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INTEGRAL FORMULAE ON QUASI-EINSTEIN MANIFOLDS AND APPLICATIONS

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Abstract. The aim of this paper is to extend for the m-quasi-Einstein metrics some integral formulae obtained in [**1**] (C. Aquino, A. Barros and E. Ribeiro Jr., Some applications of the Hodge-de Rham decomposition to Ricci solitons, *Results Math.* **60** (2011), 245–254) for Ricci solitons and derive similar results achieved there. Moreover, we shall extend the m-Bakry-Emery Ricci tensor for a vector field X on a Riemannian manifold instead of a gradient field ∇f, in order to obtain some results concerning these manifolds that generalize their correspondents to a gradient field.

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1. Introduction. One of the motivation to study quasi-Einstein metrics on a Riemannian manifold (M^n, g) is their close relation to Einstein metrics, which are warped products, see e.g. [**4**]. In this subject the m-Bakry-Emery Ricci tensor appears naturally. This tensor is given as follows:

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df, \tag{1.1}$$

where $0 < m \leq \infty$, while Ric and $\nabla^2 f$ stand for the Ricci tensor and the Hessian form, respectively. A natural generalisation for the previous tensor is to consider a vector field X instead of a gradient of a smooth function f, more exactly, we define Ric_X^m as follows:

$$Ric_X^m = Ric + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m} X^\flat \otimes X^\flat, \tag{1.2}$$

where $X \in \mathfrak{X}(M)$, $X^\flat$ is the 1-form associated to X, while $\mathcal{L}_X g$ stands for the Lie derivative of the vector field X.

A metric g on a Riemannian manifold (M^n, X) will be called m-quasi-Einstein metric, or simply a quasi-Einstein metric if the next relation

$$Ric + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m} X^\flat \otimes X^\flat = \lambda g \tag{1.3}$$

*Both partially supported by CNPq-BR.

holds for some $\lambda \in \mathbb{R}$. In particular, we have

$$Ric(X, X) + \langle \nabla_X X, X \rangle = \frac{1}{m}|X|^4 + \lambda |X|^2. \tag{1.4}$$

Moreover, taking the trace of equation (1.3), we deduce

$$R + \mathrm{div} X - \frac{1}{m}|X|^2 = \lambda n. \tag{1.5}$$

We point out that if $m = \infty$, then equation (1.3) reduces to the one associated to a Ricci soliton, as well as when m is a positive integer and X is a gradient vector field, it corresponds to warped product Einstein metrics, for more details see [**5**]. Following the terminology of Ricci soliton, a quasi-Einstein metric g on a manifold M^n will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

DEFINITION 1. A quasi-Einstein metric will be called *trivial* if $X \equiv 0$.

The triviality definition is equivalent to saying that M^n is an Einstein manifold. On the other hand, it is well known that on a compact manifold an ∞-quasi-Einstein metric with $\lambda \le 0$ is trivial, see [**6**]. The same result was proved in [**9**] for quasi-Einstein metric on compact manifold with m finite. Besides, we known that compact shrinking Ricci solitons have positive scalar curvature, see for example [**6**]. An extension of this result for shrinking quasi-Einstein metric with X a gradient vector field and $1 \le m < \infty$ was obtained in [**5**].

Before announcing the results we point out that they are generalisations of the results due to [**1**, **10**] for Ricci solitons. Firstly, we have the following theorem.

THEOREM 1. *Let (M^n, g, X), $n \ge 3$, be a compact Riemannian manifold satisfying $Ric_X^m = \lambda g$. Then M^n is an Einstein manifold provided:*

(1) $\int_M Ric(X, X)\mathrm{dM} \le \frac{2}{m}\int_M |X|^2 \mathrm{div} X \mathrm{dM}$.
(2) *X is a conformal vector field and* $\int_M Ric(X, X)\mathrm{dM} \le 0$.
(3) $|X|$ *is constant and* $\int_M Ric(X, X)\mathrm{dM} \le 0$.

In order to proceed we remember a result due to Yau [**11**], which is a generalisation of Hopf's theorem: A subharmonic function $f : M^n \to \mathbb{R}$ defined over a complete non-compact Riemannian manifold is constant, provided its gradient belongs to $\mathrm{L}^1(M^n)$. Recently, this result was extended by Camargo et al. [**3**] for a vector field X. With the aid of this extension we derive the following result.

