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ON WEAKLY ALMOST-PERIODIC FAMILIES OF LINEAR OPERATORS

BY

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ABSTRACT. In this note first the weak almost-periodicity of the action of a weakly almost-periodic family of linear operators on an almost-periodic function is established. Then an application of this result is given.

1. Let $\mathscr{J}$ be the interval $-\infty < t < \infty$, $\mathscr{X}$ a Banach space and $\mathscr{X}^*$ the dual space of $\mathscr{X}$. A continuous function $f:\mathscr{J}\to\mathscr{X}$ is said to be (strongly) almost-periodic if, for every $\varepsilon>0$, there is, on the real line, a relatively dense set of numbers $\{\tau\}_\varepsilon$ such that

$$\sup_{t\in\mathscr{J}} \|f(t+\tau)-f(t)\| \leq \varepsilon \quad \text{for all} \quad \tau\in\{\tau\}_\varepsilon$$

(see Amerio and Prouse [1]; the results of this paper are based on this reference). We say that a function $f:\mathscr{J}\to\mathscr{X}$ is weakly almost-periodic if $\langle x^*, f(t)\rangle = x^*f(t)$ is almost-periodic for each $x^*\in\mathscr{X}^*$.

A function $f\in\mathscr{L}^p_{\text{loc}}(\mathscr{J};\mathscr{X})$ for $1\leq p<\infty$ is said to be $\mathscr{S}^p$ almost-periodic if, for every $\varepsilon>0$, there is a positive real number $l=l(\varepsilon)$ such that any interval of the real line of length l contains at least one point τ for which

$$\sup_{a\in\mathscr{J}}\left[\int_a^{a+1} \|f(t+\tau)+f(t)\|^p \, dt\right]^{1/p} \leq \varepsilon.$$

Let $\mathfrak{L}(\mathscr{X},\mathscr{X})$ be the space of bounded linear operators of $\mathscr{X}$ into itself. An operator-valued function $\mathscr{G}:\mathscr{J}\to\mathfrak{L}(\mathscr{X},\mathscr{X})$ is said to be strongly (weakly) almost-periodic if $\mathscr{G}(t)x$, $t\in\mathscr{J}\to\mathscr{X}$ is strongly (weakly) almost-periodic for each $x\in\mathscr{X}$.

$\mathscr{G}:\mathscr{J}\to\mathfrak{L}(\mathscr{X},\mathscr{X})$ is called a one-parameter group if $\mathscr{G}(0)=\mathscr{I}=$the identity operator of $\mathscr{X}$ and $\mathscr{G}(t_1+t_2)=\mathscr{G}(t_1)\mathscr{G}(t_2)$ for all $t_1, t_2\in\mathscr{J}$.

Our main result is as follows.

THEOREM 1. *If $f(t)$, $t\in\mathscr{J}\to\mathscr{X}$ is almost-periodic and if $\mathscr{G}(t)$, $t\in\mathscr{J}\to\mathfrak{L}(\mathscr{X},\mathscr{X})$ is weakly almost-periodic, then the function $\mathscr{Y}(t)=\mathscr{G}(t)f(t)$ is weakly almost-periodic.*

Proof. For an arbitrary but fixed $x^*\in\mathscr{X}^*$, $\{x^*\mathscr{G}(t)\}_{t\in J}$ is a family of bounded

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linear functionals on $\mathscr{X}$. Further, by our assumption, for each $x \in \mathscr{X}$, the scalar-valued function $x^*\mathscr{G}(t)x$ is almost-periodic, and so is bounded on $\mathscr{J}$. Thus, by the uniform boundedness principle,

$$\sup_{t\in\mathscr{J}} \|x^*\mathscr{G}(t)\| = \mathscr{M} < \infty. \tag{1.1}$$

