Canad. Math. Bull. Vol. **24** (3), 1981

A CLASS OF SPACES IN WHICH COMPACT SETS ARE FINITE

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ABSTRACT. It is shown that in a dense-in-itself Hausdorff space if every set having a dense interior is open, then every compact set is finite.

In recent years, there has been considerable interest in discovering classes of Hausdorff topologies in which compact sets are finite. The general problem is to move as far away from discreteness as possible and yet to retain the property that only finite sets be compact. There are numerous interesting open problems in this area [2].

Kirch [5] proved that if a Hausdorff space X is dense-in-itself and if each dense subset of X is open, then each compact subset of X is finite. Recall that a topolical space having all of its dense sets open, is called *submaximal* [3] (and its topology is called a submaximal topology). A subset A of a topological space X is *regular-open* provided A is the interior of its closure in X. A topological space is *semi-regular* if regular-open sets in it form a base for its topology. If τ is a topology on a set X, then the collection of all regular-open sets of (X, τ) is a base for a topology τ_s on X; the space (X, τ_s) is semi-regular and τ_s is called the *semi-regularization* of τ. Call two topologies on a given set X to be *s-equivalent* provided they have the same semi-regularization. Each equivalence class of s-equivalent topologies on X will be called an *s-class*. Then, a topology τ on a set X is submaximal if and only if it is a maximal member of the s-class to which it belongs [3]. Clearly, if an s-class has a dense-in-itself member, then every member of the class is dense-in-itself. A technique for constructing the whole s-class of a given topology is detailed in [3].

In the present terminology, Kirch's result can be restated as follows: If a semi-regular Hausdorff topology is dense-in-itself then each maximal member of its s-class satisfies the condition that only compact sets are the finite ones. We prove that even with regard to the topology which is the intersection of all the maximal members of such an s-class, every compact set is finite. This result is quite surprising, because, quite often, an s-class has uncountably many

Received by the editors November 2, 1979.
AMS (1970) Subject Classifications: 54A10, 54D10, 54D20.
Key words and phrases: submaximal, regular-open set, semi-regular, s-class.

https://doi.org/10.4153/CMB-1981-057-4 Published online by Cambridge University Press

distinct maximal members. The proof of this is derived from an easily proven theorem.

THEOREM 1. *Let X be a dense-in-itself Hausdorff space. If each subset of X containing a dense open set is open then each countably compact subspace of X is finite.*

Proof. Let A be an infinite subset of X. Since X is Hausdorff, there is an infinite subset B of A such that B is a discrete set in X. Let $\bar{B}$ be the closure of B in X. Since X is dense-in-itself, it is easily seen that the open set $X-\bar{B}$ is dense in X. Therefore, the set $X-B$ which contains $X-\bar{B}$, must also be open in X. Therefore B is closed in X and hence in A. Consequently the subspace A of X cannot be countably compact.

THEOREM 2. *Let (X, τ) be a semi-regular space and let τ^* be the intersection of all the maximal members of the s-class of τ. Then*:

(i) *A subbase for τ^* is $\tau \cup \mathcal{F}^*$ where $\mathcal{F}^*$ is the filter on X based on the collection of dense open subsets of (X, τ).*

(ii) *Each set which contains a dense open subset of (X, τ^*) is open in (X, τ^*); and furthermore τ^* is the smallest member of the s-class of τ having this property.*

(iii) *If (X, τ) is dense-in-itself and Hausdorff, then each countably compact (and, in particular each compact) subspace of (X, τ^*) is finite.*

Proof. (i) If τ_1 is a maximal member of the s-class of τ then it is submaximal and, therefore, a subbase for τ_1 is $\tau \cup \mathcal{G}$ for some maximal filter $\mathcal{G}$ of dense sets in (X, τ) [3]. Clearly $\mathcal{F}^*$ is contained in $\mathcal{G}$ and therefore the topology τ_0, generated by $\tau \cup \mathcal{F}^*$, is contained in τ^*.

To show that $\tau^* \subset \tau_0$, we note that if D is dense in (X, τ) such that its interior D° in (X, τ) is not dense, then, since $E = D^\circ \cup (X-D)$ is also dense in (X, τ); we have two dense sets D and E in (X, τ) such that $D \cap E$ is not dense. Therefore there is a maximal member of the s-class of τ with regard to which D is not open [3]. In particular, D is not open in (X, τ^*). Finally, as τ^* is generated by τ together with the sets which are dense open in (X, τ^*) [3], it follows that $\tau^* \subset \tau_0$. The proofs of (ii) and (iii) are also straightforward.

EXAMPLE. Let X be a dense-in-itself locally compact Hausdorff space. Then, there is an uncountable collection of pairwise disjoint dense subsets of X ([1], [4]). Therefore the s-class of τ has uncountably many maximal members. The intersection of all these submaximal topologies has the property that the only compact sets with regard to it are the finite subsets of X. Finally, we remark that, besides local compactness, there are several other topological properties which imply the existence of at least two pairwise disjoint dense sets in a dense-in-itself space. In all such cases, we will obtain topologies much weaker than the submaximal ones where all compact sets would be finite.

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https://doi.org/10.4153/CMB-1981-057-4 Published online by Cambridge University Press