THEOREM 2. *Let (M^n, g, X) be a complete, non-compact Riemannian manifold satisfying $Ric_X^m = \lambda g$. If $n\lambda \ge R$ and $|X| \in \mathrm{L}^1(M^n)$, then M^n is an Einstein manifold.*

Before proceeding, we make an observation: When $X = \nabla f$ is a gradient field, equation (1.5) enables us to write

$$R + \Delta f = \frac{1}{m}|\nabla f|^2 + \lambda n. \tag{1.6}$$

Thereby, we derive

$$\langle \nabla f, \nabla R \rangle + \langle \nabla f, \nabla \Delta f \rangle = \frac{2}{m}\langle \nabla_{\nabla f} \nabla f, \nabla f \rangle. \tag{1.7}$$

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2. Preliminaries. In this section we shall present some preliminaries which will be useful for the establishment of desired results. First we remember Lemma 2.1 due to [**10**].

LEMMA 1. *Given a vector field X on a Riemannian manifold (M^n, g), we have*

$$\mathrm{div}(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + Ric(X, X) + D_X \mathrm{div} X. \tag{2.1}$$

In particular, if $X = \nabla f$ is a gradient field, we have for all $Z \in \mathfrak{X}(M)$

$$\mathrm{div}(\mathcal{L}_X g)(Z) = 2Ric(Z, X) + 2D_Z \mathrm{div} X, \tag{2.2}$$

or in $(1, 1)$*-tensorial notation*

$$\mathrm{div}\nabla\nabla f = Ric(\nabla f) + \nabla\Delta f. \tag{2.3}$$

Remembering that the diffusion operator is given by $\Delta_X = \Delta - D_X$, the previous lemma allows us to deduce the following one.

LEMMA 2. *Let (M^n, g, X) be a Riemannian manifold such that $Ric_X^m = \lambda g$. Then we have*

(1) $\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - Ric(X, X) + \frac{2}{m}|X|^2 \mathrm{div} X$.
(2) $\frac{1}{2}\Delta_X|X|^2 = |\nabla X|^2 - \lambda|X|^2 + \frac{1}{m}|X|^2(2\mathrm{div} X - |X|^2)$.
(3) *If M^n is compact and $\nabla X = 0$, then $X = 0$.*

Proof. Since $\mathrm{div}\, g = 0$, we deduce from the assumptions of the lemma that

$$\mathrm{div} Ric + \frac{1}{2}\mathrm{div}\mathcal{L}_X g - \frac{1}{m}\mathrm{div}(X^\flat \otimes X^\flat) = 0.$$

Next, we use the contracted second Bianchi identity, $\nabla R = 2\mathrm{div} Ric$, to arrive at

$$\nabla R + \mathrm{div}\mathcal{L}_X g - \frac{2}{m}\mathrm{div} X X^\flat - \frac{2}{m}(\nabla|X|^2)^\flat = 0.$$

In particular, for any $Z \in \mathfrak{X}(M)$ we have

$$\langle\nabla R, Z\rangle + \mathrm{div}(\mathcal{L}_X g)(Z) - \frac{2}{m}X^\flat(Z)\mathrm{div} X - \frac{1}{m}(\nabla|X|^2)^\flat(Z) = 0.$$

Therefore, for $Z = X$ we deduce

$$\mathrm{div}(\mathcal{L}_X g)(X) = -\langle\nabla R, X\rangle + \frac{2}{m}\mathrm{div} X X^\flat(X) + \frac{1}{m}\mathcal{L}_X g(X, X). \tag{2.4}$$

Next, we use the relation $\nabla R + \nabla \mathrm{div} X = \frac{1}{m}\nabla|X|^2$, jointly with equations (2.1) and (2.4) to arrive at

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - Ric(X, X) - D_X \mathrm{div} X + \frac{1}{m}\mathcal{L}_X g(X, X) + D_X \mathrm{div} X$$
$$- \frac{1}{m}X(|X|^2) + \frac{2}{m}\mathrm{div} X X^\flat(X).$$

Hence, we make use of Lemma 1 to conclude the first assertion of the lemma.

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Next, we notice that the second assertion is immediate from the first one just applying (1.4).

