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SOME FINITENESS CONDITIONS FOR ORTHOMODULAR LATTICES

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Throughout this paper L will be an orthomodular lattice and $\mathfrak{A}(L)$ the set of all maximal Boolean subalgebras, also called blocks [**4**], of L. For every $x \in L$, $C(x)$ will be the set of all elements of L which commute with x. Let $n \geqq 1$ be a natural number. In this paper we consider the following conditions for L:

A_n: L has at most n blocks,
B_n: there exists a covering of L by at most n blocks,
C_n: the set $\{C(x) | x \in L\}$ has at most n elements,
D_n: out of any $n + 1$ elements of L at least two commute.

We also consider quantified versions of these statements, namely the statements A, B, C, D defined by: $A \Leftrightarrow \exists\, nA_n$, $B \Leftrightarrow \exists\, nB_n$, $C \Leftrightarrow \exists\, nC_n$ and $D \Leftrightarrow \exists\, nD_n$. Thus A is the statement that L has only finitely many blocks, B is the statement that L can be covered by finitely many blocks etc.

It is our conjecture that the conditions A, B, C, D are pairwise equivalent but we have not been able to prove this completely. We have, indeed, the stronger conjecture that they imply each other "uniformly" in the following sense: If X and Y stand for two of A, B, C, D then for every natural number n there exists a natural number m such that every L satisfying X_n also satisfies Y_m. We prove in this paper that the conditions A and C and the conditions B and D are uniformly equivalent in this sense. Since A_n trivially implies B_n the only question left open is whether B implies A, uniformly or not. The only things we have been able to prove regarding this question are the implications $B_1 \Rightarrow A_1$, $B_2 \Rightarrow A_2$, $B_3 \Rightarrow A_4$, $B_4 \Rightarrow A_5$.

For general background information regarding orthomodular lattices the reader is referred to [**3**] and [**5**]. Throughout the paper we abbreviate "orthomodular lattice" as OML.

1. The equivalence of A and C. If a is an element of L then it is well known (and easy to prove) that the blocks of the subalgebra $C(a)$ are exactly those blocks of L which contain a. It follows from this that

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whenever an OML L satisfies A_n it also satisfies C_m with $m = 2^n - 1$. This value of m is the smallest possible for $n = 1, 2$. But it follows easily from Section 6 of [**2**] that A_3 implies C_6. Thus for $n = 3$ the value $m = 2^n - 1$ is not the best possible. We suspect that for higher values of n the bound can be improved considerably, but we have not been able to prove this.

To prove the converse we define for a subset M of L as usual $C(M)$ to be the set of all elements of L which commute with every element of M. We furthermore define

$$\sigma(L) = \{C(F) \mid \emptyset \neq F \subseteq L, F \text{ finite}\}.$$

As a first step we show:

PROPOSITION 1.1. *If* $|\sigma(L)| \leqq n$ *then* L *satisfies* A_m *with* $m = (n - 1)!$.

Proof. We prove the cases $n = 1, 2$ first and the rest by induction. If $|\sigma(L)| = 1$ then $C(x) = C(0) = L$ holds for every element x of L, hence every element of L is central, i.e., L is Boolean and satisfies A_1. If L has at least two blocks B_1 and B_2 and if $a \in B_1 - B_2, b \in B_2 - B_1$ then the sets $C(a)$, $C(b)$ and $C(0)$ are all different, i.e., $|\sigma(L)| \geqq 3$. It follows that $|\sigma(L)| = 2$ is impossible, which trivially proves the claim for $n = 2$. Assume now $n \geqq 3$ and $|\sigma(L)| \leqq n$. Let $a_1, a_2, \ldots, a_k$ be non-central elements such that $C(a_i) \neq C(a_j)$ if $i \neq j$ and such that for every non-central element x of L, $C(x)$ equals one of the $C(a_i)$. Note that $C(a_i) \neq C(0)$ $(i = 1, 2, \ldots, k)$ and hence that $k \leqq n - 1$ holds. We show that for each of the subalgebras $C(a_i)$, $\sigma(C(a_i))$ is a proper subset of $\sigma(L)$. It clearly is a subset. But since $L = C(0) \in \sigma(L)$ and $L \notin \sigma(C(a_i))$ it is a proper subset. By inductive hypothesis $C(a_i)$ has at most $(n - 2)!$ blocks. Now let B be an arbitrary block of L. Since we may assume that L is not Boolean the block B contains a non-central element x and hence is a block of one of the $C(a_i)$. It follows that L has at most $(n - 1)!$ blocks.

If L satisfies C_n then clearly $|\sigma(L)| \leqq 2^n - 1$. From the proposition and the considerations above we thus obtain:

THEOREM 1. A_n *implies* C_m *with* $m = 2^n - 1$ *and* C_n *implies* A_m *with* $m = (2^n - 2)!$. *In particular the conditions* A *and* C *are uniformly equivalent.*

The question which is the smallest possible value of m in the implication is again open.

Since A_n trivially implies B_n we obtain from Theorem 1 in particular that C_n implies B_m with $m = (2^n - 2)!$. The following proof, however, gives a much better bound.

