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MATHEMATICAL NOTES

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A NOTE ON PRIMITIVE GRAPHS

BY
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0. Let G denote a connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset C of $E(G)$ is called a *cutset* of G if the graph with vertex set $V(G)$ and edge set $E(G)-C$ is not connected, and C is minimal with respect to this property. A cutset C of G is *simple* if no two edges of C have a common vertex. The graph G is called *primitive* if G has no simple cutset but every proper connected subgraph of G with at least one edge has a simple cutset. For any edge e of G, let $G-e$ denote the graph with vertex set $V(G)$ and with edge set $E(G)-e$. An edge $e=[x, y]$ of G is a *regular* edge of G if $G-e$ is connected and has both a simple cutset no edge of which is incident with x and a simple cutset no edge of which is incident with y.

In his generalization [1] of a cube vertex assignment problem, Graham introduced the concepts of primitive graph and regular edge, and asked the following questions [1, (VII, 3 and 4)], among others: "Must all the edges of a primitive graph be regular? Must a primitive graph have a vertex of degree 2? Can a primitive graph have an even number of vertices?" We settle these questions by constructing a primitive graph P_0 on 10 vertices having no vertex of degree 2, and also no regular edge. By using a binary operation on graphs introduced in [1], we show that P_0 generates a family of primitive graphs on $2n$ vertices, $n\geq 5$. In conjunction with known results [1], and a verification by us (which we do not include here) that there is no primitive graph on 8 vertices, this implies that *a primitive graph on n vertices exists if and only if* $n=3, 5, 7$, *or* ≥ 9.

1. The graph P_0 on 10 vertices is constructed by joining each set of four alternate vertices of an octagon to a new vertex (Fig. 1, where the octagon vertices appear numbered from 1 to 8).

(1.1) P_0 *has no simple cutset.*

Proof. Any cutset C of a graph has an even number of edges in common with any circuit (see, for instance, [2, p. 32]). In particular, if C is a simple cutset, then

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P_0:

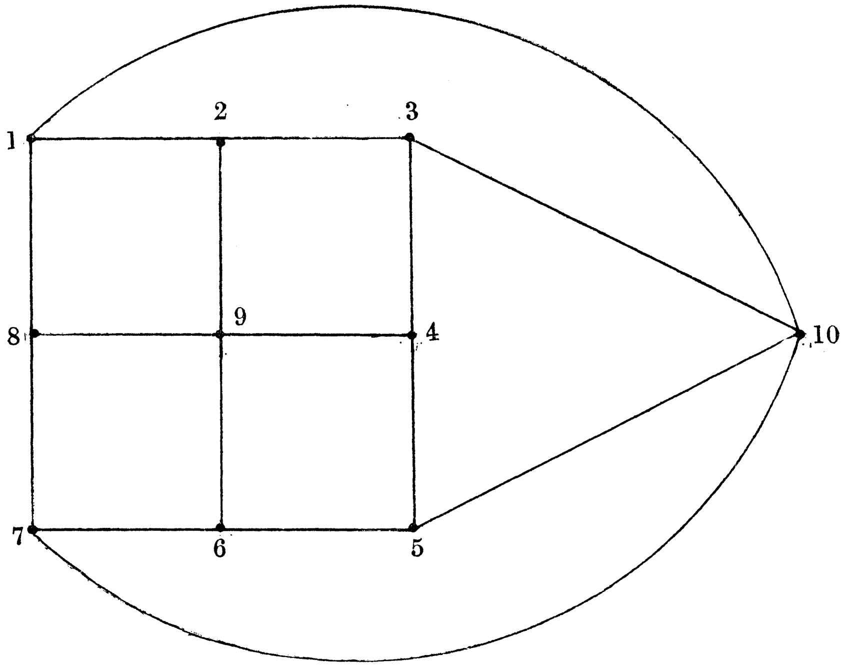

Figure 1.

C is a set S with the following property Q: If an edge e of a quadrilateral in the graph belongs to S, then the edge of the quadrilateral which is opposite to e, also belongs to S. In the graph P_0, let S be any set of edges with property Q. A direct verification shows that $S \neq \phi \Rightarrow S = E(P_0)$, and S cannot be a simple cutset of P_0.

