UNDIRECTED GRAPHS REALIZABLE AS GRAPHS OF MODULAR LATTICES

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1. Introduction. If $(L, \geqslant)$ is a lattice or partial order we may think of its Hesse diagram as a directed graph, G, containing the single edge $E(c, d)$ if and only if c covers d in $(L, \geqslant)$. This graph we shall call the graph of $(L, \geqslant)$. Strictly speaking it is the basis graph of $(L, \geqslant)$ with the loops at each vertex removed; see (**3**, p. 170).

We shall say that an undirected graph G_u can be realized as the graph of a (modular) (distributive) lattice if and only if there is some (modular) (distributive) lattice whose graph has G_u as its associated undirected graph. The main objective of this paper is to characterize those undirected graphs which can be realized as the graph of a modular lattice of finite length and to extend the result to distributive lattices of finite length. This is accomplished in Theorems 2 and 3.

In what follows G_u will always be an undirected graph, usually the associated undirected graph of the directed graph G. We shall use $u(c, d)$ $[p(c, d)]$ and $E(c, d)[P(c, d) = P(c, e_1, e_2, \ldots, e_n, d)]$ to denote respectively undirected and directed edges [arcs] from c to d. $V(G)$ $[V(G_u)]$ will be the vertex set of the graph $G[G_u]$.

2. Necessity. Throughout this section $(L, \geqslant)$ will be a modular lattice of finite length, G its graph, and G_u the associated undirected graph of G. The maximal chains in $(L, \geqslant)$ correspond in a 1–1 fashion to the directed arcs of G, and to each of these there corresponds an undirected arc in G_u. If $c, d \in L$, $c \geqslant$ d, there are two ways of thinking of the distance from c to d. One is to consider the distance from c to d as the length of a shortest maximal chain from c to d in $(L, \geqslant)$ or equivalently the length of a shortest directed arc from c to d in G. This we shall call the *directed distance* from c to d, and we shall denote it by $\Delta(c, d)$. The other way is to consider the distance from c to d as the length of a shortest undirected arc from c to d in G_u. This we shall call the *undirected distance* from c to d, and we shall denote it by $\delta(c, d)$.

We note that: (1) since $(L, \geqslant)$ is a modular lattice of finite length, $\Delta(c, d)$ is the length of any maximal chain or directed arc from c to d; (2) a simple induction argument shows that $\Delta(c, d) = \delta(c, d)$; and (3) G_u is connected and of finite diameter, so $\delta(c, d)$ is defined for all $c, d \in V(G_u)$.

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We now proceed with a succession of lemmas leading to the conclusion that G_u satisfies the following three conditions:

I. G_u is a connected graph of finite diameter which contains no loops, multiple edges, or circuits of odd length.

II. There exist two vertices, a_1 and a_2 in $V(G_u)$ such that $\operatorname{dia}(G_u) = \delta(a_1, a_2)$ and if $u(c, d)$ and $u(c, e)$ are distinct edges of G_u, and $\delta(a_i, e) = \delta(a_i, d) = \delta(a_i, c) + 1$, then there is a unique $f_i \in V(G_u)$ such that $\delta(a_i, f_i) = \delta(a_i, c) + 2$ and $u(f_i, e)$ and $u(f_i, d) \in G_u$; $i = 1, 2$.

III. If the subgraph, F_u, of the edges of a cube formed by removing one vertex and its incident edges is a subgraph of G_u, then the whole cube must be a subgraph of G_u.

LEMMA 1. *G_u is a connected graph of finite diameter which contains no loops or multiple edges.*

The proof follows directly from the definition of G_u.