PROPOSITION 1.2. C_1 *implies* B_1 *and* C_n *implies* B_{n-1} *but not* B_{n-2} *if* $n \geqq 2$.

Proof. C_1 implies A_1 and hence B_1 by Theorem 1. Assume now that L satisfies C_n with $n \geqq 2$. Define an equivalence relation $\sim$ in L by

$$x \sim y \Leftrightarrow C(x) = C(y).$$

Since there are at most n sets $C(x)$ there are at most n equivalence classes. But $x \sim y$ implies that $x \in C(x) = C(y)$, hence that x commutes with y. It follows that every equivalence class is contained in a block and hence that L can be covered by n blocks. But $x \sim 0$ is equivalent with x being central. Thus the equivalence class of 0 is contained in every block. Since we may assume that there are at least two equivalence classes it follows that L can be covered by $n - 1$ blocks. To see that this bound is best possible consider the OML consisting of $2n - 2$ pairwise incomparable elements and the bounds. It satisfies C_n but not B_{n-2}.

2. The equivalence of B **and** D**.** Clearly B_n implies D_n. To show that conversely D uniformly implies B we start out with a lemma which will turn out to be useful not only in the present context. In the proof of the following lemma and later on in this paper we will make use of the Boolean ring sum $a + b = (a \wedge b') \vee (a' \wedge b)$. This operation is, of course, not associative in an arbitrary orthomodular lattice. Whenever we use the associative law in the following we do so because the computations take place in a Boolean subalgebra of L.

LEMMA 2.1. *If L satisfies D_n and if $a_1, a_2, \ldots, a_n$ are pairwise not commuting elements of L then*

$$C(a_1, a_2, \ldots, a_n) = C(L).$$

Proof. Clearly $C(L) \subseteq C(a_1, a_2, \ldots, a_n)$. To show the inverse inclusion assume $x \in C(a_1, a_2, \ldots, a_n)$. Note first that, for distinct indices i and j, $a_i + x$ commutes with neither a_j nor $a_j + x$. In fact the relation $a_i + x \, \mathrm{C} \, a_j$ would imply

$$a_i = (a_i + x) + x \, \mathrm{C} \, a_j,$$

a contradiction. Similarly $a_i + x \, \mathrm{C} \, a_j + x$ would imply

$$a_i + x \, \mathrm{C} (a_j + x) + x = a_j,$$

which we have seen not to be the case. To show $x \in C(L)$ let y be an arbitrary element of L. We have to show that $x \, \mathrm{C} \, y$. Consider now the set $Z = \{z_1, z_2, \ldots, z_{n+1}\}$ defined by

$$z_i = \begin{cases} a_i + x & \text{if } i \leqq n \text{ and } a_i \, \mathrm{C} \, y \\ a_i & \text{if } i \leqq n \text{ and } a_i \not\mathrm{C}_o y \\ y & \text{if } i = n + 1. \end{cases}$$

Since L satisfies D_n at least two elements in Z commute. By what we have

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shown it follows that there exists an index i with $a_i \mathrm{C}\, y$ and $a_i + x \mathrm{C}\, y$. It follows

$$x = a_i + (a_i + x) \mathrm{C}\, y,$$

which was to be proved.

PROPOSITION 2.2. *D_n implies B_n if $1 \leqq n \leqq 3$.*

Proof. D_1 trivially implies that L is Boolean and hence satisfies B_1.

Assume now that L satisfies D_2 and let a_1, a_2 be non-commuting elements of L. Define

$$A_i = \{x \in L | \, x \mathrm{C}\, a_j\} \quad \text{for} \quad \{i, j\} = \{1, 2\}.$$

Clearly A_1 and A_2 consist of pairwise commuting elements and hence the the subalgebras ΓA_i generated by them are Boolean. To show the claim it is obviously enough to show that every $x \in L$ belongs to one of the ΓA_i. This is trivially true if x commutes with exactly one of the a_i. Since L satisfies D_2 the only other possibility is that x commutes with both a_1 and a_2. But in this case $a_1 + x \mathrm{C}\, a_2$ would imply

$$a_1 = (a_1 + x) + x \mathrm{C}\, a_2,$$

a contradiction. We thus obtain

$$a_1 + x \in A_1 \quad \text{and} \quad x = a_1 + (a_1 + x) \in \Gamma A_1,$$

proving the claim for $n = 2$.

To prove the claim for $n = 3$ is considerably more complicated. We again start out with three pairwise not commuting elements a_1, a_2, a_3 and define

$$A_i = \{x \in L | \, x \not\mathrm{C}\, a_j \quad \text{for all } j \neq i\}.$$

As before, the sets A_i consist of pairwise commuting elements and hence the subalgebras ΓA_i generated by them are Boolean. We show first:

(1) $C(a_1, a_2, a_3) \subseteq \Gamma A_1 \cap \Gamma A_2 \cap \Gamma A_3$.

By symmetry it is enough to show that $C(a_1, a_2, a_3) \subseteq \Gamma A_1$. But, as before, if $x \in C(a_1, a_2, a_3)$ we have $a_1 + x \in A_1$ hence and $x = a_1 + (a_1 + x) \in \Gamma A_1$.