Supposing $\nabla X = 0$, we have $|X|$ constant as well as $\mathrm{div}X = 0$. Hence, the first item of the lemma yields $Ric(X, X) = 0$. Now we use equation (1.4) to deduce

$$\frac{1}{m}|X|^4 + \lambda|X|^2 = 0. \tag{2.5}$$

If λ is non-negative we are done. Otherwise, let us assume $X \neq 0$ to arrive at a contradiction. In fact, equation (2.5) enables us to write $\lambda = -\frac{1}{m}|X|^2$. Thus, we obtain

$$Ric(X, Y) = \frac{1}{m}X^\flat(X)X^\flat(Y) - \frac{1}{m}|X|^2 g(X, Y) = 0, \tag{2.6}$$

for any Y. So, we conclude that M^n is Ricci flat. On the other hand, if we consider Y a non-zero vector orthogonal to X, we get $Ric(Y, Y) = \frac{1}{m}(\langle X, Y\rangle^2 - |X|^2|Y|^2) = -\frac{1}{m}|X|^2|Y|^2 < 0$, giving a contradiction. Then, $\lambda < 0$, also implies $X = 0$, which finishes the proof of the lemma. □

Taking $X = \nabla f$ in the previous lemma and letting $\Delta_f = \Delta_{\nabla f}$, we derive the following corollary.

Corollary 1. *Under the assumptions of Lemma 2, if in addition $X = \nabla f$, then the following are true.*

(1) $\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 - Ric(\nabla f, \nabla f) + \frac{2}{m}|\nabla f|^2\Delta f$.
(2) $\frac{1}{2}\Delta_f|\nabla f|^2 = |\nabla\nabla f|^2 - \lambda|\nabla f|^2 + \frac{1}{m}|\nabla f|^2(2\Delta f - |\nabla f|^2)$.

Writing equation (1.3) in the tensorial language

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{m}(df \otimes df)_{ij} = \lambda g_{ij}, \tag{2.7}$$

we have the following lemma.

Lemma 3. *Let $(M^n, g, \nabla f)$ be a Riemannian manifold such that $n \geq 3$ and $Ric^m_{\nabla f} = \lambda g$. Then the following formulae hold*:

(1) $\frac{1}{2}\nabla_i R = \frac{m-1}{m}R_{ij}\nabla^j f + \frac{1}{m}(R - (n-1)\lambda)\nabla_i f$.
(2) $\nabla_k R_{ij} - \nabla_j R_{ik} = R_{ijks}\nabla^s f + \frac{1}{m}(R_{ij}\nabla_k f - R_{ik}\nabla_j f) - \frac{\lambda}{m}(g_{ij}\nabla_k f - g_{ik}\nabla_j f)$.
(3) $\nabla(R + |\nabla f|^2 - 2\lambda f) = \frac{2}{m}\{\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f\}$.

Proof. For the first assertion we address the reader to formula (3.12) in Lemma 3.2 in **[5]**. Now we treat item (2). From equation (2.7) we infer

$$\begin{aligned}\nabla_k R_{ij} - \nabla_j R_{ik} &= -(\nabla_k\nabla_j\nabla_i f - \nabla_j\nabla_k\nabla_i f)\\ &\quad + \frac{1}{m}(\nabla_k\nabla_i f\nabla_j f + \nabla_k\nabla_j f\nabla_i f - \nabla_j\nabla_i f\nabla_k f - \nabla_j\nabla_k f\nabla_i f)\\ &= R_{ijks}\nabla^s f + \frac{1}{m}(R_{ij}\nabla_k f - R_{ik}\nabla_j f) - \frac{\lambda}{m}(g_{ij}\nabla_k f - g_{ik}\nabla_j f),\end{aligned}$$

where we interchanged the covariant derivatives to get item (2).

Finally, we prove the last item of the lemma. In fact, from item (1) and equation (2.7) we deduce

$$\begin{aligned}\frac{1}{2}\nabla(R+|\nabla f|^2) &= \frac{m-1}{m}Ric(\nabla f)+\frac{1}{m}(R-(n-1)\lambda)\nabla f+\nabla_{\nabla f}\nabla f\\ &= Ric(\nabla f)+\nabla_{\nabla f}\nabla f-\frac{1}{m}Ric(\nabla f)+\frac{1}{m}(R-(n-1)\lambda)\nabla f\\ &= \frac{1}{m}|\nabla f|^2\nabla f+\lambda\nabla f-\frac{1}{m}Ric(\nabla f)+\frac{1}{m}(R-(n-1)\lambda)\nabla f.\end{aligned}$$

