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MAXIMAL COPLANAR SETS OF INTERSECTION POINTS

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Let F be any set of five points in $\mathbf{R}^3$ so situated that no four of the points are coplanar, and that the line $\mathbf{xy}$ through any two $\mathbf{x}$ and $\mathbf{y}$ of the points has a unique intersection point $\mathbf{xy}^*$ with the plane determined by the other three. Let $F^\wedge$ denote the family of all such $\mathbf{xy}^*$. Let $\mathcal{S}(F)$ denote the set of all $X \subseteq F^\wedge$ which are maximal with respect to the property that X is a subset of a plane in $\mathbf{R}^3$. For $k > 2$ an integer, let $\mathcal{S}(k;F)$ denote the family of all k-membered elements in $\mathcal{S}(F)$.

A family $\mathcal{D}$ of sets is said to be ***uniformly deep of depth d*** if and only if for every $x \in \bigcup\mathcal{D}$ there are exactly d distinct $A \in \mathcal{D}$ for which $x \in A$.

We establish the following result, and extend our ideas to general Euclidean spaces.

THEOREM. *$F^\wedge$ contains exactly ten points, and no three of them are collinear. Furthermore, $\mathcal{S}(F) = \mathcal{S}(3;F) \cup \mathcal{S}(4;F)$ with $|\mathcal{S}(3;F)| = 20$ and with $|\mathcal{S}(4;F)| = 25$. Both $\mathcal{S}(3;F)$ and $\mathcal{S}(4;F)$ are uniformly deep; the depth of $\mathcal{S}(3;F)$ is 6, and the depth of $\mathcal{S}(4;F)$ is 10.*

1. INTRODUCTION

This paper considers subsets $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ of n-dimensional Euclidean space $\mathbf{R}^n$ such that each n-membered $G \subseteq E$ determines a unique hyperplane $\Pi(G)$, and every 2-membered subset $\{\mathbf{e}_i, \mathbf{e}_j\}$ of $E \setminus G$ determines a line $\mathbf{e}_i\mathbf{e}_j$ which intersects $\Pi(G)$ in exactly one point $\mathbf{e}_i\mathbf{e}_j^G$. Subjecting E to the further condition that $\mathbf{e}_i\mathbf{e}_j^G = \mathbf{e}_r\mathbf{e}_s^H$ if and only if $\{\{\mathbf{e}_i, \mathbf{e}_j\}, G\} = \{\{\mathbf{e}_r, \mathbf{e}_s\}, H\}$ we focus our attention upon the set $E^\wedge$ of all such intersection points $\mathbf{e}_i\mathbf{e}_j^G$, and we initiate a classification of those subsets X of $E^\wedge$ which under set inclusion are maximal with respect to the property that the j-plane $\Pi(X)$ determined by X is a hyperplane. Let $\mathcal{S}(E)$ denote the family of all

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such maximal X, and for k an integer let $\mathcal{S}(k;E)$ denote the family of all k-membered elements in $\mathcal{S}(E)$.

Implicit in the sort of classification announced above is a geometric enquiry: What regularities does E impose upon the configuration of the hyperplanes $\Pi(X)$ for these maximal $X \subseteq E^{\wedge}$? But our concern in this paper is at least as combinatorial as it is geometric, and centres more upon the families $\mathcal{S}(E)$ and $\mathcal{S}(k;E)$ than it centres upon the hyperplanes $\Pi(X)$ which their elements X determine.

When $m = n + 2$ then for every 2-membered $\{\mathbf{e}_i, \mathbf{e}_j\} \subseteq E$ the set $E \setminus \{\mathbf{e}_i, \mathbf{e}_j\}$ is n-membered, and so without ambiguity the expression $\mathbf{e}_i\mathbf{e}_j^*$ denotes the intersection point $\mathbf{e}_i\mathbf{e}_j^{E\setminus\{\mathbf{e}_i,\mathbf{e}_j\}}$. In passing we deal with the very easy case where $\langle m,n\rangle = \langle 4,2\rangle$. But our main concrete result is Theorem 1, which explores the evocative case where $\langle m,n\rangle = \langle 5,3\rangle$.

A family $\mathcal{D}$ of sets is said to be *uniformly deep of depth* d if and only if for every $x \in \cup\mathcal{D}$ there are exactly d distinct $A \in \mathcal{D}$ for which $x \in A$. Uniformly deep $\mathcal{D}$ are also called "regular hypergraphs", principally when all members of $\mathcal{D}$ have the same cardinal number.

It seems unknown for which triples $\langle s,d,k\rangle$ of integers there exists a uniformly deep family $\mathcal{D}$ of k-membered sets such that $d = \text{depth}(\mathcal{D})$ while $s = |\cup\mathcal{D}|$. In [2] this question receives some scrutiny; there, Theorem 2 gives the necessary condition $sd = k\,|\mathcal{D}|$ for the existence of such a $\mathcal{D}$, and Theorem 13 and Corollary 14 in [2] supply some of the sufficient conditions. However, even when the existence of such a $\mathcal{D}$ is ensured, the process of constructing it may be irksome. Furthermore, there are practical uses to which these $\mathcal{D}$ can be put; for example, in the design of experiments. The present paper proposes an application of geometry to the construction of uniformly deep families.

Let F be any set of five points in $\mathbb{R}^3$ so situated that no four of the points are coplanar, and that the line $\mathbf{xy}$ through any two $\mathbf{x}$ and $\mathbf{y}$ of the points has a unique intersection point $\mathbf{xy}^*$ with the plane determined by the other three. Let $F^{\wedge}$, $\mathcal{S}(F)$ and $\mathcal{S}(k;F)$ be as defined above. Then the following conditions are satisfied.

Theorem 1. *$F^{\wedge}$ contains exactly ten points, and no three of them are collinear. Furthermore, $\mathcal{S}(F) = \mathcal{S}(3;F) \cup \mathcal{S}(4;F)$ with $|\mathcal{S}(3;F)| = 20$ and with $|\mathcal{S}(4;F)| = 25$. Both $\mathcal{S}(3;F)$ and $\mathcal{S}(4;F)$ are uniformly deep; the depth of $\mathcal{S}(3;F)$ is 6, and the depth of $\mathcal{S}(4;F)$ is 10.*

Note that, if $\mathcal{A}$ and $\mathcal{B}$ are any two uniformly deep families with $\cup\mathcal{A} = \cup\mathcal{B}$ and with $\mathcal{A} \cap \mathcal{B} = \emptyset$, then $\mathcal{A} \cup \mathcal{B}$ is uniformly deep and moreover $\text{depth}(\mathcal{A} \cup \mathcal{B}) = \text{depth}(\mathcal{A}) + \text{depth}(\mathcal{B})$. Thus Theorem 1 implies immediately that $\mathcal{S}(F)$ is a uniformly deep 45-membered family whose depth is 16.

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In Section 2 we lay the groundwork for proving Theorem 1, and at the same time we develop the general problem suggested by the theorem. In Section 3 we prove the theorem, and in Section 4 we offer concluding remarks.

2. Obesity

Henceforth m and n are integers with $m \geqslant n+2 \geqslant 4$. A review of some elementary linear algebra may be helpful here.

For $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^n$ the expressions $X+Y$ and $X-Y$ denote the sets $\{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in X \,\&\, \mathbf{y} \in Y\}$ and $\{\mathbf{x}-\mathbf{y} \mid \mathbf{x} \in X \,\&\, \mathbf{y} \in Y\}$, respectively. Furthermore, $X+\mathbf{z} = \mathbf{z}+X := \{\mathbf{z}\}+X$ when $\mathbf{z} \in \mathbf{R}^n$. The expression $\mathsf{V}(X)$ denotes the vector subspace generated (that is, spanned) by X.

