SOME REMARKS ON THE CHARACTERS OF THE SYMMETRIC GROUP

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Introduction. In [**2**], we derived some character relations of the symmetric group S_n. These relations were also obtained in [**1**] independently. In the present paper, we shall study the properties of these character relations in some detail. In the last section, using a result obtained in [**3**], we shall further determine the number of modular irreducible representations in a p-block of S_n.

1. We shall denote by $[\alpha]$ the irreducible representation of S_n corresponding to a diagram $[\alpha]$ of n nodes, and by χ_α its character. Similarly we define the irreducible representation $[\beta_\mu]$ of S_{n-u} and its character χ_{β_u}. We denote by $m(n)$ the number of distinct irreducible representations of S_n. Then, as is well known, the number of classes of conjugate elements in S_n is equal to $m(n)$.

Let $Q = A.U$ be an element of S_n where U is a single cycle of length u, and A is any permutation on the remaining $n - u$ symbols. By the Murnaghan-Nakayama recursion formula

1.1 $$\chi_\alpha(A.U) = \sum_{\beta_u} a_{\alpha\beta_u}\chi_{\beta_u}(A).$$

Here,

$$a_{\alpha\beta_u} = (-1)^{r_i}$$

if a diagram $[\beta_u]$ of S_{n-u} is obtainable from $[\alpha]$ by the removal of a single u-hook H_i with leg length r_i, and

$$a_{\alpha\beta_u} = 0$$

otherwise. We set

1.2 $$\mu_\alpha^{(u)} = \sum_{\beta_u} a_{\alpha\beta_u}\chi_{\beta_u}.$$

$\mu_\alpha^{(u)}$ is called the (generalized) character of S_{n-u} corresponding to χ_α.

Let $A_1, A_2, \ldots, A_{m(n-u)}$ be a complete system of representatives for the classes of conjugate elements in S_{n-u}. If we set

1.3 $$Z = (\chi_\alpha(A_i.U)),$$

then

1.4 $$Z'Z = (n(A_i.U)\delta_{ij}),$$

where Z' is the transpose of Z and $n(A_i.U)$ is the order of the normalizer $N(A_i.U)$ of A_i.U in S_n. Since we have from (1.1),

Received April 7, 1952.

1.5 $$Z = (a_{\alpha\beta_u})(\chi_{\beta_u}(A_i)) = (a_{\alpha\beta_u})Z_{\beta_u},$$

(1.4) gives

1.6 $$Z'(a_{\alpha\beta_u})\, Z_{\beta_u} = (n(A_i.U)\,\delta_{ij}).$$

Hence, if we set

1.7 $$\rho_{\beta_u}^{(u)}(A_i.U) = \sum_{\alpha} a_{\alpha\beta_u}\chi_\alpha(A_i.U), \quad X = (\rho_{\beta_u}^{(u)}(A_i.U)),$$

then (1.6) becomes

1.8 $$X'Z_{\beta_u} = (n(A_i.U)\,\delta_{ij}),$$

that is,

1.9 $$\sum_{\beta_u} \rho_{\beta_u}^{(u)}(A_i.U)\ \chi_{\beta_u}(A_j) = n(A_i.U)\,\delta_{ij}.$$

If an element P of S_n possesses no u-cycle, then by [**2**]

1.10 $$\rho_{\beta_u}^{(u)}(P) = 0.$$

We shall call

$$\rho_{\beta_u}^{(u)} = \sum a_{\alpha\beta_u}\chi_\alpha$$

the (generalized) character of S_n corresponding to χ_{β_u} of S_{n-u}. If we set $T = (n(A_i.U)\delta_{ij})$, then from (1.8) we have

$$T^{-1}X'\,Z_{\beta_u} = I,$$

where I is the unit matrix. Since X and Z are square matrices,

$$Z_{\beta_u}\,T^{-1}X' = I.$$

Then, from $T^{-1} = (g(A_i.U)\delta_{ij})/n!$ we have

$$Z_{\beta_u}(g(A_i.U)\,\delta_{ij})\,X' = (n!\,\delta_{ij}),$$

which may be written

1.11 $$\sum_i g(A_i.U)\ \rho_{\beta_u}^{(u)}(A_i.U)\ \chi_{\beta'_u}(A_i) = \begin{cases} n! & \text{for } [\beta_u] = [\beta'_u], \\ 0 & \text{for } [\beta_u] \neq [\beta'_u], \end{cases}$$

where $g(A_i.U) = n!/n(A_i.U)$.