LEMMA 2. *A connected undirected graph H_u contains an odd circuit if and only if given any $h \in V(H_u)$ there exists an edge $u(h_1, h_2) \in H_u$ such that $\delta(h, h_1) = \delta(h, h_2)$.*

Proof. Assume H_u contains an odd circuit and let $h \in V(H_u)$ be arbitrary. Let $p(h_0, h_1, \ldots, h_n, h_0)$ be any odd circuit of H_u, h_0 chosen such that $\delta(h, h_0) \leqslant \delta(h, h_j)$ for all $j = 0, 1, \ldots, n$. Either $\delta(h, h_0) = \delta(h, h_n)$ or there is some $j = 0, 1, \ldots, n - 1$ such that $\delta(h, h_j) = \delta(h, h_{j+1})$, for otherwise

$$\delta(h, h_0) = \delta(h, h_1) \pm 1 = \delta(h, h_2) \pm 1 \pm 1 = \ldots = \delta(h, h_0) \pm 1 \pm 1 \ldots \pm 1, \quad n + 1 \text{ terms}$$

which is impossible since n must be even.

Now let $h \in V(H_u)$ and $u(h_1, h_2) \in H_u$ be such that $\delta(h, h_1) = \delta(h, h_2)$. There are shortest arcs $p_1(h, h_1)$ and $p_2(h, h_2)$ from h to h_1 and h_2 respectively. The path formed by going from h to h_1 on $p_1(h, h_1)$, then from h_1 to h_2 on $u(h_1, h_2)$, and then back to h by the reverse of $p_2(h, h_2)$ is a path of odd length. At least one of its components must be a cycle of odd length.

LEMMA 3. *G_u contains no odd circuits.*

Proof. If G_u contained an odd circuit, there would be some edge, $u(c, d) \in G_u$, such that $\delta(I, c) = \delta(I, d)$ where I is the largest element of the lattice. This means, however, that $\Delta(I, c) = \Delta(I, d)$ and $u(c, d)$ cannot be directed in such a way that $(L, \geqslant)$ satisfies the Jordan–Dedekind chain condition.

THEOREM 1. *The vertices c and d are complementary elements of $(L, \geqslant)$ if and only if $\delta(c, d) = \operatorname{dia}(G_u)$.*

Proof. First we show that $\delta(I, 0) = \operatorname{dia}(G_u)$. Let $e, f \in V(G_u)$ be arbitrary.

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Then

$$2\delta(e,f) \leqslant \delta(I,e) + \delta(I,f) + \delta(e,0) + \delta(f,0) = 2\delta(I,0),$$

so $\delta(e,f) \leqslant \delta(I,0)$ for all $e, f \in V(G_u)$.

Now if c and d are complementary elements of $(L, \geqslant)$ and $p(c, d)$ is any shortest arc from c to d, then to $p(c, d)$ there corresponds a sequence of directed edges of G. This sequence may be replaced by another sequence of the same length constituting two arcs, one from $c \cup d = I$ to c (this one traversed backwards) and one from I to d. Likewise it can be replaced by a sequence of the same length constituting two arcs, one from c to $c \cap d = 0$ and one from 0 to d (this one traversed backwards). We may conclude, therefore, that

$$2\delta(c, d) = \delta(c, I) + \delta(I, d) + \delta(c, 0) + \delta(0, d) = 2\delta(I, 0) = 2\,\mathrm{dia}(G_u).$$

By the above argument if $\delta(c, d) = \mathrm{dia}(G_u)$, then $\delta(c \cup d, c \cap d) = \mathrm{dia}(G_u)$, implying $c \cup d = I$ and $c \cap d = 0$.

LEMMA 4. *G_u satisfies Condition* II.

Proof. According to Theorem 1, $\delta(I, 0) = \mathrm{dia}(G_u)$. If we take $a_1 = I$ and $a_2 = 0$, then the covering conditions imply II.

LEMMA 5. *G_u satisfies Condition* III.

Proof. Using the fact that there is essentially only one way in which a rectangle of G_u can be directed, it can be shown that there are exactly four (two of which are isomorphic) non-dual directed graphs that can result from F_u being a subgraph of G_u. Each of these gives rise to the required vertex and edges by use of the covering conditions. The details are straightforward.