Thus, using $R-n\lambda=\frac{1}{m}|\nabla f|^2-\Delta f$ we achieve

$$\begin{aligned}\nabla(R+|\nabla f|^2-2\lambda f) &= \frac{2}{m}\{(|\nabla f|^2+R-n\lambda+\lambda)\nabla f-Ric(\nabla f)\}\\ &= \frac{2}{m}\left\{(|\nabla f|^2+\frac{1}{m}|\nabla f|^2-\Delta f+\lambda)\nabla f-Ric(\nabla f)\right\}\\ &= \frac{2}{m}\left\{(|\nabla f|^2-\Delta f)\nabla f+\frac{1}{m}|\nabla f|^2\nabla f+\lambda\nabla f-Ric(\nabla f)\right\}\\ &= \frac{2}{m}\{(|\nabla f|^2-\Delta f)\nabla f+\nabla_{\nabla f}\nabla f\},\end{aligned}$$

which concludes the proof of the lemma. □

It is convenient to notice that for $m=\infty$ we derive the classical Hamilton equation [7] for a gradient Ricci soliton:

$$R+|\nabla f|^2-2\lambda f=C, \tag{2.8}$$

where C is constant.

As a consequence of the preceding lemma we obtain the following corollary.

Corollary 2. *Let* $(M^n,\ g,\ \nabla f)$ *be a Riemannian manifold such that* $n\geq 3$ *and* $Ric^m_{\nabla f}=\lambda g$. *Then the following formulae hold:*

(1) $\frac{1}{2}\langle\nabla R,\nabla f\rangle=\frac{m-1}{m}Ric(\nabla f,\nabla f)+\frac{1}{m}(R-(n-1)\lambda)|\nabla f|^2$.
(2) $\frac{1}{2}|\nabla R|^2=\frac{m-1}{m}Ric(\nabla f,\nabla R)+\frac{1}{m}(R-(n-1)\lambda)\langle\nabla f,\nabla R\rangle$.

Proof. We choose $Z\in\mathfrak{X}(M)$ on item (1) of the quoted lemma to deduce

$$\frac{1}{2}\langle\nabla R,Z\rangle=\frac{m-1}{m}Ric(\nabla f,Z)+\frac{1}{m}(R-(n-1)\lambda)\langle\nabla f,Z\rangle. \tag{2.9}$$

Therefore, the corollary follows. □

Proceeding, we arrive at the main lemma of this section.

Lemma 4. *Let* $(M^n,\ g,\ \nabla f)$ *be a Riemannian manifold satisfying* $Ric^m_{\nabla f}=\lambda g$. *Then,*

$$\begin{aligned}\frac{1}{2}\Delta R &= -Ric(\nabla f,\nabla f)-\left|\nabla^2 f-\frac{(\Delta f)}{n}g\right|^2-\frac{(\Delta f)^2}{n}+\lambda\Delta f+\langle\nabla R,\nabla f\rangle\\ &\quad+\frac{1}{m}\left\{|\nabla f|^2\Delta f+\operatorname{div}(\nabla_{\nabla f}\nabla f-\nabla f\Delta f)\right\}.\end{aligned} \tag{2.10}$$

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Proof. Initially we compute the divergence of identity (3) of Lemma 3 to obtain

$$\Delta R+\Delta|\nabla f|^2-2\lambda\Delta f=\frac{2}{m}\big\{\langle\nabla(|\nabla f|^2-\Delta f),\nabla f\rangle+(|\nabla f|^2-\Delta f)\Delta f+\operatorname{div}(\nabla_{\nabla f}\nabla f)\big\}.$$

Using Bochner's formula: $\frac{1}{2}\Delta|\nabla f|^2=Ric(\nabla f,\nabla f)+|\nabla^2 f|^2+\langle\nabla f,\nabla\Delta f\rangle$, and writing $|\nabla^2 f|^2=|\nabla^2 f-\frac{(\Delta f)}{n}g|^2-\frac{1}{n}(\Delta f)^2$, we have

$$\frac{1}{2}\Delta R=-Ric(\nabla f,\nabla f)-\left|\nabla^2 f-\frac{\Delta f}{n}g\right|^2-\frac{(\Delta f)^2}{n}+\lambda\Delta f-\langle\nabla\Delta f,\nabla f\rangle$$
$$+\frac{2}{m}\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle+\frac{1}{m}\big\{(|\nabla f|^2-\Delta f)\Delta f-\langle\nabla\Delta f,\nabla f\rangle+\operatorname{div}(\nabla_{\nabla f}\nabla f)\big\}.$$