Lemma 2. *Let $\{\mathbf{x},\mathbf{y}\} \subseteq \mathbf{R}^n$ and let S and T be subspaces of $\mathbf{R}^n$. Then $\mathbf{x}+\mathsf{S} = \mathbf{y}+\mathsf{T}$ if and only if both $\mathsf{S}=\mathsf{T}$ and $\mathbf{x}-\mathbf{y} \in \mathsf{S}$.*

Proof: First, suppose that $\mathbf{x}-\mathbf{y} \in \mathsf{S} = \mathsf{T}$. Then $\mathbf{x}+\mathsf{S} = \mathbf{y}+\mathbf{x}-\mathbf{y}+\mathsf{S} = \mathbf{y}+\mathsf{S} = \mathbf{y}+\mathsf{T}$. Next, suppose that $\mathbf{x}+\mathsf{S} = \mathbf{y}+\mathsf{T}$. Then $\mathbf{x}-\mathbf{y}+\mathsf{S} = \mathbf{y}-\mathbf{y}+\mathsf{T}$, and so $\mathbf{x}-\mathbf{y} = \mathbf{x}-\mathbf{y}+\mathbf{0} \in \mathbf{x}-\mathbf{y}+\mathsf{S} = \mathsf{T}$. Therefore, $\mathbf{y}-\mathbf{x} = -(\mathbf{x}-\mathbf{y}) \in \mathsf{T}$ since T is a subspace. It follows that $\mathsf{T} = \mathbf{y}-\mathbf{x}+\mathsf{T} = -\mathbf{x}+\mathbf{x}+\mathsf{S}$. Therefore, $\mathbf{x}-\mathbf{y} \in \mathsf{S} = \mathsf{T}$. ∎

Lemma 3. *Let $\{\mathbf{y},\mathbf{z}\} \subseteq X \subseteq \mathbf{R}^n$. Then $\mathsf{V}(X-\mathbf{y}) = \mathsf{V}(X-\mathbf{z})$, and therefore the set $(X-\mathbf{y}) \setminus \{\mathbf{0}\}$ is linearly independent if and only if $(X-\mathbf{z}) \setminus \{\mathbf{0}\}$ is linearly independent.*

Proof: Choose $\mathbf{p} \in X-\mathbf{y}$. Then $\mathbf{p} = \mathbf{x}-\mathbf{y}$ for some $\mathbf{x} \in X$. It follows that $\mathbf{p} = (\mathbf{x}-\mathbf{z})-(\mathbf{y}-\mathbf{z}) \in \mathsf{V}(X-\mathbf{z})$, and hence that $X-\mathbf{y} \subseteq \mathsf{V}(X-\mathbf{z})$. Therefore $\mathsf{V}(X-\mathbf{y}) \subseteq \mathsf{V}(\mathsf{V}(X-\mathbf{z})) = \mathsf{V}(X-\mathbf{z})$. Similarly, $\mathsf{V}(X-\mathbf{z}) \subseteq \mathsf{V}(X-\mathbf{y})$. So $\mathsf{V}(X-\mathbf{y}) = \mathsf{V}(X-\mathbf{z})$.

Since $(X-\mathbf{y}) \setminus \{\mathbf{0}\}$ and $(X-\mathbf{z}) \setminus \{\mathbf{0}\}$ have the same number of elements, and span the same space $\mathsf{V}(X-\mathbf{y})$, one set is linearly independent if the other is. ∎

Corollary 4. *Let $\{\mathbf{y},\mathbf{z}\} \subseteq X \subseteq \mathbf{R}^n$. Then $X \subseteq \mathbf{y}+\mathsf{V}(X-\mathbf{z})$. Moreover, if $X \subseteq \mathbf{p}+\mathsf{S}$ where $\mathbf{p} \in \mathbf{y}+\mathsf{V}(X-\mathbf{z})$ and where S is a subspace then $\mathsf{V}(X-\mathbf{z})$ is a subspace of S.*

Proof: Let $\mathbf{x} \in X$. It follows by Lemma 3 that $\mathbf{x}-\mathbf{y} \in X-\mathbf{y} \subseteq \mathsf{V}(X-\mathbf{y}) = \mathsf{V}(X-\mathbf{z})$, and so $\mathbf{x} = \mathbf{x}-\mathbf{y}+\mathbf{y} \in \mathbf{y}+\mathsf{V}(X-\mathbf{z})$. It follows that $X \subseteq \mathbf{y}+\mathsf{V}(X-\mathbf{z})$ as claimed.

Now suppose also that $X \subseteq \mathbf{p}+\mathsf{S}$ where $\mathbf{p} \in \mathbf{y}+\mathsf{V}(X-\mathbf{z})$ and where S is a subspace. Then $X-\mathbf{p} \subseteq \mathsf{S}$ and so $\mathsf{V}(X-\mathbf{p}) \subseteq \mathsf{S}$. But $X-\mathbf{p} = X-\mathbf{y}-\mathbf{v}$ for some $\mathbf{v} \in \mathsf{V}(X-\mathbf{z}) = \mathsf{V}(X-\mathbf{y})$. Now, $-\mathbf{v} = \mathbf{y}-\mathbf{y}-\mathbf{v} \in X-\mathbf{y}-\mathbf{v} \subseteq \mathsf{V}(X-\mathbf{y}-\mathbf{v})$. Thus $\mathbf{v} \in \mathsf{V}(X-\mathbf{y}-\mathbf{v})$. So $X-\mathbf{y} = X-\mathbf{y}-\mathbf{v}+\mathbf{v} \subseteq \mathsf{V}(X-\mathbf{y}-\mathbf{v})+\mathbf{v} = \mathsf{V}(X-\mathbf{y}-\mathbf{v})$. It

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follows that $V(X-y) \subseteq V(V(X-y-v)) = V(X-y-v) = V(X-p) \subseteq S$, whence $V(X-z) \subseteq S$, as required. ∎

When S is a j-dimensional subspace of R^n and when $y \in R^n$ then the set $y+S$ is said to be a *j-plane*. An $(n-1)$-plane in R^n is called a *hyperplane* in R^n. By Corollary 4 we have for $y \in X \subseteq R^n$ that $V(X-y)$ is the unique subspace S of the smallest dimension for which $X \subseteq z+S$ when $z \in X$. So for $\emptyset \neq X \subseteq R^n$ we can define $\Pi(X) := z + V(X-y)$, where $\{y,z\} \subseteq X$. The plane $\Pi(X)$ is said to be *determined* by X. When $X = \{x_1, x_2, \ldots, x_k\}$ is finite, then $\Pi(X)$ may instead be written as $x_1 x_2 \ldots x_k$.

It is easily seen by Corollary 4 that if $X \subseteq Y \subseteq \Pi(X)$ then $\Pi(Y) = \Pi(X)$, and hence that $\Pi(\Pi(X)) = \Pi(X)$.

Of course, an n-plane in R^n is just R^n itself.

COROLLARY 5. *Let $Y \subseteq X \subseteq R^n$ with $0 < j+1 = |Y|$ and with $|X| = k+1 \leqslant n+1$ and such that $\Pi(X)$ is a k-plane. Then $\Pi(Y)$ is a j-plane.*

PROOF: Since $Y \subseteq X$ we can write $\Pi(X)$ as a translate of the k-dimensional subspace $V(X-y)$ for some $y \in Y$. Note that $|(X-y) \setminus \{0\}| = k$. Therefore the set $(X-y)\setminus\{0\}$ is linearly independent. So $(Y-y)\setminus\{0\}$ is linearly independent since $Y - y \subseteq X-y$. So $\Pi(Y)$ is a translate of the j-dimensional subspace $V((Y-y) \setminus \{0\}) = V(Y-y)$. ∎

Now we introduce our main concepts. These are motivated by Theorem 1.

DEFINITION 6: Let $E \subseteq R^n$. Then E is said to be *fat* if and only if every subset X of E satisfies the following two conditions:

6.1 if $|X| > n$ then $\Pi(X) = R^n$;

6.2 if $|X| = n$ and if y and z are two distinct elements in $E \setminus X$ then there is a unique element yz^X in the set $yz \cap \Pi(X)$.

When E is a fat subset of R^n the expression $E^\wedge$ denotes the set of all yz^X for which X is an n-membered subset of E and for which y and z are distinct elements in $E \setminus X$. Of course $E^\wedge = \emptyset$ unless $|E| \geqslant n+2$, and if $|E| \geqslant n+2$ but $n = 1$ then $E^\wedge = E$; each of these situations is uninteresting.

THEOREM 7. *Let E be a fat subset of R^n with $|E| \geqslant n+2 \geqslant 4$. Let X be a k-membered subset of E with $0 < k < n$ and let y and z be distinct elements in $E \setminus X$. Then $yz \cap \Pi(X) = \emptyset$. In particular $E^\wedge \cap E = \emptyset$.*

PROOF: Assume that there exists $x \in yz \cap \Pi(X)$. Then since $|X \cup \{z\}| = k+1 \leqslant n$ we have by Corollary 5 together with Condition 6.1 that $\Pi(X \cup \{z\})$ is a k-plane. But $y \in zx \subseteq \Pi(X \cup \{z\})$ since $x \in \Pi(X) \subseteq \Pi(X \cup \{z\})$ and $z \in \Pi(X \cup \{z\})$. Therefore $\Pi(X \cup \{y,z\}) = \Pi(X \cup \{z\})$. On the other hand $|X \cup \{y,z\}| = k+2$, and so $\Pi(X \cup \{\mathbf{y}, \mathbf{z}\})$ is a $(k+1)$-plane. We reach a contradiction. $\square$

