BULL. AUSTRAL. MATH. SOC. 34C05, 34D10
VOL. 44 (1991) [139–147]

PERTURBATIONS OF A HAMILTONIAN FAMILY OF CUBIC VECTOR FIELDS

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This paper is related with the configurations of limit cycles for cubic polynomial vector fields in two variables (χ_3).

It is an open question to decide whether every limit cycle configuration in χ_3 can be obtained by perturbation of a corresponding Hamiltonian configuration of centres and graphs.

In this work, by considering perturbations of the Hamiltonian vector field $X_H = (H_y, -H_x)$, where $H(x, y) = [a(x+h)^2 + by^2 - 1][a(x-h)^2 + by^2 - 1]$, we make a global analysis of the possible cases.

The vector field X_H has three centres (C^-, C^+ and the origin) and two saddles. By means of quadratic perturbations the centres become fine foci where C^- and C^+ have the same type of stability but opposed to that one of the origin and infinity. Further introducing cubic perturbations changes the stability of C^-, C^+ and the cycle at infinity and generates limit cycles. Lastly extra linear terms change the stability of the origin and generate another limit cycle.

Finally, we analyse the rupture of saddle connection of the Hamiltonian field under perturbation, via Melnikov's integral, in order to complete the study of the global phase portrait and to consider the possibility of new limit cycles emerging from the Hamiltonian graph.

1. INTRODUCTION

Hilbert's sixteenth problem asks for the maximum number and position of limit cycles for a differential system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y)$$

where P and Q are polynomials and at least one of them is of degree n in x and y. For $n = 2$ the distribution of limit cycles is already known and Bamon [2] has established that there are only a finite number. (See also survey papers [4, 5]).

The corresponding study for cubic vector fields is far from complete but we can mention, in connection with the possible configurations of limit cycles, the works of Holmes and Rand [6] dealing with the case of two limit cycles surrounded by a third

Received 27 August 1990

one. Then Li Ji-bin [7] found a configuration of five limit cycles consisting of the Rand distribution plus two other separate limit cycles. In 1987, Li Ji-bin and Quiming [8] gave several new configurations obtained by perturbing of Hamiltonian systems, including one with 11 limit cycles.

A very important contribution is also given by Lloyd, Blows and Kalenge [**3, 9, 10**]. They give detailed versions of the techniques employed and obtain a family of cubic fields with a fine focus of order 6 and then they generate 6 small-amplitude limit cycles. They use a particular family of cubic vector fields with non zero quadratic part. They also analised the coexistence of fine foci for cubic fields without quadratic part and established some compatibilities for the order of weakness of the foci. Finally, they obtain a family with five limit cycles, four generated from the origin and one from infinity.

In connection with the bifurcation of limit cycles at infinity, for general polynomial vector fields, there is a work by Rousseau [11]. This author establishes a duality between the above mentioned bifurcation and the Hopf bifurcation and gives examples of cubic systems with at least seven limit cycles: three around one singular point, one around another singular point and three limit cycles surrounding the four first limit cycles.

The purpose of this paper is to initiate a systematic approach to obtain new configurations of limit cycles for cubic vector fields in two variables. For this we consider quartic functions $H(x, y)$ and the associated Hamiltonian vector field $X = (P, Q)$ defined by

$$P(x, y) = \frac{\partial H}{\partial y}(x, y)$$
$$Q(x, y) = -\frac{\partial H}{\partial x}(x, y).$$

The vector field (P, Q) is tangent to the level curves of H.

In the next sections we study a family of vector fields obtained in this manner, for the case $H(x, y) = \left(a(x+h)^2 + by^2 - 1\right)\left(a(x-h)^2 + by^2 - 1\right)$.

By means of adequate perturbations of the Hamiltonian family we obtain different and new configurations of limit cycles. Moreover, the global phase portrait is analysed through the study of the behaviour at infinity and the determinations of the type of ruptures of the Hamiltonian saddle connections.