If A_i possesses t u-cycles, then we have generally:

$$g(A_i.U) = \frac{1}{t+1}\ (\text{number of conjugates of } U \text{ in } S_n) \times (\text{number of conjugates of } A_i \text{ in } S_{n-u}).$$

In case $[\beta'_u]$ is the 1-representation of S_{n-u}, (1.11) becomes

1.12 $$\sum_i g(A_i.U)\ \rho_{\beta_u}^{(u)}(A_i.U) = \sum_H \rho_{\beta_u}^{(u)}\ (H) = \begin{cases} n! & \text{for the 1-representation } [\beta_u], \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.4153/CJM-1953-039-1 Published online by Cambridge University Press

Here, H ranges over all elements of S_n which possess at least one u-cycle.

THEOREM 1. *If Q is an element of S_n with t u-cycles, then*

$$\rho_{\beta_u}^{(u)}(Q) = tu\ \chi_{\beta_u}(Q^{(u)})$$

where $Q^{(u)}$ is a permutation on the $n - u$ symbols obtained from Q by the removal of a single u-cycle.

Proof. Let $Q^{(u)}$ be conjugate with A_i. Then

$$\rho_{\beta_u}^{(u)}(Q) = \rho_{\beta_u}^{(u)}(A_i.U).$$

If we denote by $n_u(A_i)$ the order of the normalizer $N_u(A_i)$ of A_i in S_{n-u}, then

1.13 $$\sum_{\beta_u} \chi_{\beta_u}(A_i)\ \chi_{\beta_u}(A_j) = n_u(A_i)\ \delta_{ij}.$$

Since $n(A_i.U)/n_u(A_i) = tu$, we obtain

$$\sum_{\beta_u} tu\ \chi_{\beta_u}(A_i)\ \chi_{\beta_u}(A_j) = n(A_i.U)\ \delta_{ij}.$$

This, combined with (1.9), gives

$$\rho_{\beta_u}^{(u)}(A_i.U) = tu\ \chi_{\beta_u}(A_i),$$

whence

$$\rho_{\beta_u}^{(u)}(Q) = tu\ \chi_{\beta_u}(Q^{(u)}).$$

If we set

1.14 $$(b_{\beta_u\beta_u'}) = (a_{\alpha\beta_u})'(a_{\alpha\beta_u}),$$

then

$$b_{\beta_u\beta_u'} = \sum a_{\alpha\beta_u}a_{\alpha\beta_u'}$$

and

1.15 $$\rho_{\beta_u}^{(u)}(A_i.U) = \sum_{\beta_u} b_{\beta_u\beta_u'}\ \chi_{\beta_u'}(A_i).$$

THEOREM 2. *If $A_i.U$ possesses t_i u-cycles, then*

$$|b_{\beta_u\beta_u'}| = u^{m(n-u)} \prod_i t_i.$$

Proof. From (1.6) and (1.13), we have

$$Z_{\beta_u}'(a_{\alpha\beta_u})'(a_{\alpha\beta_u})Z_{\beta_u} = Z_{\beta_u}'(b_{\beta_u\beta_u'})Z_{\beta_u} = (n(A_i.U)\ \delta_{ij})$$

and

$$Z_{\beta_u}'\ Z_{\beta_u} = (n_u(A_i)\ \delta_{ij}).$$

Hence

$$|b_{\beta_u\beta_u'}| = \prod_i n(A_i.U)/\prod_i n_u(A_i) = \prod_i (t_iu) = u^{m(n-u)}\prod_i t_i.$$

https://doi.org/10.4153/CJM-1953-039-1 Published online by Cambridge University Press

Let $A = B.V$ be an element of S_{n-u}, where V is a single cycle of length v $(v \neq u)$ and B is any permutation on the remaining $n - (u + v)$ symbols. We shall denote by $[\beta_{u+v}]$ an irreducible representation of $S_{n-(u+v)}$. Then, for the character $\mu_{\beta_u}^{(v)}$ of $S_{n-(u+v)}$ corresponding to χ_{β_u}, we have

1.16 $$\chi_{\beta_u}(B.V) = \mu_{\beta_u}^{(v)}(B) = \sum_{\beta_{u+v}} a_{\beta_u\beta_{u+v}}\chi_{\beta_{u+v}}(B).$$

THEOREM 3. *Let $\rho_{\beta_{u+v}}^{(u)}$ be the character of S_{n-v} corresponding to $\chi_{\beta_{u+v}}$. Then*

$$\rho_{\beta_u}^{(u)}(Q) = \sum_{\beta_{u+v}} a_{\beta_u\beta_{u+v}}\rho_{\beta_{u+v}}^{(u)}(Q^{(v)}),$$

where Q is an element of S_n with at least one v-cycle and $Q^{(v)}$ is a permutation on the $n - v$ symbols obtained from Q by the removal of a single v-cycle.