Lemmas 1, 3, 4, and 5 show that the three conditions are necessary in order that G_u be realizable as the graph of a modular lattice of finite length.

3. Sufficiency. Throughout this section G_u will be an undirected graph satisfying Conditions I, II, and III, and $a = a_1$ and $b = a_2$ will be as in Condition II.

Since G_u is connected and contains no odd circuits, we shall direct the edges of G_u away from the vertex a by directing each edge towards the vertex farthest from a. That this can be done is assured by Lemma 2. This directed graph we denote by G, and we shall prove that G is the graph of a modular lattice of finite length. In particular we shall show that the pair $(L, \geqslant)$, $L = V(G)$, where $c \geqslant d$ if and only if there is a directed arc (possibly of zero length) from c to d in G, is a modular lattice of finite length.

LEMMA 6. (1) *If $c \geqslant d$, then $\delta(c, d) = \Delta(c, d)$.*

(2) *$(L, \geqslant)$ is a partial order of finite length satisfying the Jordan–Dedekind chain condition.*

(3) *The graph of $(L, \geqslant)$ is G, and $a \geqslant c$ for all $c \in L$.*

Proof. (1) Let $c \geqslant d$ and $P(e_0, e_1, e_2, \ldots, e_n)$, $e_0 = c$, $e_n = d$, be any arc in G from c to d. If $n = 1$, $\delta(c, d) = 1 = \Delta(c, d)$. If m is the smallest integer such that $\delta(c, e_m) \neq m$ and $\delta(c, e_k) = k$ for all $0 \leqslant k < m$, then

$$\delta(c, e_m) = (m - 1) \pm 1.$$

Since $\delta(c, e_m) = m - 2$ is impossible, $\delta(c, e_m) = m$, This yields a contradiction, so no such m exists and $\delta(c, d) = n = \Delta(c, d)$.

(2) Since G cannot contain any directed circuits, "$\geqslant$" is anti-symmetric; it is clearly reflexive and transitive. Hence, $(L, \geqslant)$ is a partial order. That $(L, \geqslant)$ is of finite length and satisfies the Jordan–Dedekind chain condition follows immediately from (1).

(3) G is a directed graph with no multiple edges; G is acyclic and transitive, so by (**3**, p. 170) G is the graph of a partial order. That $(L, \geqslant)$ is that partial order is clear. That $a \geqslant c$ for all $c \in L$ follows from the way G is directed.

LEMMA 7. *$(L, \geqslant)$ satisfies the two covering conditions of a modular lattice, and $c \geqslant b$ for all $c \in L$.*

Proof. If c covers d and e, $d \neq e$, then

$$\delta(a, c) + 1 = \delta(a, d) = \delta(a, e)$$

and Condition II implies that there is a unique $f \in L$ such that d and e cover f.

Let $c \in L$ be arbitrary and let $d \in L$ by any minimal element of $\{e | e \geqslant c$ and $e \geqslant b\}$. We shall show that $d = c$. If $d \neq c$, then there are non-intersecting (except at d) maximal chains from d to c and from d to b. We have, therefore, an edge $E(d, e)$, $e \geqslant c$, and an arc $P_1(e_0, e_1, e_2, \ldots, e_n)$, $e_0 = d$, $e_n = b$ in G. According to the first part of this lemma and the minimality of d, we can construct an arc of G, $P_2(f_0, f_1, \ldots, f_n)$, $f_0 = e$, such that $f_j \neq e_i$ for any $i, j = 0, 1, \ldots, n$, and e_j covers f_j for each $j = 0, 1, \ldots, n$. Since this gives an edge $E(b, f_n) \in G$ contradicting the choice of b, we must conclude that $d = c$ and $c \geqslant b$ for all $c \in L$.

The second covering condition now follows. Since

$$\delta(b, c) = \Delta(c, b) = \Delta(a, b) - \Delta(a, c) = \operatorname{dia}(G_u) - \delta(a, c),$$

a simple calculation shows that $E(e, f) \in G$ if and only if $\delta(b, f) = \delta(b, e) - 1$. Thus, II gives the second covering condition in the same way that II gave the first one.