2. A family of perturbed Hamiltonian vector fields

Let

$$H(x, y) = -\frac{1}{2}(x^2 + y^2) + \frac{a+b}{4}x^2y^2 + \frac{a+b}{8}\left(\frac{1}{b}x^4 + \frac{b}{a}y^4\right)$$

with $0 < a < b$.

The Hamiltonian vector field X associated with $H(x, y)$ is defined by

$$\text{(1)} \qquad \begin{aligned} \dot{x} &= P(x, y) = -y + Ax^2y + By^3 \\ \dot{y} &= Q(x, y) = x - Axy^2 - Cx^3 \end{aligned}$$

where $A = (a+b)/2$, $B = b(a+b)/2a$ and $C = a(a+b)/2b$.

The singularities of X are: centres at $(0, 0)$ and $C^{\pm} = \left(\pm\sqrt{2b/(a(a+b))}, 0\right)$, saddles at $S^{\pm} = \left(0, \pm\sqrt{2a/((a+b)b)}\right)$. The phase portrait is shown in Figure 1.

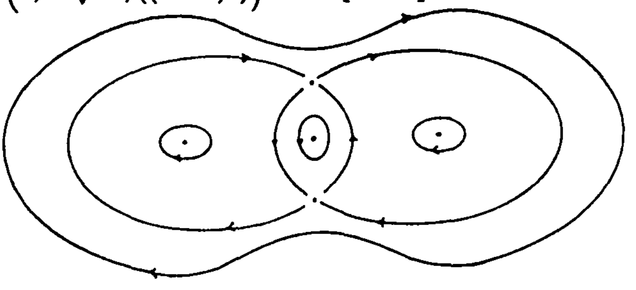


Figure 1

The saddle separatrices are determined by $H(x, y) = -a/2b(a+b)$ and they are located on the ellipses $a(x-h)^2 + by^2 = 1$ and $a(x+h)^2 + by^2 = 1$ where $h^2 = (b-a)/a(a+b)$.

Next, we consider the vector field

$$\text{(2)} \qquad X_{\varepsilon,\mu,\lambda} = X + \varepsilon Z_1 + \mu Z_2 + \lambda Z_3$$

where $Z_1(x, y) = \left(Fxy + Gy^2, Ky^2\right)$, $Z_2(x, y) = \left(xy^2, -x^2y\right)$ and $Z_3(x, y) = (0, y)$.

These perturbations preserve the singularities of (1) on the axis $y = 0$. Moreover, the divergence of $X_{\varepsilon,0,0}$ is $\varepsilon(F+2K)y$ so that the singularities 0 and $C^{\pm}$ are fine foci.

Proposition 1. *The vector field $X_{\varepsilon,0,0}$ with $\varepsilon \neq 0$ has at the origin and at the points $C^{\pm}$, centres or weak foci of order at most one.*

Proof: The values of the Poincaré derivatives [1] at the origin are

$$\alpha_1(\theta) \equiv 1, \qquad \alpha_2(2\pi) = 0, \qquad \alpha_3(2\pi) = \frac{\pi\varepsilon^2}{4}G(F+2K).$$

If the order of the fine focus were greater than one, then $G(F+2K) = 0$, that is $G = 0$ or $F + 2K = 0$. In both cases we have a centre according to the symmetry

https://doi.org/10.1017/S0004972700029531 Published online by Cambridge University Press

principle or the divergence criterion, respectively. So, the order of the fine focus $(0, 0)$ is at most one.

In order to study $C^+ = (c_1, 0)$ let $z = x - c_1$; then

$$\begin{aligned}\dot{z} &= \left(-1 + Ac_1^2 + \varepsilon F c_1\right)y + (2Ac_1 + \varepsilon F)zy + \varepsilon G y^2 + Az^2 y + By^3,\\ \dot{y} &= -2z + 3Cc_1 z^2 + (\varepsilon K - Ac_1)y^2 - Cz^3 - Azy^2.\end{aligned}$$

Since $-1 + Ac_1^2 + \varepsilon F c_1 > 0$, for $\varepsilon \ll 1$, we scale the variables so that the linear part of the vector field is in the standard form for a centre.