Next, we invoke equation (1.6) to write

$$\langle\nabla\Delta f,\nabla f\rangle=\left\langle\nabla\left(n\lambda+\frac{1}{m}|\nabla f|^2-R\right),\nabla f\right\rangle=\frac{2}{m}\langle\nabla_{\nabla f}\nabla f,\nabla f\rangle-\langle\nabla R,\nabla f\rangle.$$

Then, the last relation for $\frac{1}{2}\Delta R$ turns into

$$\frac{1}{2}\Delta R=-Ric(\nabla f,\nabla f)-\left|\nabla^2 f-\frac{\Delta f}{n}g\right|^2-\frac{(\Delta f)^2}{n}+\lambda\Delta f+\langle\nabla R,\nabla f\rangle$$
$$+\frac{1}{m}\big\{(|\nabla f|^2-\Delta f)\Delta f-\langle\nabla\Delta f,\nabla f\rangle+\operatorname{div}(\nabla_{\nabla f}\nabla f)\big\}.$$

At this point we use $\operatorname{div}(\nabla f\Delta f)=(\Delta f)^2+\langle\nabla\Delta f,\nabla f\rangle$ to achieve the formula in the statement, which finishes the proof of lemma. □

3. Proofs of the results stated in the introduction.

3.1. Proof of Theorem 1. First we integrate identity (1) of Lemma 2 to infer

$$\frac{1}{2}\int_M\Delta|X|^2\,\mathrm{dM}=\int_M|\nabla X|^2\,\mathrm{dM}-\int_M Ric(X,X)\,\mathrm{dM}+\frac{2}{m}\int_M|X|^2\operatorname{div}X\,\mathrm{dM}.$$

This yields

$$\int_M|\nabla X|^2\,\mathrm{dM}=\int_M Ric(X,X)\,\mathrm{dM}-\frac{2}{m}\int_M|X|^2\operatorname{div}X\,\mathrm{dM}. \tag{3.1}$$

Since we are assuming that the right-hand side of (3.1) is less than or equal to zero, we obtain $\nabla X=0$. So, assertion (3) of Lemma 2 allows us to conclude the first item.

Proceeding, we know that there exists a smooth function ρ on M, for which

$$\mathcal{L}_X g=2\rho g. \tag{3.2}$$

In particular, $\langle\nabla_X X,X\rangle=\rho|X|^2$. Moreover, taking the trace of both members of equation (3.2) we also obtain

$$\operatorname{div}X=n\rho. \tag{3.3}$$

On the other hand, we notice that

$$\begin{aligned}\operatorname{div}(X|X|^2) &= |X|^2\operatorname{div}X + 2\langle\nabla_X X, X\rangle\\ &= (n+2)\rho|X|^2.\end{aligned}$$

Since M^n is compact, we use Stokes' formula to obtain

$$\int_M \rho|X|^2\,\mathrm{dM} = 0. \tag{3.4}$$

Thereby, using this result jointly with relation (3.1), we conclude that $\nabla X = 0$, since we are assuming $\int_M Ric(X, X)\,\mathrm{dM} \leq 0$. Therefore, using assertion (3) of Lemma 2, we conclude that M^n is an Einstein manifold.

Finally, if $|X|$ is constant, we can apply Stokes' formula on equation (3.1) to derive

$$\int_M |\nabla X|^2\,\mathrm{dM} = \int_M Ric(X, X)\,\mathrm{dM}. \tag{3.5}$$

From here we conclude the proof of the theorem.

REMARK 1. We notice that for $n = 2$, we may write equation (3.1) as follows

$$\int_M |\nabla X|^2\mathrm{dM} = \frac{1}{2}\int_M K|X|^2\mathrm{dM} - \frac{2}{m}\int_M |X|^2\operatorname{div}X\mathrm{dM}, \tag{3.6}$$

where K stands for the Gaussian curvature. In particular we have:

- If $|X|$ is a non-null constant, then M^2 has genus zero or one.
- If X is a non-trivial conformal vector field and K is constant, then M^2 is isometric to $\mathbb{S}^2(r)$.