LEMMA 8. *Any rectangle of four edges in G is directed as the graph of the distributive lattice of length two on four elements.*

Proof. By Lemma 6 there cannot be any arcs of length four or three. If c and d cover both e and f, $c \neq d$, $e \neq f$, we have a contradiction to the preceding lemma. Therefore, the only possibility is for the edges to be directed as desired.

We shall now show in three steps that given any two elements c and d in L, the set $\{e|e \geqslant c \text{ and } e \geqslant d\}$ has a unique minimal element. This, of course, will mean that every pair of elements of L has a least upper bound, and since L has a lower bound, we shall have shown that $(L, \geqslant)$ is a lattice.

LEMMA 9. *If e covers c and d, $c \neq d$, $f > c$, $f > d$, and $\Delta(f, c) = \Delta(f, d)$, then $f \geqslant e$.*

Proof. The proof proceeds by induction on $\Delta(f, c)$. If c, d, e, f are as in the statement of the lemma and $\Delta(f, c) = 1$, then $e = f$ by Lemma 8. We now assume that for some $m > 0$ the lemma is true for all c, d, e, f as above such that $\Delta(f, c) < m$. Let $c, d, e, f \in L$ be as above and let $\Delta(f, c) = m$. Let us further assume that $f \not\geqslant e$. There are in G two arcs,

$$P_{01}(c_{00}, c_{01}, \ldots, c_{0m}) \text{ and } P_{02}(d_{00}, d_{01}, \ldots, d_{0m}),$$

where $c_{00} = f = d_{00}$, $c_{0m} = c$, and $d_{0m} = d$. Note that $c_{0j} \neq d_{0j}$ for all $1 \leqslant j < m$; otherwise $f > e$ by the inductive hypothesis. Using Lemma 7, the Jordan–Dedekind chain condition, and the inductive hypothesis, we can find $c_{10}, c_{11}, \ldots, c_{1,m-1}, c_{10} = d_{10}, d_{11}, \ldots, d_{1,m-1}$ such that

(1) $\{c_{10}, c_{11}, \ldots, c_{1,m-1}, d_{10}, d_{11}, \ldots, d_{1,m-1}\} \cap$
$\{c_{00}, c_{01}, \ldots, c_{0m}, d_{00}, d_{01}, \ldots, d_{0m}, e\} = \emptyset,$

(2) c_{1i} covers $c_{1,i+1}$ and is covered by $c_{0,i+1}$ and d_{1i} covers $d_{1,i+1}$ and is covered by $d_{0,i+1}$ for all $0 \leqslant i \leqslant m - 1$.

Now we show that $c_{1,m-1} \neq d_{1,m-1}$. Assume that $c_{1,m-1} = d_{1,m-1}$. c_{0m} and $d_{1,m-2}$ cover $c_{1,m-1} = d_{1,m-1}$ and $c_{0m} \neq d_{1,m-2}$. Hence by Lemma 7 there is a g which covers both. According to Condition III and Lemma 8 there is some $h \in L$ which covers g, $d_{0,m-1}$, and e. If $g = c_{0,m-1}$, the inductive hypothesis implies that $f > h > e$, contrary to our assumption. If $g \neq c_{0,m-1}$, the inductive hypothesis implies that $c_{01} > g$, and hence that $c_{00} = f > e$, which again is contrary to assumption. We conclude that $c_{1,m-1} \neq d_{1,m-1}$.