In general, with E an m-membered fat subset of R^n there is for $\{\mathbf{yz}^G, \mathbf{pq}^H\} \subseteq E^\wedge$ no guarantee that if $\mathbf{yz}^G = \mathbf{pq}^H$ then $\langle\{\mathbf{y}, \mathbf{z}\}, G\rangle = \langle\{\mathbf{p}, \mathbf{q}\}, H\rangle$. Indeed this implication fails in the special case where $n = 2$ and where therefore each point in $E^\wedge$ is counted at least twice; that is, $\mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}} = \mathbf{pq}^{\{\mathbf{x},\mathbf{y}\}}$ for every 4-membered subset $\{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}\}$ of E. We believe it best to confine our attention to those m-membered fat E for which $|E^\wedge|$ is as large as possible; that is, when $|E^\wedge| = \binom{m}{2}\binom{m-2}{n}$. This is our motivation for the following

Definition 8: A fat subset E of R^n is said to be *obese* if and only if

8.1. for $n = 2$, if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ and $\{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}\}$ are 4-membered subsets of E then $\mathbf{xy}^{\{\mathbf{z},\mathbf{w}\}} = \mathbf{pq}^{\{\mathbf{r},\mathbf{s}\}}$ implies either that $\langle\{\mathbf{x}, \mathbf{y}\}, \{\mathbf{z}, \mathbf{w}\}\rangle = \langle\{\mathbf{p}, \mathbf{q}\}, \{\mathbf{r}, \mathbf{s}\}\rangle$ or that $\langle\{\mathbf{x}, \mathbf{y}\}, \{\mathbf{z}, \mathbf{w}\}\rangle = \langle\{\mathbf{r}, \mathbf{s}\}, \{\mathbf{p}, \mathbf{q}\}\rangle$;

8.2. for $n > 2$, if G and H are n-membered subsets of E, if $\mathbf{x}$ and $\mathbf{y}$ are distinct elements in $E \setminus G$, and if $\mathbf{p}$ and $\mathbf{q}$ are distinct elements in $E \setminus H$ then $\mathbf{xy}^G = \mathbf{pq}^H$ implies that $\langle\{\mathbf{x}, \mathbf{y}\}, G\rangle = \langle\{\mathbf{p}, \mathbf{q}\}, H\rangle$.

The expression $\Phi(m, n)$ denotes the family of all m-membered fat subsets of R^n, and $\Omega(m, n)$ denotes the family of all m-membered obese subsets of R^n. Of course $\Omega(m, n) \subseteq \Phi(m, n)$. The following instance shows that the reverse inclusion sometimes fails.

Proposition 9. $\Phi(7, 3) \neq \Omega(7, 3)$.

Proof: Let $E = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_6\}$ where $\mathbf{a}_0 = \langle -3, -3, -3\rangle$, $\mathbf{a}_1 = \langle -1, -1, -1\rangle$, $\mathbf{a}_2 = \langle 0, 1, 0\rangle$, $\mathbf{a}_3 = \langle 0, 2, 5\rangle$, $\mathbf{a}_4 = \langle 0, 5, 9\rangle$, $\mathbf{a}_5 = \langle 3, 7, 0\rangle$, and $\mathbf{a}_6 = \langle 5, 3, 0\rangle$. We omit the lengthy sequence of routine calculations that establish the fatness of the set E. Since $\mathbf{a}_0\mathbf{a}_1^G = \langle 0, 0, 0\rangle = \mathbf{a}_0\mathbf{a}_1^H$ when $G = \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ and $H = \{\mathbf{a}_2, \mathbf{a}_5, \mathbf{a}_6\}$, we have that E is not obese. $\square$

Let $\sigma(n)$ denote the largest integer for which $\Phi(m, n) = \Omega(m, n)$ whenever $\sigma(n) \geqslant m \geqslant n + 2 \geqslant 4$. From Proposition 9 we learn that $\sigma(3) < 7$; Theorem 1 alleges that $\sigma(3)$ exists and indeed that $\sigma(3) \geqslant 5$.

Lemma 10. *Let $n > 2$. Let $E \in \Phi(m, n)$. Let G and H be n-membered subsets of E, let $\{\mathbf{x}, \mathbf{y}\}$ be a 2-membered subset of $E \setminus G$, and let $\{\mathbf{r}, \mathbf{s}\}$ be a 2-membered subset of $E \setminus H$. Suppose that $\mathbf{xy}^G = \mathbf{rs}^H$. Then $\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{r}, \mathbf{s}\}$.*

Proof: If the set $\{\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}\}$ is 4-membered then the j-plane $\mathbf{xy}^G\mathbf{xyrs}$ is determined by the two intersecting lines $\mathbf{xy} = \mathbf{xy}^G\mathbf{xy}$ and $\mathbf{rs} = \mathbf{rs}^H\mathbf{rs} = \mathbf{xy}^G\mathbf{rs}$, whence $j = 2$. But by Corollary 5 together with Condition 6.1 we have that $\mathbf{xyrs}$ is a 3-plane if $\{\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}\}$ is 4-membered. It follows that $|\{\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}\}| \leqslant 3$. On the other hand, if $|\{\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}\}| = 3$ then the distinct lines $\mathbf{xy}$ and $\mathbf{rs}$ intersect in $\{\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}\} \subseteq E$.

This implies that $\mathbf{xy}^G = \mathbf{rs}^H$ is a point in E, a violation of Theorem 7. Therefore $2 \leqslant |\{\mathbf{x},\mathbf{y}\}| \leqslant |\{\mathbf{x},\mathbf{y},\mathbf{r},\mathbf{s}\}| < 3$, and thus we conclude that $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{r},\mathbf{s}\}$. ∎

THEOREM 11. $\sigma(n) \geqslant n+3$.

PROOF: Let $m \in \{n+2, n+3\}$ and let $E \in \Phi(m,n)$.

CASE. $n = 2$. Suppose that $\mathbf{xy}^{\{\mathbf{z},\mathbf{w}\}} = \mathbf{rs}^{\{\mathbf{p},\mathbf{q}\}}$ where $\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\}$ and $\{\mathbf{r},\mathbf{s},\mathbf{p},\mathbf{q}\}$ are 4-membered subsets of E. We easily infer from Theorem 7 that either $\{\mathbf{x},\mathbf{y}\} \cap \{\mathbf{r},\mathbf{s}\} = \emptyset$ or $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{r},\mathbf{s}\}$. So, if $\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\} = \{\mathbf{r},\mathbf{s},\mathbf{p},\mathbf{q}\}$ then either $\langle\{\mathbf{x},\mathbf{y}\}, \{\mathbf{z},\mathbf{w}\}\rangle = \langle\{\mathbf{p},\mathbf{q}\}, \{\mathbf{r},\mathbf{s}\}\rangle$ or $\langle\{\mathbf{x},\mathbf{y}\}, \{\mathbf{z},\mathbf{w}\}\rangle = \langle\{\mathbf{r},\mathbf{s}\}, \{\mathbf{p},\mathbf{q}\}\rangle$ whereupon Condition 8.1 is satisfied.

Assume that $\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\} \neq \{\mathbf{r},\mathbf{s},\mathbf{p},\mathbf{q}\}$. Then since $m \leqslant 5$ implies that $|\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\} \cap \{\mathbf{r},\mathbf{s},\mathbf{p},\mathbf{q}\}| \geqslant 3$, we infer that $|\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\} \cap \{\mathbf{r},\mathbf{s},\mathbf{p},\mathbf{q}\}| = 3$. Without loss of generality we may suppose that $\{\mathbf{x},\mathbf{y},\mathbf{z}\} = \{\mathbf{r},\mathbf{s},\mathbf{p}\}$ but that $\mathbf{w} \neq \mathbf{q}$.

SUBCASE. $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{r},\mathbf{s}\}$ and $\mathbf{z} = \mathbf{p}$. Note that $\mathbf{pw} \neq \mathbf{pq}$, whence $\mathbf{pw} \cap \mathbf{pq} = \{\mathbf{p}\}$. Since $\mathbf{xy} = \mathbf{rs}$, we have that $\mathbf{xy}^{\{\mathbf{p},\mathbf{w}\}} = \mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}}$. It follows that $\mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}} \in \mathbf{pw}$. But $\mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}} \in \mathbf{pq}$. So $\mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}} \in \mathbf{pw} \cap \mathbf{pq} = \{\mathbf{p}\}$. We must infer that $\mathbf{xy}^{\{\mathbf{p},\mathbf{q}\}} = \mathbf{p} \in E$ in violation of Theorem 7.

SUBCASE. $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{r},\mathbf{p}\}$ and $\mathbf{z} = \mathbf{s}$. Then $\mathbf{xy}^{\{\mathbf{s},\mathbf{w}\}} \in \mathbf{xy} = \mathbf{rp}$. But $\mathbf{xy}^{\{\mathbf{s},\mathbf{w}\}} = \mathbf{rs}^{\{\mathbf{p},\mathbf{q}\}} = \mathbf{pq}^{\{\mathbf{r},\mathbf{s}\}}$, and so $\mathbf{xy}^{\{\mathbf{s},\mathbf{w}\}} \in \mathbf{pq}$. So $\mathbf{xy}^{\{\mathbf{s},\mathbf{w}\}} \in \mathbf{rp} \cap \mathbf{pq} = \{\mathbf{p}\}$. Therefore $\mathbf{xy}^{\{\mathbf{s},\mathbf{w}\}} = \mathbf{p} \in E$ in violation of Theorem 7.