Proof. For Q without u-cycle, we have by (1.10)

$$\rho_{\beta_u}^{(u)}(Q) = 0, \qquad \rho_{\beta_{u+v}}^{(u)}(Q^{(v)}) = 0.$$

For Q with t u-cycles, we have by Theorem 1

$$\rho_{\beta_u}^{(u)}(Q) = tu\ \chi_{\beta_u}(Q^{(u)}), \quad \rho_{\beta_{u+v}}^{(u)}(Q^{(v)}) = tu\ \chi_{\beta_{u+v}}(Q^{(u,v)})$$

where $Q = Q^{(u)}.U = Q^{(v)}.V = Q^{(u,v)}.U.V$. It follows from (1.16) that

$$\begin{aligned}\sum_{\beta_{u+v}} a_{\beta_u\beta_{u+v}}\rho_{\beta_{u+v}}^{(u)}(Q^{(v)}) &= tu\sum_{\beta_{u+v}} a_{\beta_u\beta_{u+v}}\chi_{\beta_{u+v}}(Q^{(u,v)})\\ &= tu\ \mu_{\beta_u}^{(v)}(Q^{(u,v)}) = tu\ \chi_{\beta_u}(Q^{(u)}) = \rho_{\beta_u}^{(u)}(Q).\end{aligned}$$

2. We shall consider the character of a representation $[\alpha]$ for an element $Q = B.V.U.$ where U, V are cycles of lengths u,v $(u \neq v)$, and B is a permutation on the remaining $n - (u + v)$ symbols. Applying the Murnaghan-Nakayama recursion formula twice, we obtain

$$\chi_\alpha(Q) = \sum_{\beta_u} a_{\alpha\beta_u}\chi_{\beta_u}(B.V) = \sum_{\beta_u} a_{\alpha\beta_u}\sum_{\beta_{u+v}} a_{\beta_u\beta_{u+v}}\chi_{\beta_{u+v}}(B)$$

and

$$\chi_\alpha(Q) = \sum_{\beta_v} a_{\alpha\beta_v}\chi_{\beta_v}(B.U) = \sum_{\beta_v} a_{\alpha\beta_v}\sum_{\beta_{u+v}} a_{\beta_v\beta_{u+v}}\chi_{\beta_{u+v}}(B).$$

Here, $[\beta_u]$, $[\beta_v]$, $[\beta_{u+v}]$ are representations of S_{n-u}, S_{n-v}, $S_{n-(u+v)}$ respectively. Then it follows that

2.1 $$\sum_{\beta_u} a_{\beta_u\beta_{u+v}}a_{\alpha\beta_u} = \sum_{\beta_v} a_{\beta_v\beta_{u+v}}a_{\alpha\beta_v},$$

that is, in matrix form

2.2 $$(a_{\alpha\beta_u})(a_{\beta_u\beta_{u+v}}) = (a_{\alpha\beta_v})(a_{\beta_v\beta_{u+v}}).$$

We set

2.3 $$(\overset{*}{a}_{\alpha\beta_{u+v}}) = (a_{\alpha\beta_u})(a_{\beta_u\beta_{u+v}})$$

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Then we can define the character

$$\mu_{\alpha}^{(u,v)} = \mu_{\alpha}^{(v,u)}$$

of $S_{n-(u+v)}$ corresponding to χ_α and the character

$$\rho_{\beta_{u+v}}^{(u,v)} = \rho_{\beta_{u+v}}^{(v,u)}$$

of S_n corresponding to $\chi_{\beta_{u+v}}$ as follows:

2.4 $$\mu_{\alpha}^{(u,v)} = \sum_{\beta_{u+v}} a^{*}_{\alpha\beta_{u+v}} \chi_{\beta_{u+v}},$$

2.5 $$\rho_{\beta_{u+v}}^{(u,v)} = \sum_{\alpha} a^{*}_{\alpha\beta_{u+v}} \chi_{\alpha}.$$

The character $\rho_{\beta_{u+v}}^{(u,v)}$ is called the character of type (u, v). Equation (2.1) shows that

2.6 $$\rho_{\beta_{u+v}}^{(u,v)} = \sum_{\beta_u} a_{\beta_u\beta_{u+v}} \rho_{\beta_u}^{(u)} = \sum_{\beta_v} a_{\beta_v\beta_{u+v}} \rho_{\beta_v}^{(v)}.$$