Since e covers c_{0m} and d_{0m} and $c_{0m} \neq d_{0m}$, there is an e_1 which is covered by both c_{0m} and d_{0m}. We can conclude that $e_1 \neq c_{1,m-1}$ or $d_{1,m-1}$ as follows. If $e_1 = c_{1,m-1}$, then $c_{10} > e_1$ and $c_{10} > d_{1,m-1}$. Hence $c_{10} > d_{0m}$ by the inductive hypothesis. But now $c_{01} > c_{10} > d_{0m}$ and $c_{01} > c_{0m}$. Hence $c_{01} > e$ by the inductive hypothesis, and therefore $f > e$. A similar argument applies if instead $e_1 = d_{1,m-1}$.

We now use Lemma 7 again to find c_{1m} and d_{1m} such that e_1 and c_{1m-1} cover c_{1m}, and e_1 and $d_{1,m-1}$ cover d_{1m}. If $c_{1m} = d_{1m}$, according to Condition III and Lemma 8, we first have some $g \in L$, $g \neq c_{0m}$ or d_{0m}, which covers $c_{1,m-1}$ and $d_{1,m-1}$ and is covered by e. The inductive hypothesis yields $c_{01} > c_{10} > g$. Using it again, we obtain $c_{01} > e$, so $f > e$, contrary to our assumption. Thus $c_{1m} \neq d_{1m}$; cf. Figure 1.

Next we shall show that $c_{10} \not> e_1$. If $c_{10} > e_1$, then there is some $g \in L$ such that $c_{01} > g$ and g covers e_1. If $g = c_{0m}$, then $d_{01} > c_{10} > g = c_{0m}$ and $d_{01} > d_{0m}$. Hence $d_{01} > e$ by the inductive hypothesis, and so $f > e$, which is impossible.

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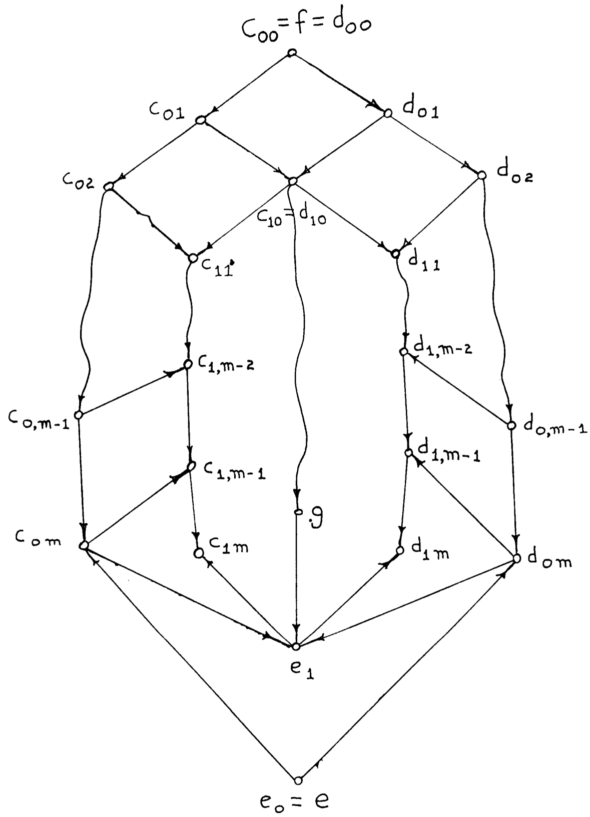

FIGURE 1

We deduce that $g \neq c_{0m}$ and similarly $g \neq d_{0m}$. Since $g \neq c_{0m}$ or d_{0m}, and g, c_{0m}, and d_{0m} cover e_1, there are h_1 and h_2 in L such that h_1 covers c_{0m} and g, and h_2 covers d_{0m} and g. Now applying the inductive hypothesis twice, we conclude that $c_{01} > h_1$ and $d_{01} > h_2$; hence neither h_1 nor h_2 is equal to e. Now Condition III and Lemma 8 yield the existence of an $h_3 \in L$ which covers h_1, h_2, and e, and the inductive hypothesis yields $f > h_3 > e$. This contradiction to our assumption that $f \not> e$ implies that $c_{10} \not> e_1$, as desired.