In both subcases the assumption fails, and thus E satisfies Condition 8.1. We conclude that $E \in \Omega(m,2)$.

CASE. $n > 2$. Suppose that $\mathbf{xy}^G = \mathbf{rs}^H$ where G and H are n-membered subsets of E, where $\{\mathbf{x},\mathbf{y}\}$ is a 2-membered subset of $E \setminus G$ and where $\{\mathbf{r},\mathbf{s}\}$ is a 2-membered subset of $E \setminus H$. Then $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{r},\mathbf{s}\}$ by Lemma 10, and so G and H are subsets of the same $(m-2)$-membered set $E \setminus \{\mathbf{x},\mathbf{y}\}$. If $m = n+2$ then $G = H$ and so Condition 8.2 is satisfied. Therefore we may take it that $m = n+3$.

Assume that $G \neq H$. Then $G \cup H = E \setminus \{\mathbf{x},\mathbf{y}\}$, and so $|G \cup H| = m-2 = n+1$ and $|G \cap H| = n-1$. So by Corollary 5 with Condition 6.1 we have that $\Pi(G \cap H)$ is an $(n-2)$-plane. It follows by Theorem 7 that $\mathbf{xy}^G \notin \Pi(G \cap H)$.

We now claim that $\Pi(G \cap H) = \Pi(G) \cap \Pi(H)$. Surely $\Pi(G \cap H) \subseteq \Pi(G) \cap \Pi(H)$. Since $\Pi(\Pi(G) \cap \Pi(H)) \subseteq \Pi(\Pi(G)) = \Pi(G)$ and since similarly $\Pi(\Pi(G) \cap \Pi(H)) \subseteq \Pi(H)$, we have that $\Pi(G) \cap \Pi(H) \subseteq \Pi(\Pi(G) \cap \Pi(H)) \subseteq \Pi(G) \cap \Pi(H)$, whence $\Pi(\Pi(G) \cap \Pi(H)) = \Pi(G) \cap \Pi(H)$. That is, as common wisdom would suggest, the intersection $\Pi(G) \cap \Pi(H)$ of two hyperplanes is a j-plane for some $j \leqslant n-1$. But if $\Pi(G) \cap \Pi(H)$ were also a hyperplane then $\Pi(G) \cap \Pi(H) = \Pi(G)$ whence $\Pi(G) = \Pi(H) = \Pi(G \cup H) = \mathbf{R}^n$ since $|G \cup H| = n+1$. Thus we infer that the

j-plane $\Pi(G) \cap \Pi(H)$ is not a hyperplane, but that $j \leqslant n-2$. So, since $\Pi(G \cap H)$ is an $(n-2)$-plane and since $\Pi(G \cap H) \subseteq \Pi(G) \cap \Pi(H)$ we infer that $\Pi(G) \cap \Pi(H)$ is an $(n-2)$-plane, and hence that $\Pi(G \cap H) = \Pi(G) \cap \Pi(H)$ as claimed. But then $\mathbf{xy}^G = \mathbf{xy}^H \in \Pi(G) \cap \Pi(H) = \Pi(G \cap H)$, and we reach a contradiction. Therefore $G = H$, and E satisfies Condition 8.2. We conclude that $E \in \Omega(m,n)$. ∎

By Proposition 9 with Theorem 11 we have that $\sigma(3) = 6$.

CONJECTURE 12. $\sigma(n) = n+3$ for every $n \geqslant 2$, and $\Phi(m,n) \neq \Omega(m,n)$ for every $m > \sigma(n)$.

CONJECTURE 13. $\Omega(m,n)$ is uncountable whenever $m \geqslant n+2 \geqslant 4$.

For $E \in \Phi(m,n)$ the expression $\mathcal{S}(E)$ denotes the family of all subsets X of $E^\wedge$ such that $\Pi(X)$ is a hyperplane in $\mathbf{R}^n$ but such that $\Pi(X \cup \{\mathbf{y}\}) = \mathbf{R}^n$ for every $\mathbf{y} \in E^\wedge \setminus X$. For each integer $k \geqslant n$ the expression $\mathcal{S}(k;E)$ denotes the family of all k-membered elements in $\mathcal{S}(E)$. Of course $\mathcal{S}(E)$ is the disjoint union of the $\mathcal{S}(k;E)$. Our principal interest resides in exactly these families $\mathcal{S}(E)$ and $\mathcal{S}(k;E)$ for $E \in \Omega(m,n)$.

OPEN QUESTION 14. If $E \in \Omega(m,n)$ then is $\mathcal{S}(k;E)$ uniformly deep for every k?

OPEN QUESTION 15. To every pair m and n of integers with $m \geqslant n+2 \geqslant 4$ is there a function $\beta(m,n;\ ) : k \mapsto \beta(m,n;k)$ such that $|\mathcal{S}(k;E)| = \beta(m,n;k)$ for every $E \in \Omega(m,n)$ and for every integer $k \geqslant n$?

By Theorem 1 for $\langle m,n\rangle = \langle 5,3\rangle$ both of the questions 14 and 15 have affirmative answers.

We consider briefly the simplest case $\langle m,n\rangle = \langle 4,2\rangle$. It is easy to confirm that whenever $E \in \Phi(4,2) = \Omega(4,2)$ then $|E^\wedge| = 3$, and $\mathcal{S}(E) = \mathcal{S}(2;E)$ is a uniformly deep 3-membered family of depth 2.

The most accessible cases yet to be studied are $\langle m,n\rangle \in \{\langle 5,2\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$.

3. PROOF OF THEOREM 1

Henceforth $F = \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}\}$ is an arbitrary fat 5-membered subset of $\mathbf{R}^3$. So F is obese by Theorem 11. Therefore $|F^\wedge| = \binom{5}{2}\binom{5-2}{3} = 10$. Since for each 2-membered $\{\mathbf{x},\mathbf{y}\} \subseteq F$ the set $F \setminus \{\mathbf{x},\mathbf{y}\}$ is 3-membered, we can without ambiguity write $\mathbf{xy}^*$ to mean the unique intersection point $\mathbf{xy}^{F\setminus\{\mathbf{x},\mathbf{y}\}}$ lying both on the line $\mathbf{xy}$ and also on the plane $\Pi(F \setminus \{\mathbf{x},\mathbf{y}\})$. Now, by Theorem 7 we have that $\mathbf{xy}^* \notin F$. Also immediately by Theorem 7 we have

LEMMA 16. *For $\{\mathbf{x},\mathbf{y}\}$ and $\{\mathbf{z},\mathbf{w}\}$ any pair of 2-membered subsets of F the following three assertions are equivalent:*

(1) $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{z},\mathbf{w}\}$;

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(2) $\mathbf{xy}^* \in \mathbf{zw}$;
(3) $\mathbf{xy}^* = \mathbf{zw}^*$.

LEMMA 17. *Whenever* $\{\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}\} = \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}\}$ *then* $\mathbf{x} \in \mathbf{vw}^*\mathbf{yz}^*$.

PROOF: Without loss of generality let $\mathbf{v} = \mathbf{a}$, $\mathbf{w} = \mathbf{b}$, $\mathbf{x} = \mathbf{c}$, $\mathbf{y} = \mathbf{d}$, and $\mathbf{z} = \mathbf{e}$; now show that $\mathbf{c} \in \mathbf{ab}^*\mathbf{de}^*$. By 6.2 we have that $\mathbf{ab}^* \in \mathbf{cde}$ and that $\mathbf{de}^* \in \mathbf{abc}$. But also $\mathbf{ab}^* \in \mathbf{ab} \subseteq \mathbf{abc}$ and $\mathbf{de}^* \in \mathbf{de} \subseteq \mathbf{cde}$. Clearly $\mathbf{c} \in \mathbf{abc} \cap \mathbf{cde}$. So now $\{\mathbf{ab}^*, \mathbf{de}^*, \mathbf{c}\} \subseteq \mathbf{abc} \cap \mathbf{cde}$. By 6.1 we have that $\mathbf{abcd} = \mathbf{R}^3$, and by Corollary 5 we have that $\mathbf{abc}$ and $\mathbf{cde}$ are planes. Therefore $\mathbf{abc} \cap \mathbf{cde}$ is a line. By Theorems 7 and 11 the set $\{\mathbf{ab}^*, \mathbf{de}^*, \mathbf{c}\}$ has three distinct elements. So $\mathbf{ab}^*\mathbf{de}^*$ is a line, and $\mathbf{c} \in \mathbf{ab}^*\mathbf{de}^*$. ∎

Although our identification of the families $\mathcal{S}(k;F)$ is geometric in its conception, it will be convenient to organise this work graph theoretically. Furthermore, our subsequent arguments establishing the uniform depth of the $\mathcal{S}(k;F)$ depend basically upon graph theory, and moreover will require a subtle departure from some of the standard terminology codified in [**1**].