To show the claim it is obviously enough to show that every $x \in L$ belongs to at least one of the ΓA_i. This is trivially true if x commutes with exactly one of the a_i and is true by (1) if it commutes with all the a_i. By symmetry it is thus enough to show that

(2) $C(a_1, a_2) \subseteq \Gamma A_1 \cup \Gamma A_2$.

Assume $x \in C(a_1, a_2)$. We have to show that $x \in \Gamma A_1 \cup \Gamma A_2$. Since by an earlier argument $a_1 + x \not\mathrm{C}\, a_2$ and $a_2 + x \not\mathrm{C}\, a_1$ the claim follows easily

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if $a_1 + x \not\mathrm{C}\, a_3$ or $a_2 + x \not\mathrm{C}\, a_3$. We may thus assume without loss of generality that

$$(3) \quad \begin{cases} (a_1 \vee x) \wedge (a_1' \vee x') \mathrel{C} a_3 \\ (a_1 \vee x') \wedge (a_1' \vee x) \mathrel{C} a_3 \\ (a_2 \vee x) \wedge (a_2' \vee x') \mathrel{C} a_3 \\ (a_2 \vee x') \wedge (a_2' \vee x) \mathrel{C} a_3 \end{cases}$$

From (3) and the orthomodular law we obtain:

$$(4) \quad \begin{cases} a_1 \vee x \mathrel{C} a_3 \Leftrightarrow a_1' \vee x' \mathrel{C} a_3 \\ a_1 \vee x' \mathrel{C} a_3 \Leftrightarrow a_1' \vee x \mathrel{C} a_3 \\ a_2 \vee x \mathrel{C} a_3 \Leftrightarrow a_2' \vee x' \mathrel{C} a_3 \\ a_2 \vee x' \mathrel{C} a_3 \Leftrightarrow a_2' \vee x \mathrel{C} a_3 \end{cases}$$

Assume now first that at least one of $a_1 \vee x \mathrel{C} a_3$ or $a_1 \vee x' \mathrel{C} a_3$ and at least one of $a_2 \vee x \mathrel{C} a_3$ or $a_2 \vee x' \mathrel{C} a_3$ hold. We assume that

$$a_1 \vee x \mathrel{C} a_3 \quad \text{and} \quad a_2 \vee x \mathrel{C} a_3.$$

The remaining cases follow similarly. By (4) we have

$$(a_1 \wedge a_2) \vee x \mathrel{C} a_3 \quad \text{and} \quad (a_1' \wedge a_2') \vee x' \mathrel{C} a_3.$$

Since $(a_1 \wedge a_2) \vee x$ and $(a_1' \wedge a_2') \vee x'$ clearly commute with a_1 and a_2 we obtain from (1) that $(a_1 \wedge a_2) \vee x$ and $(a_1' \wedge a_2') \vee x'$ belong to ΓA_1. It follows that

$$a_1' \wedge x = a_1' \wedge ((a_1 \wedge a_2) \vee x) \in \Gamma A_1 \quad \text{and}$$
$$a_1 \wedge x' = a_1 \wedge ((a_1' \wedge a_2') \vee x') \in \Gamma A_1,$$

hence that

$$a_1 + x = (a_1 \wedge x') \vee (a' \wedge x) \in \Gamma A_1 \quad \text{and}$$
$$x = a_1 + (a_1 + x) \in \Gamma A_1.$$

We may thus assume by symmetry that none of the elements $a_1 \vee x$, $a_1 \vee x'$, $a_1' \vee x$, $a_1' \vee x'$ commutes with a_3. If $a_1 \vee x \not\mathrm{C}\, a_2$ we also have $a_1' \vee x \not\mathrm{C}\, a_2$ since $a_1' \vee x \mathrel{C} a_2$ would imply

$$a_1' \wedge x' = x' \wedge (a_1' \vee x) \mathrel{C} a_2$$

and hence $a_1 \vee x \mathrel{C} a_2$. It follows that $a_1 \vee x, a_1' \vee x \in A_1$ and hence that

$$x = (a_1 \vee x) \wedge (a_1' \vee x) \in \Gamma A_1.$$

We are thus left with the case that $a_1 \vee x \mathrel{C} a_2$. Since

$$a_1 = (a_1 \vee x) \wedge (a_1 \vee x') \not\mathrm{C}\, a_2$$

we then have $a_1 \vee x' \not\mathrm{C}\, a_2$ and, as before, $a_1' \vee x' \not\mathrm{C}\, a_2$. It follows that

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$a_1 \vee x', a_1' \vee x' \in A_1$ and that

$$x = ((a_1 \vee x') \wedge (a_1' \vee x'))' \in \Gamma A_1,$$

completing the proof.