To see that $\mathscr{G}(t)f(t)$ is weakly continuous, let $t'_n, t' \in \mathscr{J}$ and $t'_n \to t'$. Then the function $x^*\mathscr{G}(t)f(t')$ is continuous, and so, by (1.1) and the continuity of f,

$$|x^*\mathscr{G}(t'_n)f(t'_n) - x^*\mathscr{G}(t')f(t')| \leq \|x^*\mathscr{G}(t'_n)\| \cdot \|f(t'_n)-f(t')\| + |x^*\mathscr{G}(t'_n)f(t') - x^*\mathscr{G}(t')f(t')| \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\mathscr{R}_f$ denote the range of the function f. As is well known, $\mathscr{R}_f$ is a relatively compact set in $\mathscr{X}$. So, given $\varepsilon > 0$, there exists a finite set $\{f(t_1), f(t_2), \ldots, f(t_{n_\varepsilon})\}$ which is an ε-net for $\mathscr{R}_f$. We observe that the $(n_\varepsilon + 1)$ functions

$$x^*\mathscr{G}(t)f(t_1), x^*\mathscr{G}(t)f(t_2), \ldots, x^*\mathscr{G}(t)f(t_{n_\varepsilon}), f(t)$$

are almost-periodic, and hence admit a common relatively dense set $\{\tau\}_\varepsilon$ of ε-almost-periods. Consequently, we have

$$\sup_{t\in\mathscr{J}} |x^*\mathscr{G}(t+\tau)f(t_k) - x^*\mathscr{G}(t)f(t_k)| \leq \varepsilon, \sup_{t\in\mathscr{J}} \|f(t+\tau)-f(t)\| \leq \varepsilon \tag{1.2}$$

for all $\tau \in \{\tau\}_\varepsilon$ and $k = 1, 2, \ldots, n_\varepsilon$.

For an arbitrary but fixed $\bar{t} \in \mathscr{J}$, there is $f(t_k)$ in the ε-net for $\mathscr{R}_f$ such that

$$\|f(\bar{t}) - f(t_k)\| < \varepsilon. \tag{1.3}$$

Now, if $\tau \in \{\tau\}_\varepsilon$, then by (1.1)–(1.3), we have

$$\begin{aligned} |x^*\mathscr{G}(\bar{t}+\tau)f(\bar{t}+\tau) - x^*\mathscr{G}(\bar{t})f(\bar{t})| &\leq \|x^*\mathscr{G}(\bar{t}+\tau)\| \cdot \|f(\bar{t}+\tau)-f(\bar{t})\| \\ &+ \|x^*\mathscr{G}(\bar{t}+\tau)\| \cdot \|f(\bar{t})-f(t_k)\| + |x^*\mathscr{G}(\bar{t}+\tau)f(t_k) - x^*\mathscr{G}(\bar{t})f(t_k)| + \|x^*\mathscr{G}(\bar{t})\| \\ &\times \|f(t_k)-f(\bar{t})\| \leq \mathscr{M}\varepsilon + \mathscr{M}\varepsilon + \varepsilon + \mathscr{M}\varepsilon = (3\mathscr{M}+1)\varepsilon. \end{aligned} \tag{1.4}$$

So it follows that

$$\sup_{t\in\mathscr{J}} |x^*\mathscr{G}(t+\tau)f(t+\tau) - x^*\mathscr{G}(t)f(t)| \leq (3\mathscr{M}+1)\varepsilon \quad \text{for all} \quad \tau \in \{\tau\}_\varepsilon,$$

which completes the proof of the theorem.

REMARKS. (i) From (1.1), again by the uniform boundedness principle, we obtain

$$\sup_{t\in\mathscr{J}} \|\mathscr{G}(t)\| < \infty. \tag{1.5}$$

(ii) From the proof of Theorem 1, it is obvious that Theorem 1 remains valid if $f(t)$, $t \in \mathscr{J} \to \mathscr{X}$ is almost-periodic, (1.5) holds and $\mathscr{G}(t)x$, $t \in \mathscr{J} \to \mathscr{X}$ is weakly almost-periodic for each $x \in \mathscr{R}_f$.

(iii) Let $\mathscr{G}^*(t)$ be the conjugate of the operator $\mathscr{G}(t)$. If $\mathscr{G}^*(t)$, $t \in \mathscr{J} \to \mathfrak{L}(\mathscr{X}^*, \mathscr{X}^*)$

is strongly almost-periodic, and if $f(t)$, $t \in \mathcal{J} \to \mathcal{X}$ is weakly almost-periodic, then $\mathcal{G}(t)f(t)$ is weakly almost-periodic.

Proof. By our assumption, for each $x^* \in \mathcal{X}^*$, $\mathcal{G}^*(t)x^*$ is almost-periodic from $\mathcal{J}$ to $\mathcal{X}^*$. So, by an argument similar to that of Theorem 1, the scalar-valued function

$$x^*[\mathcal{G}(t)f(t)] = [x^*\mathcal{G}(t)]f(t) = [\mathcal{G}^*(t)x^*]f(t)$$

is almost-periodic, here making use of the relative compactness of the range of $\mathcal{G}^*(t)x^*$ in $\mathcal{X}^*$.