Let $\tau = 2\sigma t$, $u = \sigma y$ and $v = z$ where $\sigma = \left[\left(-1 + Ac_1^2 + \varepsilon F c_1\right)/2\right]^{1/2}$.

Then the system becomes

$$\begin{aligned}\dot{u} &= -v - \frac{3}{2}Cc_1 v^2 + \frac{\varepsilon K - Ac_1}{2\sigma^2}u^2 - \frac{C}{2}v^3 - \frac{A}{2\sigma^2}u^2 v\\ \dot{v} &= u + \frac{2Ac_1 + \varepsilon F}{2\sigma^2}uv + \frac{\varepsilon G}{2\sigma^3}u^2 + \frac{A}{2\sigma^2}uv^2 + \frac{B}{2\sigma^4}u^3\end{aligned}$$

and the values of the first three Poincaré derivatives at C^+ are

$$\alpha_1(\theta) \equiv 1, \qquad \alpha_2(2\pi) = 0 \quad \text{and} \quad \alpha_3(2\pi) = -\frac{\pi\varepsilon^2}{16\sigma^5}G(F + 2K).$$

For $C^- = (-c_1, 0)$ we have a similar procedure and we obtain the same results. □

Observe that the fine foci $C^\pm$ and the origin have opposite stabilities.

3. Behaviour at infinity

The Hamiltonian vector field X has no singularities at infinity, because

$$\begin{aligned}A_3(x, y) &= -y\left(Ax^2 y + By^3\right) + x\left(-Axy^2 - Cx^3\right)\\ &= -\frac{a+b}{2ab}\left(ax^2 + by^2\right)^2 < 0 \qquad \forall (x, y) \neq (0, 0).\end{aligned}$$

The same is true for the vector field $X_{\varepsilon,0,0}$ since Z_1 is quadratic.

Let $u = x/\sqrt{A}$, $v = y/\sqrt{C}$, in order to have $X_{\varepsilon,0,0}(u, v)$ in normal form at infinity. So the Poincaré derivatives at infinity are

$$\alpha_1^\infty \equiv 1, \qquad \alpha_3^\infty(-2\pi) = \frac{\pi}{8}\frac{\overline{A}\varepsilon^2 G(F + 2K)}{\overline{B}^6}$$

where $\overline{A} = \sqrt{a/b}$ and $\overline{B} = \sqrt{(a+b)/2}$.

So infinity is a non hyperbolic periodic orbit with the same type of stability as the origin.

4. Generation of limit cycles

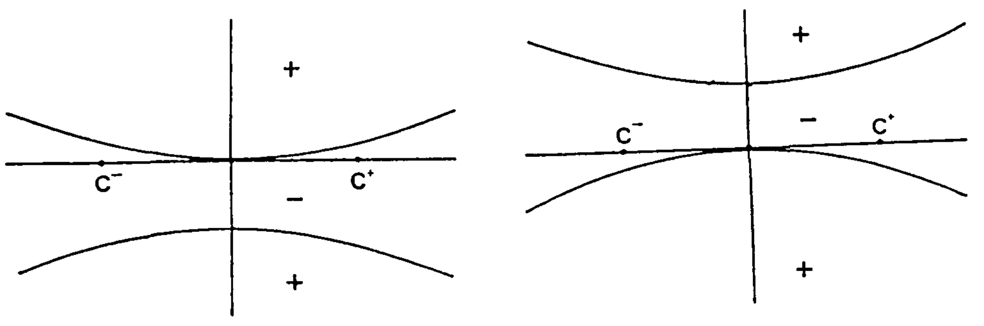
In order to generate simultaneously limit cycles from the fine foci and from infinity, we consider the vector field $X_{\varepsilon,\mu,\lambda}$, that is,

$$\dot{x} = -y + Ax^2y + By^3 + \varepsilon y(Fx + Gy) + \mu xy^2,$$
$$\dot{y} = x - Axy^2 - Cx^3 + \varepsilon Ky^2 - \mu x^2y + \lambda y.$$