Generally we can define by the same way, the character

$$\rho_{\beta_{u+v+\ldots+w}}^{(u,v,\ldots,w)}$$

of type $(u, v, \ldots, w)$ of S_n corresponding to

$$\chi_{\beta_{u+v+\ldots+w}}$$

of $S_{n-(u+v+\ldots+w)}$. Let $G_1, G_2, \ldots, G_z$ $(z = m(n - (u + v + \ldots + w)))$ be a complete system of representatives for the classes of conjugate elements in $S_{n-(u+v+\ldots+w)}$. Corresponding to (1.10), we can prove by the method used in [**2**], the following

THEOREM 4. *If an element P of S_n is not conjugate to $G_i.W \ldots V.U$ $(i = 1, 2, \ldots, z)$, then*

$$\rho_{\beta_{u+v+\ldots+w}}^{(u,v,\ldots,w)}(P) = 0.$$

3. Let p be a rational prime and let

3.1 $$n = r + wp, \qquad 0 \leqslant r < p.$$

A p-singular element of S_n has at least one cycle of length p or a multiple of p while a p-regular element is simply a permutation, the lengths of whose cycles are all prime to p. If a p-singular element P of S_n has only a λp-cycle as cycle of length a multiple of p, then P will be called an element of type (λ). Generally we may define by a similar way an element of type $(\lambda_1, \lambda_2, \ldots, \lambda_t)$ where $\lambda_1 < \lambda_2 < \ldots < \lambda_t$ and $\sum \lambda_i \leqslant w$. We denote by $b(\lambda_1, \lambda_2, \ldots, \lambda_t)$ the number of classes of conjugate elements in S_n which contain the elements of type $(\lambda_1, \lambda_2, \ldots, \lambda_t)$. If

$$\tfrac{1}{2}q(q + 1) \leqslant w < \tfrac{1}{2}(q + 1)(q + 2),$$

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then the maximal value of t which satisfies $\lambda_1 < \lambda_2 < \ldots < \lambda_t$, $\sum \lambda_i \leqslant w$, is q. We set

3.2 $$\sum_{\lambda_1<\lambda_2<\ldots<\lambda_t} b(\lambda_1, \lambda_2, \ldots, \lambda_t) = h_t$$

and

3.3 $$\sum_{i=t}^{q} h_i = k_t \qquad (t = 1, 2, \ldots, q).$$

Denote by $m'(n)$ the number of p-singular classes in S_n. Then we see easily that

3.4 $$m'(n) = k_1.$$

Let $m(n)$ be the number of classes of conjugate elements in S_n as in §1. We set

3.5 $$\sum_{\lambda_1<\lambda_2<\ldots<\lambda_t} m(n - (\lambda_1 + \lambda_2 + \ldots + \lambda_t)p) = s_t \qquad (t = 1, 2, \ldots, q).$$

Then (3.2) and (3.5) yield

3.6 $$s_t = h_t + \tbinom{t+1}{t}h_{t+1} + \ldots + \tbinom{q}{t}h_q \qquad (t = 1, 2, \ldots, q).$$

We obtain readily from (3.6)

3.7 $$h_t = s_t - \tbinom{t+1}{t}s_{t+1} + \ldots + (-1)^{q-t}\tbinom{q}{t}s_q \qquad (t = 1, 2, \ldots, q).$$

Theorem 5. *Let $m'(n)$ be the number of p-singular classes in S_n. Then*

$$m'(n) = s_1 - s_2 + s_3 - \ldots + (-1)^{q-1}s_q.$$

Proof. From (3.6) we have

$$s_1 - s_2 + s_3 - \ldots + (-1)^{q-1}s_q = \sum_{t=1}^{q} \left(\tbinom{t}{1} - \tbinom{t}{2} + \ldots + (-1)^{t-1}\tbinom{t}{t}\right)h_t$$
$$= \sum_t h_t = k_1 = m'(n).$$

Corollary.

$$s_2 - s_3 + s_4 - \ldots + (-1)^q s_q = \sum_{t=2}^{q} k_t.$$

Proof.

$$s_2 - s_3 + s_4 - \ldots + (-1)^q s_q = s_1 - \sum_t h_t$$
$$= h_2 + 2h_3 + 3h_4 + \ldots + (q-1)h_q = \sum_{t=2}^{q} k_t.$$