We suspect that Proposition 2.2 is true without any restriction on the number n, but we have not been able to prove this. We establish the uniform equivalence of B and D by proving:

THEOREM 2. *B_n implies D_n and D_n implies B_m with $m = \frac{1}{2}n!$ for $n \geqq 3$. In particular the conditions B and D are uniformly equivalent.*

Proof. The first claim is obvious and has already been mentioned. We prove the second claim by induction on n. For $n = 3$ the claim is contained in Proposition 2.2. Assume that $n \geqq 4$ and that L satisfies D_n but does not satisfy D_{n-1}. Let $a_1, a_2, \ldots, a_n$ be pairwise non-commuting elements of L. We claim that each of the subalgebras $C(a_i)$ satisfies D_{n-1}. If this was not the case there would exist n pairwise non-commuting elements $x_1, x_2, \ldots, x_n$ in $C(a_i)$. By Lemma 2.1 it would follow that

$$a_i \in C(x_1, x_2, \ldots, x_n) = C(L),$$

which is impossible since any two of the a_i do not commute. It follows by inductive hypothesis that each of the subalgebras $C(a_i)$ can be covered by at most $\frac{1}{2}(n - 1)!$ blocks. Since $L = \bigcup_{i=1}^{n} C(a_i)$ it follows that L can be covered by at most $\frac{1}{2}n!$ blocks, completing the proof.

It is clear that the inductive argument used in the proof of Theorem 2 yields the bound $m = n!$ without the elaborate considerations in the proof of Proposition 2.2. But since we feel that the suspected bound $m = n$ might be obtainable by a refinement of the arguments used in the proof of Proposition 2.2 (which we could not find) we felt justified in giving this proof.

3. The implications of B, preliminaries. As we have mentioned before, we have not been able to settle the question whether B implies A, uniformly or not. We develop in this section some preliminary material which will be helpful in the next section to prove the known implications of B_n.

PROPOSITION 3.1. *If $B_1, B_2, \ldots, B_n$ are Boolean subalgebras of an* OML *L, if $L = B_1 \cup B_2 \cup \ldots \cup B_n$ and if $a \in (B_1 \cap B_2 \cap \ldots \cap B_{n-1}) - B_n$ then $C(a) = B_1 \cup B_2 \cup \ldots \cup B_{n-1}$, and in particular $B_1 \cup B_2 \cup \ldots \cup B_{n-1}$ is a subalgebra of L.*

Proof. Clearly $B_1 \cup B_2 \cup \ldots \cup B_{n-1} \subseteq C(a)$. To show the converse suppose that there is an element $x \in C(a)$, $x \in B_n - (B_1 \cup B_2 \cup \ldots \cup B_{n-1})$. Then $a + x \in B_i$ for some $i \leqq n - 1$ would imply $x = a +$

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$(a + x) \in B_i$, and $a + x \in B_n$ would imply $a = (a + x) + x \in B_n$, both contradictions. It follows that no such element x exists and hence that $C(a) \subseteq B_1 \cup B_2 \cup \ldots \cup B_{n-1}$, completing the proof.

In Section 2 of [**1**] it was shown that if there exists a finite subset F of L such that $C(F) = C(L)$ then L is the direct product of a Boolean algebra and an OML without non-trivial Boolean factors. Since B_n trivially implies D_n, Lemma 2.1 gives:

PROPOSITION 3.2. *Every* OML *satisfying* B *is the direct product of a Boolean algebra and an* OML *without non-trivial Boolean factor.*

It is well known that a Boolean algebra is never the union of two proper subalgebras. We need here some results about the way a Boolean algebra can be represented as the union of three or four subalgebras. We say that a Boolean algebra B is the irredundant union of n subalgebras $B_1, B_2, \ldots, B_n$ if and only it if is the union of all the B_i but is not the union of $n - 1$ of them. The following proposition is probably well known.

PROPOSITION 3.3. *If a Boolean algebra* B *is the irredundant union of three subalgebras* B_1, B_2, B_3 *and if* $B_1 \cap B_2 \cap B_3 = \{0, 1\}$ *then* B *is an eight-element Boolean algebra and each* B_i *is a four-element Boolean algebra.*

Proof. Let $\{i, j, k\} = \{1, 2, 3\}$. Since $B_i \cap B_j \not\subseteq B_k$ would by Proposition 3.1 imply that $B = B_i \cup B_j$, which is impossible, the assumptions of 3.3 imply that

$$B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{0, 1\}$$

and hence that

$$B = (B_1 - (B_2 \cup B_3)) \cup (B_2 - (B_1 \cup B_3)) \cup (B_3 - (B_1 \cup B_2)) \cup \{0, 1\}.$$

Pick $x_i \in B_i - (B_j \cup B_k)$. Then $x_1 + x_2 \in B_1$ would imply

$$x_2 = x_1 + (x_1 + x_2) \in B_1,$$

a contradiction. We thus have $x_1 + x_2 \notin B_1$ and, by symmetry, also $x_1 + x_2 \notin B_2$. It follows that $x_1 + x_2 \in B_3$ and hence that $x_1 + x_2 + x_3 \in B_3$. By symmetry we obtain

$$x_1 + x_2 + x_3 \in B_1 \cap B_2 \cap B_3 = \{0, 1\},$$

which shows that for each of the x_i there are at most two choices and hence proves the proposition.