We now have constructed two arcs $P_{11}(c_{10}, c_{1m})$ and $P_{12}(c_{10}, d_{1m})$ of length m and a vertex e_1 which covers c_{1m} and d_{1m}. Since $c_{10} \not> e_1$, the c_{1j}'s must be distinct from the d_{1k}'s (except for $c_{10} = d_{10}$), so the situation with respect to these arcs and the vertex e_1 is the same as it was with respect to $P_{01}(f, c_{0m})$, $P_{02}(f, d_{0m})$, and e. We may, therefore, continue the above construction indefinitely, producing subsets of L, $V_0, V_1, V_2, \ldots$ such that for every $k = 0, 1, 2, \ldots$:

(1) $V_k = \{c_{k0}, c_{k1}, \ldots, c_{km}, d_{k0}, d_{k1}, \ldots, d_{km}, e_k\}$, $e_0 = e$, and $c_{k0} = d_{k0}$;

(2) $V_{k-1} \cap V_k = \emptyset$,

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(3) $c_{k-1,j+1}$ covers c_{kj} and c_{kj} covers $c_{k,j+1}$ for each $j = 0, 1, \ldots, m-1$, and e_k covers c_{km} and d_{km},

(4) $c_{k0} \not\geqslant e_k$.

We can, therefore, construct arcs and hence maximal chains

$$P_n(e_0, c_{0m}, e_1, c_{1m}, \ldots, e_{n-1}, c_{n-1,m}, e_n)$$

of arbitrary length, contradicting the fact that $(L, \geqslant)$ is of finite length. This contradiction proves that $f > e$, as desired.

LEMMA 10. *If e and f are greater than c and d, $c \neq d$, then there is some $g \in L$ such that $e \geqslant g \geqslant c$ and $f \geqslant g \geqslant d$.*

Proof. The proof of this lemma proceeds by induction on

$$R = \tfrac{1}{2}[\Delta(e,c) + \Delta(e,d) + \Delta(f,c) + \Delta(f,d)].$$

For $R = 2$ the preceding lemma yields the result.

Now assume inductively that if $R < s$, $s > 2$, the lemma is true. Let $e, f, c, d \in L$ satisfy the hypotheses of the lemma; $R = s$. Suppose that no $g \in L$ exists such that $e \geqslant g \geqslant c$ and $f \geqslant g \geqslant d$ (Assumption A). We may assume the vertices have been named such that

$$\Delta(e,c) = \min\{\Delta(e,c), \Delta(e,d), \Delta(f,c), \Delta(f,d)\},$$

and we may assume $\Delta(e,c)$ is minimal for c, d, e, and f satisfying Assumption A.

Case I, $\Delta(e,c) = \Delta(e,d) = m$. A calculation based on Lemma 6 shows that $\Delta(f,c) = \Delta(f,d)$, and the preceding lemma shows that $m > 1$. Let

$$P_{00}(c_{00}, c_{01}, \ldots, c_{0m}) \quad \text{and} \quad P_{10}(d_{00}, d_{01}, \ldots, d_{0m}),$$

$c_{00} = e = d_{00}$, $c_{0m} = c$, $d_{0m} = d$ be any two maximal chains from e to c and e to d respectively. By Lemma 6, Assumption A, and the inductive hypothesis, it follows that $c_{0k} \neq d_{0n}$ for every $k, n = 1, 2, \ldots, m$. Thus there exists $c_{10} = d_{10} \in L$ covered by c_{01} and d_{01}. If $c_{10} \geqslant c$ or d, then Lemma 6 and the inductive hypothesis yield a contradiction to Assumption A. A sequence of similar arguments gives rise to two maximal chains,

$$P_{01}(c_{10}, c_{11}, \ldots, c_{1,m-1}) \quad \text{and} \quad P_{11}(d_{10}, d_{11}, \ldots, d_{1,m-1}),$$

such that c_{0k} covers $c_{1,k-1}$ and d_{0k} covers $d_{1,k-1}$ for each $k = 1, 2, \ldots, m$, and neither $c_{1k} \geqslant c$ or d nor $d_{1k} \geqslant c$ or d holds for any $k = 0, 1, \ldots, m-1$.