3.2. Proof of Theorem 2. Taking into account that $Ric_X^m = \lambda g$, then by equation (1.5) we arrive at

$$m\operatorname{div}X = |X|^2 + m(n\lambda - R). \tag{3.7}$$

Thus, if $(n\lambda - R) \geq 0$, then we have $m\operatorname{div}X \geq 0$. On the other hand, if $|X| \in L^1(M)$, we may invoke Proposition 1 in [**3**] to derive that $\operatorname{div}X = 0$. Next, we may use equation (3.7) to conclude that $X \equiv 0$ as well as $n\lambda = R$. Therefore, M is an Einstein manifold and we finish the proof of the theorem.

4. Integral formulae for quasi-Einstein manifolds. In this section we shall show some integral formulae for a compact quasi-Einstein manifold M^n, which are generalisation of the formulae obtained for Ricci solitons in [**1**]. Those formulae enable us to derive some rigidity results for such a class of manifolds.

THEOREM 3. *Let* $(M^n, g, \nabla f)$ *be a Riemannian manifold satisfying* $Ric^m_{\nabla f} = \lambda g$. *Then we have*

$$\frac{1}{2}\Delta_f R = -\left|\nabla^2 f - \frac{(\Delta f)}{n}g\right|^2 - \frac{(\Delta f)^2}{n} + \lambda\Delta f + \frac{1}{2}\langle\nabla f, \nabla R\rangle + \frac{1}{2}\langle\nabla f, \nabla\Delta f\rangle$$
$$+ \frac{1}{m}\mathrm{div}(\nabla_{\nabla f}\nabla f - \Delta f\nabla f).$$

Proof. First of all we use Lemma 4 to obtain the following equation

$$\frac{1}{2}\Delta R - \frac{1}{2}\langle\nabla R, \nabla f\rangle = -Ric(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{(\Delta f)}{n}g\right|^2 - \frac{(\Delta f)^2}{n} + \lambda\Delta f + \frac{1}{2}\langle\nabla R, \nabla f\rangle$$
$$+ \frac{1}{m}|\nabla f|^2\Delta f + \frac{1}{m}\mathrm{div}(\nabla_{\nabla f}\nabla f - \nabla f\Delta f). \tag{4.1}$$

Now, using the definition of diffusion operator and substituting identity (1) of Corollary 2 in the preceding equation, we obtain

$$\frac{1}{2}\Delta_f R = -Ric(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{(\Delta f)}{n}g\right|^2 - \frac{(\Delta f)^2}{n} + \lambda\Delta f$$
$$+ \frac{m-1}{m}Ric(\nabla f, \nabla f) + \frac{1}{m}(R - (n-1)\lambda)|\nabla f|^2 + \frac{1}{m}|\nabla f|^2\Delta f$$
$$+ \frac{1}{m}\mathrm{div}(\nabla_{\nabla f}\nabla f - \nabla f\Delta f).$$

From here we deduce

$$\frac{1}{2}\Delta_f R = -\left|\nabla^2 f - \frac{(\Delta f)}{n}g\right|^2 - \frac{(\Delta f)^2}{n} + \lambda\Delta f - \frac{1}{m}Ric(\nabla f, \nabla f)$$
$$+ \frac{1}{m}(R + \Delta f - n\lambda)|\nabla f|^2 + \frac{1}{m}\lambda|\nabla f|^2 + \frac{1}{m}\mathrm{div}(\nabla_{\nabla f}\nabla f - \nabla f\Delta f).$$

Next, using $R + \Delta f - n\lambda = \frac{1}{m}|\nabla f|^2$, we infer

$$\frac{1}{2}\Delta_f R = -\left|\nabla^2 f - \frac{(\Delta f)}{n}g\right|^2 - \frac{(\Delta f)^2}{n} + \lambda\Delta f$$
$$+ \frac{1}{m}\left\{-Ric(\nabla f, \nabla f) + \frac{1}{m}|\nabla f|^4 + \lambda|\nabla f|^2 + \mathrm{div}(\nabla_{\nabla f}\nabla f - \nabla f\Delta f)\right\}.$$

On the other hand, using equation (1.4) with $X = \nabla f$, we have

$$-Ric(\nabla f, \nabla f) + \frac{1}{m}|\nabla f|^4 + \lambda|\nabla f|^2 = \langle\nabla_{\nabla f}\nabla f, \nabla f\rangle = \frac{m}{2}(\langle\nabla f, \nabla R\rangle + \langle\nabla f, \nabla f\rangle), \tag{4.2}$$

where for the last equality we have used equation (1.7). Substituting this in the above formula for $\Delta_f R$, we get the expression in the statement, which completes the proof of the theorem. □

As a consequence of this theorem, we deduce the following integral formulae.