By a *graph* we mean an ordered pair $\mathcal{G} = \langle A, B \rangle$, where A is a set and where B is a family of 2-membered subsets of A; the elements in A are called *vertices* of $\mathcal{G}$, and the elements in B are called *edges* of $\mathcal{G}$. The expression $\mathcal{V}(\mathcal{G})$ denotes the set of all vertices of $\mathcal{G}$, and is called the *vertex set* of $\mathcal{G}$; the expression $\mathcal{E}(\mathcal{G})$ denotes the set of all edges of $\mathcal{G}$, and is called the *edge set* of $\mathcal{G}$. Thus $\mathcal{G} = \langle \mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}) \rangle$ whenever $\mathcal{G}$ is a graph. Finally, a graph $\mathcal{H}$ is said to be a *subgraph* of $\mathcal{G}$ if and only if both $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{G})$. In the present paper, whenever $\mathcal{H}$ a subgraph of $\mathcal{G}$ then in fact $\mathcal{V}(\mathcal{H}) = \mathcal{V}(\mathcal{G})$.

For graphs $\mathcal{G}$ and $\mathcal{H}$, a bijection f from $\mathcal{V}(\mathcal{G})$ onto $\mathcal{V}(\mathcal{H})$ is a *graph isomorphism* if and only if $\mathcal{E}(\mathcal{H}) = \{\{f(\mathbf{x}), f(\mathbf{y})\} \mid \{\mathbf{x},\mathbf{y}\} \in \mathcal{E}(\mathcal{G})\}$. The expression Graphs($\mathcal{G}$) denotes the family of all graphs $\mathcal{H}$ for which $\mathcal{V}(\mathcal{H}) = \mathcal{V}(\mathcal{G})$. The expression Type($\mathcal{G}$) denotes the subfamily of all $\mathcal{H} \in$ Graphs($\mathcal{G}$) such that $\mathcal{H}$ is isomorphic to $\mathcal{G}$. Finally, the expression Edgesets($\mathcal{G}$) denotes $\{\mathcal{E}(\mathcal{H}) \mid \mathcal{H} \in \text{Type}(\mathcal{G})\}$.

It will be illuminating to associate with each subset of $F^\wedge$ a corresponding graph. Thus, recalling that each element in $F^\wedge$ lies on exactly one line $\mathbf{xy}$ with $\{\mathbf{x},\mathbf{y}\}$ a 2-membered subset of F, we see that each k-membered subset $K = \{\mathbf{x}_1\mathbf{y}_1^*, \ldots, \mathbf{x}_k\mathbf{y}_k^*\}$ of $F^\wedge$ is represented by exactly one k-edged graph $\mathcal{G}(K)$ on the vertex set F; the edge set of this graph is just $\mathcal{E}(\mathcal{G}(K)) = \{\{\mathbf{x}_1, \mathbf{y}_1\}, \ldots, \{\mathbf{x}_k, \mathbf{y}_k\}\}$. It turns out that when $k = 4$ then whether or not $\Pi(K)$ is a plane is decided by the isomorphism type of $\mathcal{G}(K)$.

Having classified each 4-membered subset X of $F^\wedge$ according to the isomorphism type of its associated graph $\mathcal{G}(X)$, we will have for each 3-membered subset Y of $F^\wedge$

that $Y \in \mathcal{S}(3;F)$ if and only if the 3-edged graph $\mathcal{G}(Y)$ is the subgraph of no 4-edged graph $\mathcal{G}(X)$ for which $X \in \mathcal{S}(4;F)$.

There are exactly 6 isomorphism types of 4-edged graphs on a 5-membered vertex set; these are displayed for future reference in Figure 1 below where they bear the Roman-numeral labels I to VI. There are exactly 4 isomorphism types of 3-edged graphs on a 5-membered vertex set; these are labelled VII to X in Figure 1.

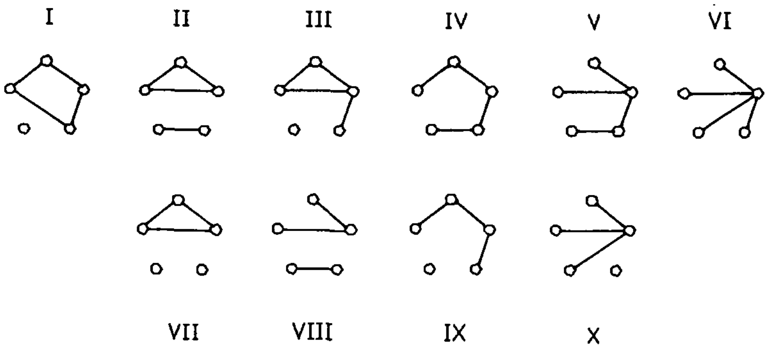

Figure 1

Theorem 18. *Let* $\{\mathbf{x},\mathbf{y}\}$, $\{\mathbf{z},\mathbf{w}\}$ *and* $\{\mathbf{u},\mathbf{v}\}$ *be any three distinct 2-membered subsets of* F. *Then* $\mathbf{xy^*zw^*uv^*}$ *is a plane.*

Proof: There are four cases to consider, corresponding to graph types VII to X in Figure 1.

Case 1. The situation represented by graph-type X. Without loss of generality we specify that $\mathbf{x}=\mathbf{z}=\mathbf{u}=\mathbf{a}$, that $\mathbf{y}=\mathbf{b}$, that $\mathbf{w}=\mathbf{c}$ and that $\mathbf{v}=\mathbf{d}$.

Now assume that $\mathbf{ab^*ac^*ad^*}$ is a line. Then $\mathbf{ab^*ac^*ad^*a}$ is a plane. But $\mathbf{b} \in \mathbf{ab^*a} \subseteq \mathbf{ab^*ac^*ad^*a}$. Similarly we see that $\mathbf{c}$ and $\mathbf{d}$ are elements in $\mathbf{ab^*ac^*ad^*a}$. Thus, $\mathbf{abcd} \subseteq \mathbf{ab^*ac^*ad^*a}$. But $\mathbf{abcd} = \mathsf{R}^3$. Therefore, $\mathbf{ab^*ac^*ad^*a} = \mathsf{R}^3$, a contradiction. We infer that $\mathbf{ab^*ac^*ad^*}$ is not a line; instead, $\mathbf{xy^*zw^*uv^*} = \mathbf{ab^*ac^*ad^*}$ is a plane.

Case 2. The situation represented by graph-type IX. Without loss of generality we specify that $\mathbf{x}=\mathbf{a}$, that $\mathbf{y}=\mathbf{z}=\mathbf{b}$, that $\mathbf{w}=\mathbf{u}=\mathbf{c}$, and that $\mathbf{v}=\mathbf{d}$.

Assume that $\mathbf{ab^*bc^*cd^*}$ is a line. Then $\mathbf{ab^*bc^*cd^*a}$ is a plane. Arguing as in Case 1 we have that $\mathbf{b} \in \mathbf{ab^*bc^*cd^*a}$ and hence that $\mathbf{c} \in \mathbf{ab^*bc^*cd^*a}$ whereupon also $\mathbf{d} \in \mathbf{ab^*bc^*cd^*a}$. It follows that $\mathsf{R}^3 = \mathbf{abcd} \subseteq \mathbf{ab^*bc^*cd^*a}$, again a contradiction. So we conclude that $\mathbf{xy^*zw^*uv^*} = \mathbf{ab^*bc^*cd^*}$ is not a line, but is instead a plane.

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CASE 3. The situation represented by graph-type VIII. Without loss of generality we specify that $\mathbf{x} = \mathbf{z} = \mathbf{a}$, that $\mathbf{y} = \mathbf{b}$, that $\mathbf{w} = \mathbf{c}$, that $\mathbf{u} = \mathbf{d}$, and that $\mathbf{v} = \mathbf{e}$. By Lemma 17 we have that both $\mathbf{ab^*de^*c}$ and $\mathbf{ac^*de^*b}$ are lines. Assume that $\mathbf{ab^*ac^*de^*}$ is a line. Then Lemma 17 implies that $\mathbf{ac^*} \in \mathbf{ab^*ac^*de^*} = \mathbf{ab^*de^*ac^*de^*} = \mathbf{ab^*de^*cac^*de^*b} = \mathbf{bc}$ contrary to Lemma 16. Therefore $\mathbf{ab^*ac^*de^*}$ is not a line; instead, it is a plane.