If $c_{1,m-1} = d_{1,m-1}$, then Lemmas 7 and 9 show that there is some $g \in L$ which covers c_{0m} and d_{0m} such that $e \geqslant g \geqslant c_{0m}$ and $f \geqslant g \geqslant d_{0m}$, which contradicts Assumption A. Thus since $c_{1,m-1} \neq d_{1,m-1}$ and $\Delta(e,c)$ is minimal, there is some $g' \in L$ such that $c_{10} \geqslant g' \geqslant c_{1,m-1}$ and $f \geqslant g' \geqslant d_{1,m-1}$; cf. Figure 2. If $g' = c_{0m}$, then d_{01}, c_{0m}, d_{0m}, and f satisfy the inductive hypothesis. This gives some $g \in L$ such that $e > d_{01} \geqslant g \geqslant c_{0m} = c$ and $f \geqslant g \geqslant d_{0m} = d$, which contradicts Assumption A.

If $g' = c_{1,m-1}$, then by Lemma 6, $g' = d_{1,m-1}$, contrary to the fact that

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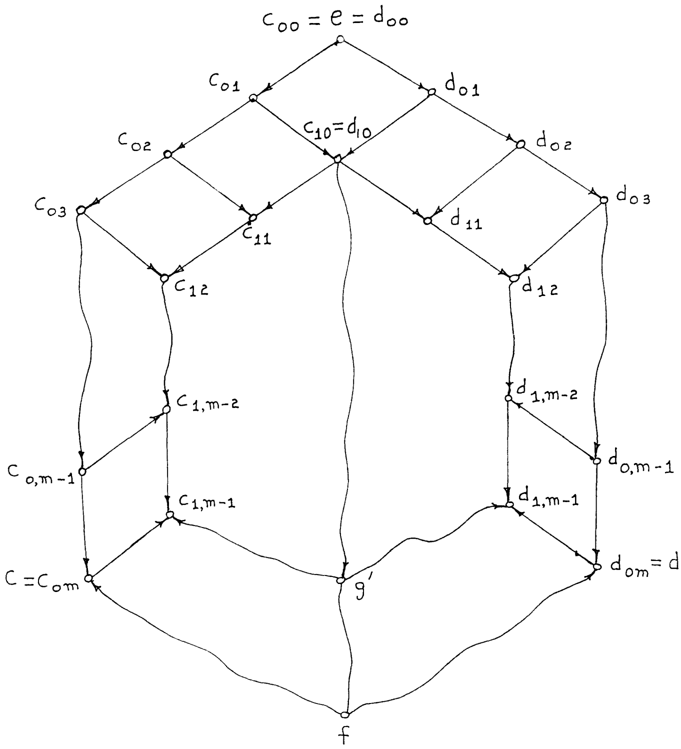

FIGURE 2

$c_{1,m-1} \not\geqslant d_{1,m-1}$. Now a calculation based on Lemma 6 and this observation shows that c_{01}, f, c_{0m}, and g' satisfy the inductive hypothesis. Hence there is some $g'' \in L$ such that $c_{01} \geqslant g'' \geqslant c_{0m}$ and $f \geqslant g'' \geqslant g'$. Now if $g'' = c_{0m}$, then

$$\Delta(g'', c_{1,m-1}) = 1 = \Delta(g'', g') + \Delta(g', c_{1,m-1});$$

hence $g' = g'' = c_{0m}$, which is impossible. This means that

$$\Delta(c_{00}, g'') + \Delta(f, g'') + \Delta(c_{00}, d) + \Delta(f, d) < s.$$

If $g'' = d$, there is nothing more to prove. If not, the inductive hypothesis applies, and we have some $g \in L$ such that $e = c_{00} \geqslant g \geqslant g'' \geqslant c$ and $f \geqslant g \geqslant d$, contradicting Assumption A. This concludes the proof of case I.