COROLLARY 3. *Let* $(M^n, g, \nabla f)$ *be a compact orientable Riemannian manifold satisfying* $Ric^m_{\nabla f} = \lambda g$. *Then we have*

(1) $\int_M |\nabla^2 f - \frac{(\Delta f)}{n} g|^2 \, dM = \frac{3}{2}\int_M \langle \nabla f, \nabla R\rangle \, dM + \frac{n+2}{2n}\int_M \langle \nabla f, \nabla \Delta f\rangle \, dM.$

(2) $\int_M |\nabla^2 f - \frac{(\Delta f)}{n} g|^2 \, dM + \frac{n+2}{2n}\int_M (\Delta f)^2 \, dM = \frac{3}{2}\int_M \langle \nabla f, \nabla R\rangle \, dM.$

(3) $\int_M Ric(\nabla f, \nabla f) \, dM + \frac{3}{2}\int_M \langle \nabla f, \nabla R\rangle \, dM = \frac{3}{2}\int_M (\Delta f)^2 \, dM.$

(4) M^n *is an Einstein manifold, if* $\int_M \langle \nabla R, \nabla f\rangle \, dM \leq 0$.

(5) *Suppose that* f *is not constant and there exists* $\mu : M^n \to \mathbb{R}$ *solution of the equation* $\frac{n+2}{2n}\Delta f + \frac{3}{2}R = \mu$, *such that* $\mu \perp \Delta f$, *in the* L^2 *inner product. Then* M^n *is conformally equivalent to a unit sphere* $\mathbb{S}^n$, *but not isometric.*

Proof. Since M^n is compact, we can use Theorem 3 and Stokes' formula to infer

$$\int_M \left|\nabla^2 f - \frac{(\Delta f)}{n} g\right|^2 dM = \int_M \left(\lambda - \frac{\Delta f}{n}\right)\Delta f \, dM + \int_M \langle \nabla f, \nabla R\rangle \, dM + \frac{1}{2}\int_M \langle \nabla f, \nabla(R + \Delta f)\rangle \, dM.$$

Next, we use relation (1.6) to write $\int_M \left(\lambda - \frac{\Delta f}{n}\right)\Delta f \, dM = \frac{1}{n}\int_M \left(R - \frac{1}{m}|\nabla f|^2\right)\Delta f \, dM$. Then, Stokes' formula gives

$$\frac{1}{n}\int_M \left(R - \frac{1}{m}|\nabla f|^2\right)\Delta f \, dM = -\frac{1}{n}\int_M \langle \nabla f, \nabla R\rangle \, dM + \frac{1}{nm}\int_M \langle \nabla f, \nabla |\nabla f|^2\rangle \, dM.$$

On the other hand, we notice that equation (1.6) yields $\nabla(R + \Delta f) = \frac{1}{m}\nabla(|\nabla f|^2)$. By using this datum on the previous equation, we have

$$\int_M \left|\nabla^2 f - \frac{(\Delta f)}{n} g\right|^2 dM = \frac{3}{2}\int_M \langle \nabla f, \nabla R\rangle \, dM + \frac{n+2}{2n}\int_M \langle \nabla f, \nabla \Delta f\rangle \, dM, \tag{4.3}$$

which ends the first assertion.

Proceeding, since $\int_M \langle \nabla f, \nabla \Delta f\rangle \, dM = -\int_M (\Delta f)^2 \, dM$, we obtain from equation (4.3) that

$$\int_M \left|\nabla^2 f - \frac{(\Delta f)}{n} g\right|^2 dM = \frac{3}{2}\int_M \langle \nabla f, \nabla R\rangle \, dM - \frac{n+2}{2n}\int_M (\Delta f)^2 \, dM, \tag{4.4}$$

which gives the second item.