CASE 4. The situation represented by graph-type VII. Without loss of generality we specify that $\mathbf{x} = \mathbf{v} = \mathbf{a}$, that $\mathbf{y} = \mathbf{z} = \mathbf{b}$ and that $\mathbf{w} = \mathbf{u} = \mathbf{c}$. By definition $\mathbf{de^*} \in \mathbf{abc}$. However $\mathbf{de^*} \notin \mathbf{ab} \cup \mathbf{bc} \cup \mathbf{ac}$ by Theorem 7.

Now assume that $\mathbf{ab^*bc^*ca^*}$ is a line. By Lemma 17 we have that $\mathbf{a} \in \mathbf{bc^*de^*}$, that $\mathbf{b} \in \mathbf{ac^*de^*}$, and that $\mathbf{c} \in \mathbf{ab^*de^*}$. Therefore if $\mathbf{de^*}$ were an element in the line $\mathbf{ab^*bc^*ca^*}$ then the points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ would be collinear, which they are not. It follows that $\mathbf{de^*}$ does not lie on the line $\mathbf{ab^*bc^*ca^*}$.

Without loss of generality we specify that $\mathbf{bc^*}$ is between $\mathbf{ab^*}$ and $\mathbf{ca^*}$. It readily follows that exactly one of the following two equivalent situations occurs:

(i) $\mathbf{b}$ is between $\mathbf{ac^*}$ and $\mathbf{de^*}$, but $\mathbf{c}$ is not between $\mathbf{ab^*}$ and $\mathbf{de^*}$.

(ii) $\mathbf{c}$ is between $\mathbf{ab^*}$ and $\mathbf{de^*}$, but $\mathbf{b}$ is not between $\mathbf{ac^*}$ and $\mathbf{de^*}$.

Again without loss of generality we can suppose that the situation (i) actually obtains, and we refer the reader to Figure 2 for the argument which follows.

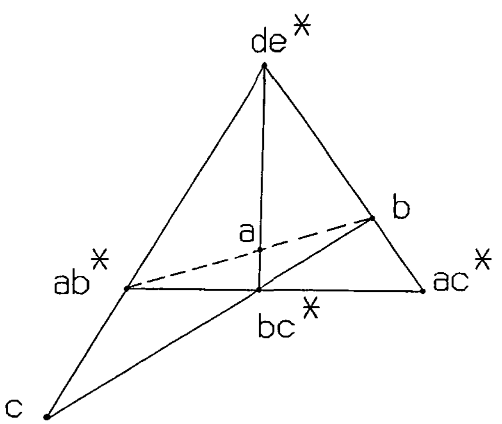

Figure 2

Now, $\mathbf{a} \in \mathbf{ab^*b}$, and by Lemma 17 also $\mathbf{a} \in \mathbf{de^*bc^*}$. So $\mathbf{a} \in \mathbf{ab^*b} \cap \mathbf{de^*bc^*}$, placing $\mathbf{a}$ inside the triangle $\Delta(\mathbf{ab^*}, \mathbf{ac^*}, \mathbf{de^*})$. But then $\mathbf{c}$, which is similarly seen to be the only element in $\mathbf{ac^*a} \cap \mathbf{ab^*de^*}$, would have to lie between $\mathbf{ab^*}$ and $\mathbf{de^*}$. This is a contradiction. So $\mathbf{xy^*zw^*uv^*} = \mathbf{ab^*bc^*ca^*}$ is a plane.

In each of the four cases considered above, we have that $\mathbf{xy^*zw^*uv^*}$ is a plane. ▯

A useful rephrasing of Theorem 18 is that no three distinct elements in $F^\wedge$ are collinear.

Our next task is to characterise the families $\mathcal{S}(k;F)$. To this end, we shall examine the $\binom{10}{4} = 210$ distinct 4-membered subsets X of the 10-membered set $F^\wedge$, and then we shall examine the $\binom{10}{3} = 120$ distinct 3-membered subsets Y of $F^\wedge$. For many of the X it happens that $\Pi(X)$ is a plane while for others $\Pi(X) = \mathbb{R}^3$. In those cases where $\Pi(X)$ is a plane we shall see that $X \in \mathcal{S}(4;F)$ and hence that $\mathcal{S}(k;F) = \emptyset$ for all integers $k > 4$. Henceforth X denotes a 4-membered subset of $F^\wedge$. The next eleven results, Lemma 19 to Corollary 29, refer to Figure 1 above.

LEMMA 19. *When the graph $\mathcal{G}(X)$ is of isomorphism type I then $\Pi(X)$ is a plane.*

PROOF: We may suppose that $X = \{\mathbf{ab^*}, \mathbf{bc^*}, \mathbf{cd^*}, \mathbf{da^*}\}$. By Lemma 17 then $\mathbf{ab^*cd^*e}$ and $\mathbf{bc^*da^*e}$ are lines. They are obviously subsets of $\Pi(X)$, and they share a common point $\mathbf{e}$. Moreover $\mathbf{ab^*cd^*e} \neq \mathbf{bc^*da^*e}$ by Theorem 18. Therefore $\Pi(X) = \mathbf{ab^*cd^*bc^*da^*} = \mathbf{ab^*cd^*ebc^*da^*e}$ is a plane. ▯

LEMMA 20. *When the graph $\mathcal{G}(X)$ is of isomorphism type II then $\Pi(X)$ is a plane.*

PROOF: We may suppose that $X = \{\mathbf{ab^*}, \mathbf{bc^*}, \mathbf{ca^*}, \mathbf{de^*}\}$. Surely $\mathbf{ab^*bc^*ca^*} \subseteq \mathbf{abc}$. But Theorem 18 implies that $\mathbf{ab^*bc^*ca^*}$ is a plane. Furthermore $\mathbf{de^*} \in \mathbf{abc}$. So $\Pi(X) = \mathbf{ab^*bc^*ca^*de^*} = \mathbf{abcde^*} = \mathbf{abc}$. ▯

LEMMA 21. *When the graph $\mathcal{G}(X)$ is of isomorphism type III then $\Pi(X) = \mathbb{R}^3$.*

PROOF: We may suppose that $X = \{\mathbf{ab^*}, \mathbf{bc^*}, \mathbf{ca^*}, \mathbf{da^*}\}$. As in the proof of Lemma 20 we see that $\mathbf{ab^*bc^*ca^*} = \mathbf{abc}$. Since we have by Theorem 7 that $\mathbf{da^*} \neq \mathbf{a}$, it follows that $\mathbf{d} \in \mathbf{da} = \mathbf{da^*a} \subseteq \mathbf{ab^*bc^*ca^*da^*}$. Thus $\mathbf{abcd} \subseteq \mathbf{ab^*bc^*ac^*da^*}$, whence $\mathbf{ab^*bc^*ac^*da^*} = \mathbb{R}^3$. ▯

LEMMA 22. *When the graph $\mathcal{G}(X)$ is of isomorphism type IV then $\Pi(X) = \mathbb{R}^3$.*

PROOF: We may suppose that $X = \{\mathbf{ab^*}, \mathbf{bc^*}, \mathbf{cd^*}, \mathbf{de^*}\}$. Obviously $\{\mathbf{ab^*}, \mathbf{cd^*}, \mathbf{de^*}\} \subseteq \mathbf{cde}$. It follows by Theorem 18 that $\mathbf{ab^*cd^*de^*} = \mathbf{cde}$. But then, as in the proof of Lemma 21, we see that $\mathbf{b} \in \mathbf{bc^*c} \subseteq \mathbf{cdebc^*} = \mathbf{ab^*cd^*de^*bc^*}$. Thus $\mathbf{bcde} \subseteq \mathbf{ab^*cd^*de^*bc^*}$, whence $\mathbf{ab^*bc^*cd^*de^*} = \mathbb{R}^3$. ▯

LEMMA 23. *When the graph $\mathcal{G}(X)$ is of isomorphism type V then $\Pi(X) = \mathbb{R}^3$.*

PROOF: We may suppose that $X = \{\mathbf{ab^*}, \mathbf{bc^*}, \mathbf{cd^*}, \mathbf{ce^*}\}$. Since $\{\mathbf{ab^*}, \mathbf{cd^*}, \mathbf{ce^*}\} \subseteq \mathbf{cde}$ we have as above that $\mathbf{ab^*cd^*ce^*} = \mathbf{cde}$ and that $\mathbf{b} \in \mathbf{bc^*c} \subseteq \mathbf{ab^*bc^*cd^*ce^*}$. The lemma follows. ▯

Lemma 24. *When the graph $\mathcal{G}(X)$ is of isomorphism type VI then $\Pi(X) = \mathbf{R}^3$.*

