l \Psi_{a_j \circ m \dot{+} b_j \circ l}(x)$$

is a polynomial on G. In fact, using the notation of Theorem 2, we see that

$$(10) \quad c_k^{(j)} = \sum \{\gamma_m \delta_l : k = a_j \circ m + b_j \circ l\}$$

for $k = 0, 1, \dots$.

Condition (3) is therefore satisfied since

$$\sum_{k=0}^{\infty} |c_k^{(j)}| \leqq \sum_{m=0}^{M} |\gamma_m| \cdot \sum_{l=0}^{L} |\delta_l| < \infty.$$

To verify condition (4) for large j, which is all that is required, we set

$$T = \sum \{\gamma_m \delta_l : 0 = a_j \circ m \dot{+} b_j \circ l \text{ but } |m| + |n| \neq 0\},$$

and observe that since (a_j, b_j) is p-normal, the sum T is empty for large j.

However, by (10), $c_0^{(j)} = \gamma_0\delta_0 + T$. Since $\gamma_0 = m(J_1)$ and $\delta_0 = m(J_2)$ are both positive, we see that $c_0^{(j)} = \gamma_0\delta_0 > 0$, for large j.

Condition (5) is similarly verified. Indeed, if $k = a_j \circ m \dot{+} b_j \circ l$ is non-zero, then the vector (m, l) is necessarily non-zero. For such vectors (m, l), however, we have

$$a_j \circ m \dot{+} b_j \circ l \rightarrow \infty \text{ as } j \rightarrow \infty.$$

It follows from (10) that $c^{(j)}$ is identically zero, for j large. This completes the proof that the functions λ_j satisfy the hypotheses of Theorem 2, and, therefore, that E is a p-set of uniqueness.

§ **4. Examples.** It is clear (see Example 1 below) that the Cantor set $C(1/p)$ is a $pH^{(1)}$-set, and thus a p-set of uniqueness, for each prime $p \geqq 2$. However, it seems difficult to decide whether $C(1/q)$ is a $pH^{(1)}$-set when $p \neq q$. In particular, a problem open since 1949 [**3**] is that of determining whether the usual Cantor set $C(1/3)$ is a set of uniqueness for Walsh-Paley series.

We close with some examples for the case $p = 2$. We shall abbreviate "$2H^{(1)}$-set" by "$\dot{H}$-set". Sneider [**3**] has shown that the set $C(1/2)$ is a set of uniqueness for Walsh-Paley series. Our first example shows that this result follows from Theorem 3.

(1) Let C_1 denote the set whose complement is given by the union of intervals of the form $(1/4, 3/4)$; $(1/16, 3/16)$, $(13/16, 15/16)$; $(1/64, 3/64)$, $(13/64, 15/64)$, $(49/64, 51/64)$; $(61/64, 63/64)$; .... It is clear that the dyadic expansion of a point in the complement of C_1 consists of n pairs of 0's of 1's ($n \geqq 0$) followed by a 01 or a 10. It follows that a necessary and sufficient condition for a point $x = \sum_{i=0}^{\infty} x_i/2^{i+1}$ to belong to C_1 is that $x_{2j+1} = x_{2j}$ for $j = 1, 2, \ldots$. Thus, if $n_j = 2^{2j} + 2^{2j+1}$ for $j = 0, 1, \ldots$, then $n_j \circ x \notin (1/2, 1)$ for $x \in C_1$ and $j \geqq 0$. In particular, C_1 is an $\dot{H}$-set.

Minor variations on this technique can be used to show that each of the following sets is an $\dot{H}$-set. Note that C_2 contains C_1, and that C_3 and C_4 are unsymmetric.

(2) $C_2 = \{x = \sum_{i=0}^{\infty} x_i/2^{j+1}$: for each $j = 0, 1, \ldots$, the set $\{x_{4_j+1}, x_{4_j+2}\}$ contains an even (possibly 0) number of 1's$\}$. The complement of C_2 is the union of intervals $(1/4, 3/4)$; $(1/64, 3/64)$, $(5/64, 7/64)$, $(9/64, 11/64)$, $(13/64, 15/64)$, $(49/64, 51/64)$, $(53/64, 55/64)$, $(57/64, 59/64)$, $(61/64, 63/64)$; $(1/1024, 3/1024)$, ...

(3) $C_3 = \{x = \sum_{i=1}^{\infty} x_i/2^{i+1}$: for each integer $j \geqq 0$, the set $\{x_{3_j+1}, x_{3_j+2}, x_{3_j+3}\}$ contains an even (possibly zero) number of 1's$\}$. The complement of C_3 is the union of intervals $(1/8, 3/8)$, $(4/8, 5/8)$, $(7/8, 1)$; $(1/64, 3/64)$, $(4/64, 5/64)$, $(7/64, 8/64)$, $(25/64, 27/64)$, $(28/64, 29/64)$, $(31/64, 32/64)$, $(41/64, 43/64)$, $(44/64, 45/64)$, $(47/64, 48/64)$, $(49/64, 51/64)$, $(52/64, 53/64)$, $(55/64, 56/64)$; ....

(4) $C_4 = \{\sum_{i=0}^{\infty} x_i/2^{i+1}$: for each integer $j \geqq 0$, the set $\{x_{4_j+1}, x_{4_j+2}, x_{4_j+3}, x_{4_j+4}\}$ always contains an odd number of 1's$\}$. The complement of C_4 is the union of intervals $(0, 1/16)$, $(3/16, 4/16)$, $(5/16, 7/16)$, $(9/16, 10/16)$, $(11/16, 13/16)$, $(15/16, 1)$; $(16/256, 17/256)$, $(19/256, 20/256)$, $(21/256, 23/256), \ldots; \ldots.$

References

1. N. J. Fine, *On the Walsh functions*, Trans, A. M. S. *65* (1949), 372–414.
2. R. J. Lindahl, *A differentiation theorem for functions defined on the dyadic rationals*, Proc. A. M. S. *30* (1971), 349–352.
3. A. Šneider, *On uniqueness of expansions in Walsh functions*, Mat. Sbornik N. S. *24* (1949), 379–400.
4. M. H. Taibleson, *Fourier analysis on local fields* (Mathematical Notes, Princeton University Press, Princeton, 1975).
5. W. R. Wade, *Uniqueness and α-capacity on the group 2^ω*, Trans. A. M. S. *208* (1975), 309–315.
6. A. Zygmund, *Trigonometric series* (2nd ed. Vol. I, Cambridge University Press, New York, 1959).

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