4. Let λ_i $(i = 1, 2, \ldots, t)$ be positive integers such that $\lambda_1 < \lambda_2 < \ldots < \lambda_t$ and $u = \sum \lambda_i \leqslant w$. In the following we shall denote by

$$\chi_i^{(u)} \qquad (i = 1, 2, \ldots, m(n - up))$$

the characters of distinct irreducible representations of S_{n-up}, and by

$$\rho_i^{\lambda_1\lambda_2\ldots\lambda_t}$$

the characters of type $(\lambda_1 p, \lambda_2 p, \ldots, \lambda_t p)$ of S_n corresponding to $\chi_i^{(u)}$ of S_{n-up}. If P is not conjugate to

$$V.P_{\lambda_1}.P_{\lambda_2}\ldots P_{\lambda_t}$$

where P_{λ_i} is a cycle of length $\lambda_i p$, and V is any permutation on the remaining $n - up$ symbols, then we have, by Theorem 4,

4.1 $$\rho_i^{\lambda_1\lambda_2\ldots\lambda_t}(P) = 0.$$

Further, if V is a p-regular element of S_{n-up}, then

4.2 $$\rho_i^{\lambda_1\lambda_2\ldots\lambda_t}(V.P_{\lambda_1}.P_{\lambda_2}\ldots P_{\lambda_t}) = \lambda_1\lambda_2\ldots\lambda_t p^t \chi_i^{(u)}(V).$$

In particular, we obtain

THEOREM 6. *If H is a p-regular element of S_n, then for any type* $(\lambda_1, \lambda_2, \ldots \lambda_t)$

$$\rho_i^{\lambda_1\lambda_2\ldots\lambda_t}(H) = 0.$$

Let $P_1, P_2, \ldots, P_{m'(n)}$ be a complete system of representatives for the p-singular classes in S_n. If we set

4.3 $$R_1 = (\rho_i^{\lambda}(P_j))$$

(j, row index; λ, i, column indices; where $\lambda = 1, 2, \ldots, w$; $i = 1, 2, \ldots, m(n - \lambda p)$; $j = 1, 2, \ldots, m'(n)$, then R_1 is a matrix of type $(m'(n), s_1)$ and we have proved in [**2**]

4.4 $$r(R_1) = m'(n) = k_1,$$

where $r(R_1)$ denotes the rank of R_1. Generally we set

4.5 $$R_t = (\rho_i^{\lambda_1\lambda_2\ldots\lambda}(P_j))$$

(j, row index: $(\lambda_1, \lambda_2, \ldots, \lambda_t)$, i, column indices). Then R_t is a matrix of type $(m'(n), s_t)$ and we can prove, as in [**2**], the following

THEOREM 7. *Let $r(R_t)$ be the rank of R_t. Then $r(R_t) = k_t$ where k_t is the number defined in* (3.3).

5. Let $[\alpha_0]$ be a p-core of S_{n-up}. Then $[\alpha_0]$ determines uniquely a p-block $B[\alpha_0]$ of S_n. We call u the weight of $B[\alpha_0]$. As in [**2**], we define $l^*(u)$ by

5.1 $$l^*(u) = \sum_{\nu_1, \nu_2, \ldots, \nu_{p-1}} m(\nu_1)m(\nu_2)\ldots m(\nu_{p-1}),$$

where the ν_i are the positive integers or zero, and the summation extends over all sets $(\nu_1, \nu_2, \ldots, \nu_{p-1})$ which satisfy $\sum \nu_i = u$. Let $c(n)$ be the number of p-cores of n nodes. We set $c(0) = 1$. Then we have by [**2**]

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5.2 $$m^*(n) = \sum_{u=0}^{w} c(n - up)l^*(u)$$

where $m^*(n)$ is the number of p-regular classes in S_n, i.e., the number of modular irreducible representations of S_n.

THEOREM 8. *The number of modular irreducible representations in a p-block of weight v is $l^*(v)$.*

Proof. By [**3**], the number of modular irreducible representations in any p-block of weight v is independent of the p-core. Hence we denote this number by $f(v)$. We have

5.3 $$m^*(n) = \sum_{u=0}^{w} c(n - up)f(u).$$

Since $l^*(0) = f(0) = 1$ and $l^*(1) = f(1) = p - 1$, we assume that $l^*(u) = f(u)$ for $u < v$. We set $n = vp$ in (5.2) and (5.3). Then

$$m^*(n) = \sum_{u=0}^{v} c(vp - up)l^*(u) = l^*(v) + \sum_{u=0}^{v-1} c(vp - up)l^*(u)$$

and

$$m^*(n) = \sum_{u=0}^{v} c(vp - up)f(u) = f(v) + \sum_{u=0}^{v-1} c(vp - up)f(u).$$

By our assumption, $l^*(u) = f(u)$ $(u = 1, 2, \ldots, v - 1)$. Hence we obtain $l^*(v) = f(v)$.

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