(1.2) *If H is a connected proper subgraph of P_0 with at least one edge, then H has a simple cutset.*

Proof. Let e be any edge of P_0 which is not an edge of H. We shall use the vertex numbering as given in Fig. 1. We may assume $e=[4, 9]$ or $[1, 2]$, since it is seen that any other edge of P_0 may be transformed, by an automorphism of P_0, to one of these two. For $e=[4, 9]$, we let $C=\{[2, 9], [1, 8], [7, 10], [5, 6]\}$, and for $e=[1, 2]$, we let $C=\{[1, 8], [7, 10], [5, 6], [9, 4], [2, 3]\}$. In each case C is found to be a simple cutset of $P_0 - e$. Thus, if H has an edge in common with C, then $C \cap E(H)$ contains a simple cutset of H. Otherwise H must be a subgraph of a connected component T of $P_0 - C - e$, and a direct check shows that each edge of T belongs to a simple cutset of T.

(1.1) and (1.2) state that P_0 is primitive.

Using the property Q (see proof of (1.1)) of a simple cutset of a graph, it is easy

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to verify that neither of the edges [4, 9] and [1, 2] is regular, and since they represent the different edge orbits of P_0, we have:

(1.3) *No edge of* P_0 *is regular.*

2. Graham [1] introduced the following binary operation on graphs: Let G be a graph and $e=[x, y]$ an edge of G. Let H be a graph with a vertex z of degree 2. Assume $V(G) \cap V(H)=\{x, y\}$, and $[z, x]$ and $[z, y]$ are the two edges of H incident with z. Let G' be the graph formed from G by deleting the edge e, and let H' be the graph formed from H by deleting the vertex z and the two edges incident with it. Then a graph K is constructed by letting $V(K)=V(G') \cup V(H')$ and $E(K)=E(G') \cup E(H')$. Theorem 1 of [1] states that if G and H are primitive, and e is a regular edge of G, then K is primitive.

We shall be interested in the case where H is the complete bipartite graph $K(2, 3)$ (which is a primitive graph [1]). In this case K has two more vertices than G, and it is not difficult to show that if e is a regular edge of G, then any one of the four edges in $E(K)-E(G')$ is a regular edge of K. Thus, if a primitive graph can be found on an even number $2n$ of vertices and with at least one regular edge, then Graham's result with $H=K(2, 3)$ implies the existence of a primitive graph on $2m$ vertices, for each $m \geq n$.

We show that for the choice $H=K(2, 3)$, $G=P_0$, $e=[4, 9]$ (Fig. 1), the resulting graph K on 12 vertices is primitive and has a regular edge. The primitivity may be shown from the following theorem. Here, with $H=K(2, 3)$, we let K' denote the graph obtained from K by deleting one of the two vertices in $V(K)-V(G')$, and the two edges incident with it.

(2.1) *Let* G *be primitive,* e *a (non-regular) edge of* G*, and* $H=K(2, 3)$. *Then* K *is primitive if and only if* K' *has a simple cutset.*

Proof. The necessity of the condition is immediate, since K' is a proper connected subgraph of K. For the sufficiency, we note that the first part of the proof of [1, Theorem 1] does not use the assumption there that e is regular in G, and the argument given (up to the end of (i) in the proof) shows that K does not have a simple cutset and that any connected subgraph of K which does not contain all of G' as a subgraph (and which has at least one edge), has a simple cutset. It remains for us to show that if P is a proper connected subgraph of K containing all of G' as a subgraph, then P has a simple cutset. We note that H' is a quadrilateral. Its two vertices not in $V(G')$ will be denoted by a, b. If P has no edge in common with H', then P is a proper subgraph of G and has a simple cutset. If one of the vertices a and b is of degree 1 in P, then the edge with which this vertex is incident, will form a simple cutset of P. Not both of the vertices a and b can be of degree 2 in P, for then P would be equal to K. The only remaining case is where exactly one of the vertices a and b is a vertex of P, and of degree 2 in P, in which case $P=K'$, and K' has a simple cutset, by hypothesis. This proves (2.1).

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For the choice $G=P_0$ and $e=[4, 9]$ in (2.1), the graph K' is found to have the simple cutset $\{[2, 9], [1, 8], [7, 10], [5, 6], [4, a]\}$, where a denotes the vertex of K' not in $V(P_0)$, so that K is primitive. A routine check shows that any edge in $E(K)-E(G')$ is regular in K. Thus we can state:

(2.2) *The graph P_0 generates a family of primitive graphs on $2n$ vertices for all $n\geq 5$.*

References

1. R. L. Graham, *On primitive graphs and optimal vertex assignments*, Proc. Internat. Conf. on Combinatorial Mathematics, New York, April 1970; New York Academy of Sciences, 1970.

2. S. Seshu and M. Reed, *Linear graphs and electrical networks*, Addison-Wesley, Reading, Mass. 1961.

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