Case II, $\Delta(e, c) \neq \Delta(e, d)$. Thus $\Delta(e, c) < \Delta(e, d)$. Let

$$P_{00}(c_{00}, \ldots, c_{0m}) \quad \text{and} \quad P_{10}(d_{00}, \ldots, d_{0m}),$$

$c_{c0} = e = d_{00}$, $c_{0m} = c$, $d_{0n} = d$ be any two maximal chains from e to c and e to d respectively. As in the preceding we can find a maximal chain $P_{01}(c_{10}, \ldots, c_{1,m-1})$ such that c_{0k} covers $c_{1,k-1}$ and $c_{1,k-1}$ is not greater than or equal to c or d for each $k = 1, 2, \ldots, m$.

Suppose there is a $g' \in L$ such that $d_{01} \geqslant g' \geqslant d_{0n} = d$ and $f \geqslant g' \geqslant c_{1,m-1}$. Then g' is not greater than or equal to c and either $\Delta(g', d) = 0$ or $\Delta(g', d) > 0$. $\Delta(g', d) = 0$ would imply that $g' = d = d_{0n}$. Thus Lemma 6 and the fact that $c_{1,m-1} \neq d$ imply that

$$\Delta(d_{01}, c_{1,m-1}) = m > \Delta(d_{01}, g') = n - 1$$

and that $m \geqslant n$, which is impossible. If $\Delta(g', d) > 0$, then we may apply the inductive hypothesis to $e = c_{00}$, g', $c = c_{0m}$, and f to obtain a $g \in L$ such that $e \geqslant g \geqslant c$ and $f \geqslant g \geqslant g' \geqslant d$, contrary to Assumption A.

We now have d_{01}, $c_{1,m-1}$, d, and f satisfying the same hypothesis as c, d, e, and f, but now $\Delta(d_{01}, c_{1,m-1}) = m$ and $\Delta(d_{01}, d) = n - 1$. By repeating the above argument $p = n - m$ times, we can find d_{0p}, $c_{p,m-1}$, d, and f contradicting case I. Since this is impossible, our Assumption A must be false for case II also, and the proof of the lemma is complete.

LEMMA 11. *$(L, \geqslant)$ is a modular lattice of finite length whose graph is G.*

Proof. Let $c, d \in L$ be arbitrary. By Lemma 10 there can be only one minimal element of $\{e | e \geqslant c \text{ and } e \geqslant d\}$. This is $c \cup d$. Since $e \geqslant b$ for all $e \in L$ and $c \cup d$ is defined for all $c, d \in L$, $(L, \geqslant)$ is a lattice. It is of finite length and its graph is G by Lemma 6. By Lemma 7 $(L, \geqslant)$ satisfies the two covering conditions, so it is modular.

The proof of the following theorem is now complete.

THEOREM 2. *G_u can be realized as the graph of a finite modular lattice if and only if G_u satisfies Conditions* I, II, *and* III.

THEOREM 3. *G_u can be realized as the graph of a finite distributive lattice if and only if G_u satisfies Conditions* I, II, III, *and:*

IV. *If G_u contains the rectangle of edges, $u(c, d)$, $u(d, e)$, $u(e, f)$, $u(f, c)$, there is no vertex g such that $u(c, g)$ and $u(g, e) \in G_u$.*

Proof. The proof is immediate from Theorem 2 above and (**2**, p. 134, corollary 2). (Two misprints should be noted in that corollary. The figure which is referred to is the figure of the first edition (**1**) and $(x^* \cap v) \cup u$ should read $(x^* \cup v) \cap u$.)

Using Theorems 1 and 3, it can be shown that for G_u satisfying I, II, III, and IV, it does not matter which diametrically opposite vertices are chosen. Any choice results in a distributive lattice.

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