Assume now for the remainder of this section that a Boolean algebra B is the irredundant union of four subalgebras B_1, B_2, B_3, B_4 and that $B_1 \cap B_2 \cap B_3 \cap B_4 = \{0, 1\}$. We say that the decomposition of B as the union is of *first kind* if and only if there exists a three-element subset I of $\{1, 2, 3, 4\}$ such that $B_i \cap B_j \neq \{0, 1\}$ if $i, j \in I$ and $B_i \cap B_j = \{0, 1\}$ if $\{i, j\} \not\subseteq I$ and $i \neq j$,

second kind if and only if exactly one of the intersections $B_i \cap B_j$ is different from $\{0, 1\}$.

To simplify notation let us introduce some abbreviations. We define

$$B_i^- = B_i - (B_j \cup B_k \cup B_l) \quad \text{and}$$

$$B_{ij}^- = (B_i \cap B_j) - (B_k \cup B_l) \quad \text{whenever } \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

PROPOSITION 3.4. *Under the assumptions described the decomposition of B is either of first or of second kind. In both cases B has 2^4 elements. In the first case each of the sets B_i^- $(1 \leqq i \leqq 4)$, B_{ij}^- $(i, j \in I, i \neq j)$ has two elements. In the second case, if $B_i \cap B_j \neq \{0, 1\}$ and k, l are the remaining indices, then each of the sets B_{ij}^-, B_k^-, B_l^- has two elements and each of the sets B_i^-, B_j^- has four elements.*

Proof. Since every finitely generated Boolean algebra is finite, it is enough to prove the proposition under the assumption that B is finite. A simple counting argument then shows that not all the intersections $B_i \cap B_j$ can be equal $\{0, 1\}$. It follows from Proposition 3.1 that the intersection of any three of the B_i equals $\{0, 1\}$ and in particular that $B_i \cap B_j = \{0, 1\}$ is equivalent with $B_{ij}^- = \emptyset$. Assume now first that exactly one of the intersections $B_i \cap B_j$ is different from $\{0, 1\}$, say that $B_1 \cap B_2 \neq \{0, 1\}$ and $B_i \cap B_j = \{0, 1\}$ whenever $\{i, j\} \neq \{1, 2\}$, $i \neq j$. Pick $x \in B_{12}^-$, $x_i \in B_i^-$. Then, as in the proof of Proposition 3.3,

$$x + x_3 + x_4 \in B_3 \cap B_4 = \{0, 1\}.$$

It follows that for each of x, x_3, x_4 there are exactly two choices and hence that

$$|B_{12}^-| = |B_3^-| = |B_4^-| = 2.$$

Also, again by the same argument,

$$x_1 + x_2 = (B_3 \cup B_4) - (B_1 \cup B_2).$$

It follows that B_1^- and B_2^- have at most four elements each. A simple counting argument shows that they have exactly four elements, i.e., that B has 2^4 elements and the decomposition is of second kind.

Note next that if $B_i \cap B_j \neq \{0, 1\}$ and if k, l are the remaining indices, then $B_k \cap B_l = \{0, 1\}$. This follows from the fact that $B_i \cap B_j \neq \{0, 1\} \neq B_k \cap B_l$ would imply the existence of elements $x \in B_{ij}^-$, $y \in B_{kl}^-$. As in the proof of Proposition 3.1 it would follow that $x + y$ does not belong to any of the B_i. Using this remark and what we have already

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proved we may assume by symmetry that

$$B_1 \cap B_2 \neq \{0, 1\} \neq B_1 \cap B_3.$$

Pick $x \in B_{12}^-$, $x_i \in B_i^-$. Then, as before,

$$x + x_3 + x_4 \in B_3 \cap B_4 = \{0, 1\}.$$

It follows that $|B_{12}^-| = |B_3^-| = |B_4^-| = 2$ and, by symmetry, also that $|B_{13}^-| = |B_2^-| = 2$. But

$$2 = |B_3^-| = |B_3| - |B_1 \cap B_3| - |B_2 \cap B_3| + 2$$

implies because $|B_1 \cap B_3| < B_3$ and $|B_2 \cap B_3| < |B_3|$ that

$$|B_2 \cap B_3| = |B_1 \cap B_3| = 4 \quad \text{and} \quad |B_{23}^-| = 2.$$

This gives in particular that

$$B_1 \cap B_4 = \{0, 1\}.$$

But $x_1 + x_2 \in (B_3 \cup B_4) - (B_1 \cup B_2)$ implies as before that $|B_1^-| \leqq 4$. A simple counting argument shows that $|B_1^-| = 2$, that B has 2^4 elements and that the decomposition is of first kind. This completes the proof.

It would be interesting to know whether a similar result holds for decompositions by more than four subalgebras.

4. The implications of B**.** The result of this section is:

THEOREM 3. *B_1 implies A_1, B_2 implies A_2, B_3 implies A_4 and B_4 implies A_5*

The first of these implications is, of course, obvious. The second implication is a consequence of the fact, mentioned in the last section, that no Boolean algebra is the union of two proper subalgebras.