Proof: We may suppose that $X = \{\mathbf{ab}^*, \mathbf{ac}^*, \mathbf{ad}^*, \mathbf{ae}^*\}$. Indeed, since incidence properties and parallelism are preserved under those transformations of $\mathbf{R}^3$ which are the composition of translations, shears, dilations, rotations and reflections, we may suppose for convenience that $\mathbf{a} = \langle 0,0,0\rangle$, that $\mathbf{b} = \langle 1,0,0\rangle$, that $\mathbf{c} = \langle 0,1,0\rangle$ and that $\mathbf{d} = \langle 0,0,1\rangle$. For each $\mathbf{x} \in \{\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}\}$ we write $\mathbf{x}^*$ as an abbreviation for $\mathbf{ax}^*$. Then, since points can also be treated as vectors, there exist real numbers β, γ, δ, and ε such that $\mathbf{b}^* = \beta\mathbf{b} = \langle \beta,0,0\rangle$, such that $\mathbf{c}^* = \gamma\mathbf{c} = \langle 0,\gamma,0\rangle$, such that $\mathbf{d}^* = \delta\mathbf{d} = \langle 0,0,\delta\rangle$, and such that $\mathbf{e}^* = \varepsilon\mathbf{e} = \varepsilon\langle e_1,e_2,e_3\rangle$. By Theorem 7 we have that $\{\beta,\gamma,\delta,\varepsilon\}\cap\{0,1\} = \emptyset$ since the set F is fat. Moreover, the fatness of F implies that $\mathbf{e}$ lies in no plane whose equation is $\xi x + \eta y + \zeta z = \lambda$ for which $\langle 0,0,0,\lambda\rangle \neq \langle \xi,\eta,\zeta,\lambda\rangle \in 2\times 2\times 2\times 2$ where as usual $2 := \{0,1\}$. Thus our constants obey the following conditions:

24.1. *$\{\beta,\gamma,\delta,\varepsilon,e_i,\ e_i+e_j,\ e_1+e_2+e_3\}\cap\{0,1\} = \emptyset$ where $\{i,j\}$ is a 2-membered subset of $\{1,2,3\}$.*

We define $E := e_1+e_2+e_3$ and $D := (1+e_1-E)(1+e_2-E)(1+e_3-E)$. Note that Conditions 24.1 imply that $E \neq 0 \neq D$.

It suffices to prove that $\mathbf{b}^*\mathbf{c}^*\mathbf{d}^*\mathbf{e}^* = \mathbf{R}^3$. This condition is equivalent to the linear independence of the set $\{\mathbf{b}^*-\mathbf{e}^*,\ \mathbf{c}^*-\mathbf{e}^*,\ \mathbf{d}^*-\mathbf{e}^*\}$. We will first express ε, β, γ and δ in terms of e_1, e_2, and e_3. Note that $\mathbf{e}^* \in \mathbf{bcd}$, and that the equation of the plane $\mathbf{bcd}$ is $x+y+z = 1$. It follows that $\varepsilon(e_1+e_2+e_3) = 1$, whence $\varepsilon = 1/E$.

Next, we obtain a vector $\mathbf{p}$ normal to the plane $\mathbf{ced}$ by applying the ordinary cross product thus: $\mathbf{p} = (\mathbf{e}-\mathbf{c})\times(\mathbf{e}-\mathbf{d}) = \langle e_1,e_2-1,e_3\rangle\times\langle e_1,e_2,e_3-1\rangle = \langle 1-e_2-e_3,e_1,e_1\rangle$. Therefore, since $\langle\beta,0,0\rangle = \mathbf{b}^* \in \mathbf{ced}$, we have that the vector $\mathbf{b}^*-\mathbf{e}$ is perpendicular to $\mathbf{p}$, and hence that $(\mathbf{b}^*-\mathbf{e}).\mathbf{p} = 0$. By routine substitution and calculation we then infer that $\beta = e_1/(1-e_2-e_3) = e_1/(1+e_1-E)$. Similarly one can solve for γ and δ in terms of the e_i, and thus get that

$$\beta = e_1/(1+e_1-E),$$
$$\gamma = e_2/(1+e_2-E),$$
$$\delta = e_3/(1+e_3-E).$$

So we have that

$$\begin{aligned}\mathbf{b}^*-\mathbf{e}^* &= \langle\beta-\varepsilon e_1,-\varepsilon e_2,-\varepsilon e_3\rangle\\ &= \langle e_1/(1+e_1-E)-e_1/E,-e_2/E,-e_3/E\rangle\\ &= (1/E)\langle e_1(E/(1+e_1-E)-1),-e_2,-e_3\rangle.\end{aligned}$$

Likewise,

$$\mathbf{c}^*-\mathbf{e}^* = (1/E)\langle -e_1,e_2(E/(1+e_2-E)-1),-e_3\rangle \text{ and}$$
$$\mathbf{d}^*-\mathbf{e}^* = (1/E)\langle -e_1,-e_2,e_3(E/(1+e_3-E)-1)\rangle.$$

Now, the set $\{\mathbf{b}^* - \mathbf{e}^*, \mathbf{c}^* - \mathbf{e}^*, \mathbf{d}^* - \mathbf{e}^*\}$ is linearly independent if and only if the matrix $\mathbf{M}$ defined by

$$\mathbf{M} := E \begin{bmatrix} \mathbf{b}^* - \mathbf{e}^* \\ \mathbf{c}^* - \mathbf{e}^* \\ \mathbf{d}^* - \mathbf{e}^* \end{bmatrix}$$

is nonsingular. Of course then

$$\mathbf{M} = \begin{bmatrix} e_1(E/(1+e_1-E)-1) & -e_2 & -e_3 \\ -e_1 & e_2(E/(1+e_2-E)-1) & -e_3 \\ -e_1 & -e_2 & e_3(E/(1+e_3-E)-1) \end{bmatrix}.$$

Now we multiply the three columns of $\mathbf{M}$ by $-1/e_1$, $-1/e_2$, and $-1/e_3$ respectively, to obtain a matrix $\mathbf{N}$ that is singular if and only if $\mathbf{M}$ is singular. Here,

$$\mathbf{N} := \begin{bmatrix} 1-E/(1+e_1-E) & 1 & 1 \\ 1 & 1-E/(1+e_2-E) & 1 \\ 1 & 1 & 1-E/(1+e_3-E) \end{bmatrix}.$$

It is "straightforward" to verify that $\det(\mathbf{N}) = 3E^2(1-E)/D$. Since Conditions 24.1 imply also that $1 - E \neq 0$, we have that $\mathbf{N}$ is nonsingular. It finally follows that $\mathrm{ab^*ac^*ad^*ae^*} = \mathsf{R}^3$. ∎

The following is a summary of Lemmas 19 to 24.

Theorem 25. *If the graph $\mathcal{G}(X)$ is of isomorphism type I or II then $\Pi(X)$ is a plane, but if $\mathcal{G}(X)$ is of isomorphism type III or IV or V or VI then $\Pi(X) = \mathsf{R}^3$.*

Corollary 26. *Let X be any 4-membered subset of $F^\wedge$. Then $X \in \mathcal{S}(4;F)$ if and only if the graph $\mathcal{G}(X)$ is of isomorphism type I or II.*

Proof: It is immediate from Theorem 25 that $X \notin \mathcal{S}(F)$ when $\mathcal{G}(X)$ is not of type I or of type II. So, if $X \in \mathcal{S}(4;F)$ then $\mathcal{G}(X)$ is either of type I or of type II.

Note that every 5-edged graph of 5 vertices has a subgraph of at least one of the types: III, IV, V, VI. Therefore if Z is a 5-membered subset of $F^\wedge$ then $\Pi(Z) = \mathsf{R}^3$. So, if $\Pi(X)$ is a plane then $X \in \mathcal{S}(4;F)$. Thus, by Theorem 25 we have that $X \in \mathcal{S}(4;F)$ if $\mathcal{G}(X)$ is either of type I or of type II. ∎

By the proof of Corollary 26, we also have

Corollary 27. *$\mathcal{S}(k;F) = \emptyset$ for every integer $k > 4$.*

Corollary 28. *Let Y be any 3-membered subset of $F^\wedge$. Then $Y \in \mathcal{S}(3;F)$ if and only if the graph $\mathcal{G}(Y)$ is of isomorphism type X.*

Proof: If the 3-edged graph $\mathcal{G}(Y)$ is of type VII or of type VIII then $\mathcal{G}(Y)$ is a subgraph of a graph of type II. And if $\mathcal{G}(Y)$ is of type IX then $\mathcal{G}(Y)$ is a subgraph

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of a graph of type I. In each of these cases, therefore, Theorem 25 implies that Y is a proper subset of some $X \subseteq F^\wedge$ for which $\Pi(X)$ is a plane. That is, $Y \notin \mathcal{S}(F)$. So $Y \notin \mathcal{S}(3;F)$.