2. As an application of our Theorem 1, we demonstrate the following result.

THEOREM 2. *Suppose $\mathcal{X}$ is a Banach space, $\mathcal{G}(t)$, $t \in \mathcal{J} \to \mathfrak{L}(\mathcal{X}, \mathcal{X})$ is a one-parameter group with $\mathcal{G}^*(t)$, $t \in \mathcal{J} \to \mathfrak{L}(\mathcal{X}^*, \mathcal{X}^*)$ being strongly almost-periodic, for $1 \le p < \infty$, a continuous function $f(t)$, $t \in \mathcal{J} \to \mathcal{X}$ is $\mathcal{S}^p$ almost-periodic, and a function $u(t)$, $t \in \mathcal{J} \to \mathcal{X}$ has the representation*

$$u(t) = \mathcal{G}(t)u(0) + \int_0^t \mathcal{G}(t-s)f(s)\,ds. \tag{2.1}$$

Then, if

$$\sup_{t \in \mathcal{J}} \|u(t)\| < \infty, \tag{2.2}$$

$u(t)$ is weakly almost-periodic.

Proof. Consider the function

$$f_h(t) = \frac{1}{h}\int_0^h f(t+s)\,ds \quad \text{for any} \quad h > 0. \tag{2.3}$$

Since f is $\mathcal{S}^p$ almost-periodic (and hence $\mathcal{S}^1$ almost-periodic), it follows easily that $f_h(t)$ is almost-periodic for each fixed $h>0$. It can be proved, as for scalar-valued functions (see Besicovitch [2], pp. 80–81), that $f_h \to f$ as $h \to 0$ in the $\mathcal{S}^1$ sense, that is,

$$\sup_{t \in \mathcal{J}} \int_t^{t+1} \|f(s)-f_h(s)\|\,ds \to 0 \quad \text{as} \quad h \to 0.$$

Under the assumption made on $\mathcal{G}^*$, it is easy to see that $\mathcal{G}(t)$, $t \in \mathcal{J} \to \mathfrak{L}(\mathcal{X}, \mathcal{X})$ is weakly almost-periodic, and so, as shown in the proof of Theorem 1, $\mathcal{G}(t)f(t)$ is weakly continuous. Now, for an arbitrary but fixed $x^* \in \mathcal{X}^*$, we have

$$x^*\mathcal{G}(t)f(t) = x^*\mathcal{G}(t)[f(t)-f_h(t)] + x^*\mathcal{G}(t)f_h(t), \tag{2.4}$$

and, by (1.1),

$$\sup_{t \in \mathcal{J}} \int_t^{t+1} |x^*\mathcal{G}(s)[f(s)-f_h(s)]|\,ds \le \mathcal{M} \sup_{t \in \mathcal{J}} \int_t^{t+1} \|f(s)-f_h(s)\|\,ds \to 0 \quad \text{as} \quad h \to 0. \tag{2.5}$$

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Since, by Theorem 1, the functions $x^*\mathcal{G}(t)f_h(t)$ are almost-periodic from $\mathcal{J}$ to the (field of) scalars, it follows that $x^*\mathcal{G}(t)f(t)$ is $\mathcal{S}^1$ almost-periodic from $\mathcal{J}$ to the scalars.

Since $\mathcal{G}(-s)$, $s \in \mathcal{J} \to \mathfrak{L}(\mathcal{X}, \mathcal{X})$ is weakly almost-periodic, it follows that $x^*\mathcal{G}(-s)f(s)$, $s \in \mathcal{J} \to$ the scalars is continuous and $\mathcal{S}^1$ almost-periodic.

If we write

$$v(t) = \int_0^t \mathcal{G}(-s)f(s)\,ds, \tag{2.6}$$

then we have

$$\mathcal{G}(-t)u(t) = u(0)+v(t). \tag{2.7}$$

So, by (1.1) and (2.2), $x^*v(t)=\int_0 x^*\mathcal{G}(-s)f(s)\,ds$ is (uniformly) bounded on $\mathcal{J}$. Now the $\mathcal{S}^1$ almost-periodicity of $x^*\mathcal{G}(-s)f(s)$ and the boundedness of $x^*v(t)$ imply that $x^*v(t)$, $t \in \mathcal{J} \to$ the scalars is almost-periodic. Then, by Remark (iii), $\mathcal{G}(t)v(t)$ is weakly almost-periodic from $\mathcal{J}$ to $\mathcal{X}$. Since $\mathcal{G}(t)u(0)$ is weakly almost-periodic from $\mathcal{J}$ to $\mathcal{X}$, the desired conclusion follows.