Proof of $B_3 \Rightarrow A_4$. By Proposition 3.2 we may assume that the OML L satisfying B_3 has no non-trivial Boolean factor. Since, as is easily seen, the product of two OMLs with at least two blocks each can not be covered by three blocks, we may even assume that L is irreducible. Assume then that B_1, B_2, B_3 are three blocks covering L. Since L is irreducible we have $B_1 \cap B_2 \cap B_3 = \{0, 1\}$. Assume now that A and B are further blocks of L. It follows from Proposition 3.3 that A and B are eight-element Boolean algebras. Clearly the atoms of A and B are also atoms of L since every element smaller than an atom of a block commutes with every element of that block. Let a_i be the atom of A belonging to $A \cap B_i$ and b_i be the atom of B belonging to $B \cap B_i$. Note that $a_i, b_i \notin B_j$ if $i \neq j$. The element $a_1 \vee b_2$ commutes with a_1, a_2, b_1, b_2, hence belongs to $A \cap B$. Since

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$a_1, b_2 < a_1 \vee b_2$ it follows that

$$a_1 \vee b_2 \in \{a_2', a_3', 1\} \cap \{b_1', b_3', 1\} \subseteq \{b_3', 1\}.$$

From this we obtain

$$b_1 = b_1 \wedge (a_1 \vee b_2) = a_1 \wedge b_1,$$

hence, a_1 and b_1 being atoms, that $a_1 = b_1$. By symmetry we obtain $a_i = b_i$ for all i and hence $A = B$, completing the proof.

Proof of $B_4 \Rightarrow A_5$. By the same argument as in the last proof we may assume that the OML L satisfying B_4 has no non-trivial Boolean factor. Assume now that L was the direct product of two non-Boolean OMLs L_1 and L_2. If both of them have only two blocks then L has four blocks and there is nothing left to prove. Assume then that one of the factors, say L_2, has at least three blocks. Since $D_2 \Rightarrow B_2 \Rightarrow A_2$ it would follow that there exist three non-commuting elements b_1, b_2, b_3 in L_2. Since L_1 is not Boolean it contains two non-commuting elements a_1, a_2. But then no two of the six elements (a_i, b_j) commute, contradicting the assumption that L satisfies B_4. We may thus assume without loss of generality that L is irreducible.

Let B_1, B_2, B_3, B_4 be four blocks covering L. We then have by the irreducibility of L that $B_1 \cap B_2 \cap B_3 \cap B_4 = \{0, 1\}$. Assume next that the union of three of the blocks is a subalgebra, say that $B_1 \cup B_2 \cup B_3$ is a subalgebra. Then, since no Boolean algebra is the union of two proper subalgebras, every further block of L would be contained in $B_1 \cup B_2 \cup B_3$. Since we have already proved that B_3 implies A_4, there is at most one such block and we obtain that L satisfies A_5. We may thus also assume that the union of no three of the blocks B_i is a subalgebra. By Proposition 3.1 this implies in particular that

$$(1) \quad B_1 \cap B_2 \cap B_3 = B_1 \cap B_2 \cap B_4 = B_1 \cap B_3 \cap B_4 = B_2 \cap B_3 \cap B_4 = \{0, 1\}.$$

Now let B be a further block. We say that B is

of ***third kind*** if it is contained in the union of three of the B_i,

of ***first kind*** if it is not of third kind and the decomposition $B = (B \cap B_1) \cup (B \cap B_2) \cup (B \cap B_3) \cup (B \cap B_4)$ is of first kind,

of ***second kind*** if it is not of third kind and the above decomposition is of second kind.

It follows from Proposition 3.4 that every block B is of first or second or third kind. It furthermore follows that every block of first or second kind is a sixteen-element Boolean algebra and that every block of third kind is an eight-element Boolean algebra. We prove the claim now in several steps.

4.1. If B is of first kind, say $B \cap B_1 \cap B_2$, $B \cap B_1 \cap B_3$ and $B \cap B_2 \cap B_3$ are different from $\{0, 1\}$ and $B \cap B_1 \cap B_4 = B \cap B_2 \cap B_4 = B \cap B_3 \cap B_4 = \{0, 1\}$ then each of the unions $B \cup B_i$ $(i = 1, 2, 3)$ is a subalgebra of L.

To see this pick

$$x_i \in (B \cap B_j \cap B_k) - (B_i \cup B_4), \{i, j, k\} = \{1, 2, 3\},$$

and

$$y_i \in (B \cap B_i) - (B_j \cup B_k \cup B_l), \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

It then follows from Proposition 3.4 that the elements x_i, y_i, their orthocomplements and 0, 1 form the whole Boolean algebra B. By symmetry and duality it is enough to show that $x \in B_1 - B$ and $y \in B - B_1$ imply $x \vee y \in B$; and, we may furthermore assume that y is one of the elements x_1, y_2, y_3, y_4. But, as is easily seen, $x \vee x_1$ commutes with x_1, x_2, x_3, y_1 and $x \vee y_2$ commutes with x_2, x_3, y_1, y_2. Since each of the subsets $\{x_1, x_2, x_3, y_1\}$, $\{x_2, x_3, y_1, y_2\}$ generates B we obtain that $x \vee x_1, x \vee y_2 \in B$. By symmetry we obtain $x \vee y_3 \in B$. Finally,

$$x \vee y_4 \, C \, x_2, x_3, y_1, y_4$$

which, by the same argument, gives $x \vee y_4 \in B$, proving 4.1.