It is easy to see that if $\mathcal{G}(Y)$ is of type X then $\mathcal{G}(Y)$ is a subgraph neither of a type-I graph nor of a type-II graph. It follows by Theorem 25 that $\Pi(X) = \mathsf{R}^3$ for every 4-membered superset $X \subseteq F^\wedge$ of Y. Furthermore $\Pi(Y)$ is a plane by Theorem 18, whence $Y \in \mathcal{S}(3;F)$. ∎

Corollary 29. *$|\mathcal{S}(3;F)| = 20$ and $|\mathcal{S}(4;F)| = 25$.*

Proof: On the vertex set F there are exactly 20 distinct graphs of type X. Therefore Corollary 28 implies that $|\mathcal{S}(3;F)| = 20$.

On the vertex set F there are exactly 15 distinct graphs of type I, and there are exactly 10 graphs of type II. Thus Theorem 25 implies that $|\mathcal{S}(4;F)| = 15 + 10$. ∎

Once it has been established that $\mathcal{S}(3;F)$ and $\mathcal{S}(4;F)$ are uniformly deep, the proof of Theorem 1 will be finished: Theorem 11 and Corollary 29, in conjunction with Proposition 8 and Theorem 2 both of [2], will imply that the depth of $\mathcal{S}(3;F)$ equals $3\,|\mathcal{S}(3;F)|\,/\,|F^\wedge| = 60/10 = 6$, and similarly that the depth of $\mathcal{S}(4;F)$ equals 10, as required in Theorem 1.

Our final task is to prove that $\mathcal{S}(3;F)$ and $\mathcal{S}(4;F)$ are uniformly deep.

For X a set the expression $\mathrm{Sym}(X)$ denotes the *symmetric group on* X; the elements in $\mathrm{Sym}(X)$ are the permutations of X. A subgroup G of $\mathrm{Sym}(X)$ is called *transitive* if and only if for every $\langle \mathrm{x}, \mathrm{y}\rangle \in X \times X$ there exists $g \in G$ such that $\mathrm{y} = g(\mathrm{x})$. A subgroup H of $\mathrm{Sym}(X)$ is said to *preserve* a given family $\mathcal{F}$ of subsets of X if and only if $\mathcal{F} = \{h[Y] \mid Y \in \mathcal{F} \,\&\, h \in H\}$ where $h[Y] := \{h(\mathrm{y}) \mid \mathrm{y} \in Y\}$. We omit proving here the following paraphrase of Theorem 6 in [**2**].

Lemma 30. *For X a finite set, and for $k \leqslant |X|$ a positive integer, let $\mathcal{F}$ be a family of k-membered subsets of X. If $\mathrm{Sym}(X)$ has a transitive subgroup which preserves $\mathcal{F}$ then $\mathcal{F}$ is uniformly deep.*

For X a v-membered set the expression $[k;X]$ will denote the family of all k-membered subsets of X. For each $g \in \mathrm{Sym}(X)$ we define ${}_kg\colon [k;X] \to [k;X]$ by ${}_kg(Y) = g[Y]$ for all $Y \in [k;X]$, and let ${}_k\mathrm{Sym}(X)$ denote $\{{}_kg \mid g \in \mathrm{Sym}(X)\}$. Note that ${}_k\mathrm{Sym}(X)$ is a transitive subgroup of $\mathrm{Sym}([k;X])$. Furthermore, ${}_2\mathrm{Sym}(X)$ preserves the family Edgesets($\mathcal{G}$) when $\mathcal{G} = \langle X, \mathcal{E}(\mathcal{G})\rangle$ is a graph. The following result is of independent interest since it provides a purely graph-theoretic method for producing uniformly deep families. It is an immediate consequence of Lemma 30 in conjunction with Theorem 2 of [**2**].

Theorem 31. *Let $\mathcal{G}$ be any graph on a v-membered vertex set X where v is a positive integer. Then the family Edgesets($\mathcal{G}$) is uniformly deep and its depth is*

$d = |\mathcal{E}(\mathcal{G})|\,|Edgesets(\mathcal{G})|\,/\binom{v}{2}$.

The proof of Theorem 1 is now complete.

Again let X be a finite, v-membered, set. An isomorphism from a graph $\mathcal{G} = \langle X, \mathcal{E}(\mathcal{G})\rangle$ onto $\mathcal{G}$ itself is called an *automorphism* of $\mathcal{G}$. The set $\mathrm{Aut}(\mathcal{G})$ of all automorphisms of $\mathcal{G}$ is a subgroup of $\mathrm{Sym}(X)$. We remark that $|\mathrm{Edgesets}(\mathcal{G})| = |\mathrm{Type}(\mathcal{G})| = |\mathrm{Sym}(X)|\,/\,|\mathrm{Aut}(\mathcal{G})| = v!/\,|\mathrm{Aut}(\mathcal{G})|$, and hence by Theorem 31 that $d = 2\,|\mathcal{E}(\mathcal{G})|\,(v-2)!/\,|\mathrm{Aut}(\mathcal{G})|$ where d is the depth of the family $\mathrm{Edgesets}(\mathcal{G})$.

There is a straightforward generalisation of Theorem 31 to k-hypergraphs. By a k-hypergraph we mean an ordered pair $\mathcal{H} = \langle X, \mathcal{K}(\mathcal{H})\rangle$ where X is a vertex set $\mathcal{V}(\mathcal{H})$ but where $\mathcal{K}(\mathcal{H})$ is a family of k-membered subsets of X; that is, the elements in $\mathcal{K}(\mathcal{H})$ are the "k-edges" of $\mathcal{H}$.

4. Slimming down

One might prefer Open Question 15 to be answered eventually in the affirmative. That is, one might hope that each pair $\langle m, n\rangle$ of integers with $m \geqslant n+2 \geqslant 4$ determines a unique sequence s_n, s_{n+1}, $s_{n+2}, \ldots$ of nonnegative integers such that $|\mathcal{S}(n+j; E)| = s_{n+j}$ for every nonnegative integer j and for every $E \in \Omega(m,n)$. One would then hope to characterise this sequence numerically.

However, even if the answer to Open Question 15 turned out to be "No!", one would still have a situation worthy of study. For, given a particular pair $\langle m, n\rangle$ there are obviously at most finitely many distinct such sequences s_n, $s_{n+1}, \ldots$. So the family of such sequences induces a natural and interesting finite partition of the (probably uncountable) family $\Omega(m,n)$. What would the geometric meaning of this partition be?

We close with the curiosity that for $\langle m, n\rangle = \langle 6, 4\rangle$ the most plausible analogue of Theorem 18 is false.

Theorem 32. *Let* $E = \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\} \in \Phi(6,4)$. *Then* $\mathbf{ab}^*\mathbf{cd}^*\mathbf{ef}^*$ *is a line.*

Proof: By Theorem 11 we have that E is obese, and hence that $\mathbf{xy}^* = \mathbf{zw}^*$ if and only if $\{\mathbf{x},\mathbf{y}\} = \{\mathbf{z},\mathbf{w}\}$ when $\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\} \subseteq E$. Now note that $\{\mathbf{ab}^*, \mathbf{cd}^*, \mathbf{ef}^*\} \subseteq \mathbf{abcd} \cap \mathbf{abef} \cap \mathbf{cdef}$. It follows that the j-plane $\mathbf{ab}^*\mathbf{cd}^*\mathbf{ef}^*$ is a subset of $\mathbf{abcd} \cap \mathbf{abef} \cap \mathbf{cdef}$. Of course $1 \leqslant j \leqslant 2$. Since $\mathbf{c} \notin \mathbf{abef}$, and since $\mathbf{abef}$ and $\mathbf{abcd}$ are 3-planes by Condition 6.1, we have that $\mathbf{abcd} \cap \mathbf{abef}$ is an i-plane for some $i \leqslant 2$. Since $\mathbf{cd}^* \notin \mathbf{ab}$ by Theorem 7, we have that $\mathbf{cd}^*\mathbf{ab}$ is a 2-plane. Therefore $\mathbf{cd}^*\mathbf{ab} = \mathbf{abcd} \cap \mathbf{abef}$ since $\{\mathbf{a},\mathbf{b},\mathbf{cd}^*\} \subseteq \mathbf{abcd} \cap \mathbf{abef}$. But $\mathbf{a} \notin \mathbf{cdef}$ while $\mathbf{a} \in \mathbf{cd}^*\mathbf{ab}$. So $\mathbf{abcd} \cap \mathbf{abef} \cap \mathbf{cdef} = \mathbf{cd}^*\mathbf{ab} \cap \mathbf{cdef}$ is a 1-plane; that is, $\mathbf{ab}^*\mathbf{cd}^*\mathbf{ef}^*$ is a subset of a line. Therefore $j = 1$. ∎

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