Notes. (i) Towards the end of the proof of Theorem 2, we used the following result: If, for $1 \leq p < \infty$, a function $\phi(s)$ is $\mathcal{S}^p$ almost-periodic from $\mathcal{J}$ to the scalars, and if $\Phi(t)=\int_0^t \phi(s)\,ds$ is bounded on $\mathcal{J}$, then $\Phi(t)$ is almost-periodic from $\mathcal{J}$ to the scalars.

Proof. Consider a sequence $\{\rho_n(t)\}_{n=1}^{\infty}$ of infinitely differentiable positive functions, null for $|t| \geq 1/n$ with integral $=1$. The convolution between ϕ and ρ_n is defined by

$$(\phi * \rho_n)(t) = \int_{-\infty}^{\infty} \phi(t-s)\rho_n(s)\,ds = \int_{-\infty}^{\infty} \phi(s)\rho_n(t-s)\,ds.$$

It is easy to see that

$$(\Phi * \rho_n)'(t) = (\phi * \rho_n)(t) \quad \text{for all} \quad t \in \mathcal{J};$$

$$\sup_{t\in\mathcal{J}} \|(\Phi * \rho_n)(t)\| \leq \sup_{t\in\mathcal{J}} \|\Phi(t)\| < \infty \qquad \text{(by our assumption)}.$$

As shown in the proof of Theorem VII, p. 78, Amerio and Prouse [1], $(\phi * \rho_n)(t)$ is almost-periodic from $\mathcal{J}$ to the scalars. Hence, by Bohl-Bohr's theorem, $(\Phi * \rho_n)(t)$ is almost-periodic from $\mathcal{J}$ to the scalars $(n=1, 2, \ldots)$.

Further, by Theorem VIII, p. 79, Amerio and Prouse [1], $\Phi(t)$ is uniformly continuous on $\mathcal{J}$. By the uniform continuity of $\Phi(t)$, the sequence of convolutions $(\Phi * \rho_n)(t)$ converges uniformly to $\Phi(t)$ for $n \to \infty$. Consequently, $\Phi(t)$ is almost-periodic.

(ii) Theorem 2 remains valid if the function f is weakly almost-periodic instead of continuous and $\mathcal{S}^p$ almost-periodic, with $\mathcal{G}^*$, u satisfying the conditions imposed on them.

Proof. By Remark (iii), $\mathcal{G}(-s)f(s)$, $s \in \mathcal{J} \to \mathcal{X}$ is weakly almost-periodic.

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By (1.5), (2.2), and (2.7), $v(t)$ is bounded on $\mathscr{J}$. So, by Bohl-Bohr's theorem, $v(t)$ is weakly almost-periodic. Now the result stated is obvious.

(iii) From the proof of Theorem 2, the following result is obvious.

THEOREM 3. *If* $\mathscr{G}(t)$, $t \in \mathscr{J} \to \mathfrak{L}(\mathscr{X}, \mathscr{X})$ *is weakly almost-periodic, for* $1 \leq p < \infty$, (t), $t \in \mathscr{J} \to \mathscr{X}$ *is continuous and* $\mathscr{S}^p$ *almost-periodic, and* $\mathscr{F}(t) = \int \mathscr{G}(s)f(s)\, ds$ *is bounded on* $\mathscr{J}$, *then* $\mathscr{F}(t)$ *is weakly almost-periodic* ($\mathscr{X}$ *a Banach space*).

(iv) If $\mathscr{A}$ is the infinitesimal generator of a strongly continuous one-parameter group $\mathscr{G}: \mathscr{J} \to \mathfrak{L}(\mathscr{X}, \mathscr{X})$, and if $f: \mathscr{J} \to \mathscr{X}$ is a continuous function, then any solution of the inhomogeneous operator differential equation

$$u'(t) = \mathscr{A}u(t) + f(t) \quad \text{on} \quad \mathscr{J}$$

has the representation (2.1) (see Dunford and Schwartz [3]).

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