4.2. Under the assumptions of 4.1, $B \cup B_1 \cup B_2 \cup B_3$ is a subalgebra of L.

We show first that $x \in B_1 - (B \cup B_2 \cup B_3)$, $y \in B_2 - (B \cup B_1 \cup B_3)$ and $x \vee y \in B_4 - (B \cup B_1 \cup B_2 \cup B_3)$ is impossible. With x_i, y_i having the same meaning as in the proof of 4.1 it would imply $x_3, y_4 \, C \, x \vee y$ and there would exist a block A containing $x_3, y_4, x \vee y$. Since $x_3 + y_4, x_3, y_4, x \vee y$ are different elements and none is the complement of another, the block A would have sixteen elements and hence would not be of third kind. But since

$$|(A \cap B_4) - (B_1 \cup B_2 \cup B_3)| \geqq 4 \quad \text{and}$$
$$|(A \cap B_1 \cap B_2) - (B_3 \cup B_4)| \geqq 2$$

it can by Proposition 3.4 not be of first or second kind. The assumptions thus lead to a contradiction. By (4.1), symmetry and duality it is thus enough to show that $x \in B_1 - (B \cup B_2 \cup B_3)$ and $y \in (B_2 \cap B_3)$ imply $x \vee y \in B \cup B_1$. But this is trivially so since $y \in B_2 \cap B_3$ implies $y \, C \, x_1, x_2, x_3, y_2$, hence $y \in B$. The result thus follows from (4.1).

4.3. If L has a block B of first kind then it has no further blocks.

By symmetry we may assume that B satisfies the assumptions of 4.1. Since $B \cap B_i \cap B_j \not\subseteq B_k$ whenever $\{i, j, k\} = \{1, 2, 3\}$ it follows from 4.2 and Proposition 3.1 that the unions $B \cup B_i \cup B_j$ are also subalgebras of L. Assume now that A is an arbitrary block different from the B_i. Since

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$B \cup B_1 \cup B_2 \cup B_3$ is a subalgebra and

$$A = (A \cap (B \cup B_1 \cup B_2 \cup B_3)) \cup (A \cap B_4)$$

it follows that $A \subseteq B \cup B_1 \cup B_2 \cup B_3$. Since $B \cup B_1 \cup B_2$ is a subalgebra it follows by the same argument that

$$A \subseteq B \cup B_1 \cup B_2.$$

Repeating the same argument twice again we obtain $A \subseteq B$ and hence $A = B$.

4.4. L has at most one block of third kind.

Assume first that there were blocks A, B of third kind, both contained in the union of the same three B_i, say $A, B \subseteq B_1 \cup B_2 \cup B_3$. It follows by the same argument as in the proof that B_3 implies A_4 that $A = B$. By symmetry we may thus assume that

$$A \subseteq B_1 \cup B_2 \cup B_3 \quad \text{and} \quad B \subseteq B_2 \cup B_3 \cup B_4.$$

By what we have shown A and B are eight-element Boolean algebras and there exist atoms

$$p_i \in (A \cap B_i) - (B_j \cup B_k), \{j, i, k\} = \{1, 2, 3\}, \quad \text{and}$$
$$q_i \in (B \cap B_i) - (B_j \cup B_k), \{i, j, k\} = \{2, 3, 4\}.$$

As before $p_2 \vee q_3$ commutes with p_2, p_3, q_2, q_3, hence belongs to $A \cap B$. Since $p_2, q_3 < p_2 \vee q_3$ it follows that

$$p_2 \vee q_3 \in \{p_1', p_3', 1\} \cap \{q_2', q_4', 1\} \subseteq \{p_1', 1\}.$$

This implies that

$$p_3 = p_3 \wedge (p_2 \vee q_3) = p_3 \wedge q_3,$$

hence $p_3 = q_3$. By symmetry we obtain $p_2 = q_2$ hence $A = B$.

4.5. If A, B are blocks of second kind and if $A \cap B_1 \cap B_2 \neq \{0, 1\} \neq B \cap B_1 \cap B_2$, then $A = B$.