Next, we integrate Bochner's formula to get

$$\int_M Ric(\nabla f, \nabla f) \, dM + \int_M |\nabla^2 f|^2 \, dM + \int_M \langle \nabla f, \nabla \Delta f\rangle \, dM = 0. \tag{4.5}$$

Since $\int_M |\nabla^2 f - \frac{(\Delta f)}{n} g|^2 \, dM = \int_M |\nabla^2 f|^2 \, dM - \frac{1}{n}\int_M (\Delta f)^2 \, dM$, we can use once more Stokes' formula to arrive at

$$\int_M Ric(\nabla f, \nabla f) \, dM + \int_M \left|\nabla^2 f - \frac{(\Delta f)}{n} g\right|^2 dM = \frac{n-1}{n}\int_M (\Delta f)^2 \, dM. \tag{4.6}$$

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Now, comparing equation (4.6) with the second item we arrive at

$$\int_M \left\{ Ric(\nabla f, \nabla f) + \frac{3}{2}\langle \nabla f, \nabla R\rangle \right\} \mathrm{dM} = \frac{3}{2}\int_M (\Delta f)^2 \,\mathrm{dM},$$

as we want.

On the other hand, if $\int_M \langle \nabla R, \nabla f\rangle \,\mathrm{dM} \leq 0$, in particular this occurs if R is constant, we deduce, from the second item, that

$$\int_M \langle \nabla R, \nabla f\rangle \,\mathrm{dM} = 0 \tag{4.7}$$

and

$$\int_M \left| \nabla^2 f - \frac{(\Delta f)}{n} g \right|^2 \mathrm{dM} + \frac{n+2}{2n}\int_M (\Delta f)^2 \,\mathrm{dM} = 0. \tag{4.8}$$

This implies that $\nabla^2 f = \frac{1}{n}(\Delta f)g$ and $\Delta f = 0$. Hence, we can apply Hopf's theorem to deduce that f is constant, which implies that M^n is an Einstein manifold.

Finally, we notice that $\int_M |\nabla^2 f - \frac{(\Delta f)}{n} g|^2 \,\mathrm{dM} = \int_M \langle \nabla f, \nabla\big(\frac{n+2}{2n}\Delta f + \frac{3}{2}R\big)\rangle \,\mathrm{dM}$. So, if $\frac{n+2}{2n}\Delta f + \frac{3}{2}R = \mu$, with $\int_M \mu \Delta f \,\mathrm{dM} = 0$, we have $\nabla^2 f = \frac{1}{n}(\Delta f)g$. Since f is not constant, this allows us to apply Theorem 2 due to Ishara and Tashiro [**8**] to conclude that M^n is conformally equivalent to a unit sphere $\mathbb{S}^n$. Moreover, if we have an isometry between M^n and $\mathbb{S}^n$, then its scalar curvature R would be constant. From assertion (2), we conclude that $\int_M |\nabla^2 f - \frac{(\Delta f)}{n} g|^2 \,\mathrm{dM} + \frac{n+2}{2n}\int_M (\Delta f)^2 \,\mathrm{dM} = 0$. Then, the previous assertion yields that f must be constant, which contradicts our assumption on f. Hence, we complete the proof of the corollary. □

As a consequence of this corollary, we derive the next result.

COROLLARY 4. *Let $(M^n, g, \nabla f)$ be an orientable compact Riemannian manifold satisfying $Ric^m_{\nabla f} = \lambda g$. Then ∇f can not be a non-trivial conformal vector field.*

Proof. Let us suppose that ∇f is a non-trivial conformal vector field, i.e. $\mathcal{L}_{\nabla f} g = 2\rho g$ with ρ not constant. Therefore, we can apply Theorem $II.9$ from [**2**] to deduce that

$$\int_M \mathcal{L}_{\nabla f} R \,\mathrm{dM} = \int_M \langle \nabla f, \nabla R\rangle \,\mathrm{dM} = 0. \tag{4.9}$$

Then, the previous corollary enables us to finish the proof. □

REMARK 2. We point out that $\int_M \langle \nabla f, \nabla R\rangle \mathrm{dM} = 0$ in dimension two for m finite is always valid. In fact, since $\nabla(e^{-\frac{f}{m}})$ is a conformal field and the Dirichlet integral is a conformal invariant, the claim follows from Theorem II.9 from [**2**]. Therefore, if $(M^2, g, \nabla f)$ is a compact quasi-Einstein manifold, then it is trivial by Corollary 3, see also [**5**] and [**9**].

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