Choose

$$x \in (B \cap B_1 \cap B_2) - (B_3 \cup B_4),$$
$$y \in (A \cap B_1 \cap B_2) - (B_3 \cup B_4),$$
$$x_i \in (B \cap B_i) - (B_j \cup B_k \cup B_l),$$
$$y_i \in (A \cap B_i) - (B_j \cup B_k \cup B_l)$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Since $x \mathrel{C} y, y_1, y_2$ it follows that $x \in A$ and hence that

$$A \cap B_1 \cap B_2 = B \cap B_1 \cap B_2.$$

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The element $x + x_3$ belongs to B_4 and hence commutes with y_4. Since $x \in A$, y_4 also commutes with x and hence

$$x_3 = x + (x + x_3) \text{ C } y_4.$$

The element $x_3 + y_4$ commutes with y_4, y_3, y (since $y \in B$) and y_1 or y_2 (since $x_3 + y_4 \in B_1 \cup B_2$), hence belongs to A. It follows that

$$x_3 = (x_3 + y_4) + y_4 \in A$$

and hence that

$$A \cap B_3 = B \cap B_3.$$

By symmetry we also have

$$A \cap B_4 = B \cap B_4.$$

The element $x_2 + x_3 + x_4$ commutes with y (since $y \in B$), y_2 (since $x_3, x_4 \in A$ and $x_2, y_2 \in B_2$), y_3 and y_4 (since $y_3, y_4 \in B$), hence belongs to A. Since also $x_3 + x_4 \in A$ we obtain $x_2 \in A$ and hence

$$A \cap B_2 = B \cap B_2,$$

from which the claim follows easily.

4.6. L has at most one block of second kind.

Assume first that there were blocks A, B of second kind such that

$$A \cap B_1 \cap B_2 \neq \{0, 1\} \neq B \cap B_2 \cap B_3.$$

Then every element $x \in A \cap B_1 \cap B_2$ would commute with every element of the set $B \cap (B_1 \cup B_2)$, hence would belong to B. It would follow that $A \cap B_2 \cap B_3 \neq \{0, 1\}$, contradicting the assumption that A is of second kind. By 4.5 and symmetry it is thus enough to show that there are no blocks A, B of second kind satisfying

$$A \cap B_1 \cap B_2 \neq \{0, 1\} \neq B \cap B_3 \cap B_4.$$

If there were such blocks pick

$$x \in (A \cap B_1 \cap B_2) - (B_3 \cup B_4),$$

$$y_i \in (B \cap B_i) - (B_j \cup B_k \cup B_l),\ \{i, j, k, l\} = \{1, 2, 3, 4\}$$

and

$$y \in (B \cap B_3 \cap B_4) - (B_1 \cup B_2).$$

Then x would commute with y_1 and y_2 and hence with $y_1 + y_2$. As before $y_1 + y_2 + y$ equals 0 or 1, hence

$$y = y_1 + y_2 \quad \text{or} \quad y = (y_1 + y_2)'.$$

In any case x commutes with y and there would exist a block C containing

x and y and hence satisfying

$$C \cap B_1 \cap B_2 \neq \{0, 1\} \neq C \cap B_3 \cap B_4,$$

which, by an argument used earlier (see (1) of Section 3), is impossible. This proves 4.6.

By what we have proved so far, L can have at most six blocks and if it has six blocks it must have a block of second kind and a block of third kind. We assume for the remainder of the proof that A is a block of third kind and that B is a block of second kind. By symmetry we may assume that $A \subseteq B_1 \cup B_2 \cup B_3$ and that $p_i \in A \cap B_i$ $(i = 1, 2, 3)$ are the atoms of A.

4.7. It is impossible that there exist distinct indices $i, j \in \{1, 2, 3\}$ and distinct atoms $q_i \in B \cap B_i$ and $q_j \in B \cap B_j$.

By symmetry we may assume that $i = 1$ and $j = 2$. Since $p_1 = q_1$ and $p_2 = q_2$ would imply $A \subseteq B$ we may assume that $p_2 \neq q_2$. The element $p_1 \vee q_2$ commutes with p_1 and p_2 and hence belongs to A. If also $p_1 \neq q_1$ we have

$$p_1 < p_1 \vee q_2 \leqq p_2', q_1',$$

hence, since $p_1 \vee q_2 \in A$, $p_2' = p_1 \vee q_2 \leqq q_1'$, i.e., $p_2 = q_1$, which is impossible since $p_2 \notin B_1$. We may thus assume that $p_1 = q_1$. Then there exists an atom $q \neq q_1, q_2$ of B such that $p_1, q_2 \leqq q'$. By the same argument as before we obtain $p_2' = p_1 \vee q_2 \leqq q'$, hence $p_2 = q \in B$. But $p_1, p_2 \in B$ imply $A \subseteq B$, which is impossible.

No three of the atoms of B belong to the same B_i since this would imply that $B \subseteq B_i$. In view of 4.7 we may thus assume without loss of generality that two atoms of B belong to $B_3 - (B_1 \cup B_2 \cup B_4)$ and that the remaining two atoms of B belong to $B_4 - (B_1 \cup B_2 \cup B_3)$. It follows from Proposition 3.4 that there exists an element $x \in (B \cap B_2) - (B_1 \cup B_3 \cup B_4)$. Replacing, if necessary, x by x', we may assume that $p_2 < x$. Since x is neither an atom nor a co-atom there exists a co-atom $q > x$ in B and this belongs to either B_3 or B_4. The chain $\{p_2, x, q\}$ belongs to some block C. Since $p_2 \notin B_1 \cup B_3$, $x \notin B_4$ and $q \notin B_2$ this block can not be one of the B_i. Since $x \notin A$ and $p_2 \notin B$ it can not be A or B. We would thus obtain a seventh block of L which, as we already know, does not exist. The theorem is thus completely proved.

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