Proposition 2. *If* $0 < |\lambda| \ll |\mu| \ll |\varepsilon|$, $G(F+2K)\mu < 0$ *and* $G(F+2K)\lambda < 0$ *then the vector field* $X_{\varepsilon,\mu,\lambda}$ *has the configuration* (1, 1, 1) *of infinitesimal limit cycles surrounded by a big amplitude one, obtained by bifurcating the infinity.*

Proof: Div $X_{\varepsilon,\mu,\lambda}(x, y) = \lambda + \varepsilon(F+2K)y + \mu(y^2 - x^2)$.

Let us suppose that $G(F+2K) < 0$, $\mu > 0$; then div $X_{\varepsilon,\mu,0} = 0$ is a hyperbola (see Figure 2).



$\varepsilon(F+2K) > 0$ $\varepsilon(F+2K) < 0$

Figure 2

In this case, the fine foci $C^{\pm}$ have been hyperbolised and change stability and we obtain two repelling hyperbolic limit cycles, one around C^+ and the other around C^-.

If we take $\lambda > 0$, small enough $(\lambda - \mu(2b/a(a+b)) < 0)$, the origin becomes a repelling hyperbolic focus, then we have an attracting hyperbolic limit cycle surrounding the origin.

In relation to infinity, if we denote by $\tilde{\alpha}_i^{\infty}(-2\pi)$ the Poincaré quantities for the field $X_{\varepsilon,\mu,\lambda}$ and by $R_3(\mu)$, and $A_3(\mu)$ the radial and angular components of its cubic part, we have

$$A_3(\mu) = -\mu xy(x^2+y^2) - \frac{a+b}{2ab}(ax^2+by^2)^2$$
$$R_3(\mu) = R_3(0) = \frac{b^2-a^2}{2ab}xy(ax^2+by^2)$$

Then as

$$\int_0^{-2\pi} \frac{R_3(0)}{A_3(0)} d\theta = 0$$

we have

$$\int_0^{-2\pi} -\frac{R_3(\mu)}{A_3(\mu)} d\theta = \int_0^{-2\pi} -\frac{R_3(\mu)}{A_3(\mu)} d\theta + \int_0^{-2\pi} \frac{R_3(0)}{A_3(0)} d\theta$$
$$= \mu \frac{b^2 - a^2}{2ab} \int_{-2\pi}^{0} f(\theta) d\theta \quad \text{with } f(\theta) > 0$$

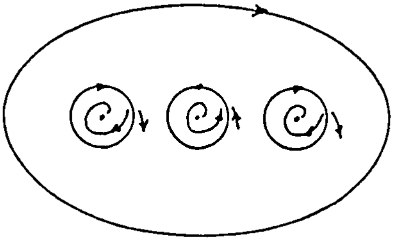
that is,

$$\text{sign} \int_0^{-2\pi} -\frac{R_3(\mu)}{A_3(\mu)} d\theta = \text{sign } \mu.$$

Thus, as we choose $\mu > 0$, we have that $\widetilde{\alpha}_1^{\infty}(-2\pi) = \exp\left(-\int_0^{-2\pi} R_3(\mu)/A_3(\mu)\, d\theta\right)$ > 1 and then we have changed the stability of the infinity and an attracting hyperbolic limit cycle is generated.

For $G(F+2K) > 0$, $\mu < 0$, the same configuration is obtained but with the natural change in the stability of the critical elements.

The configuration of limit cycles is shown in Figure 3.



$G(F+2K) < 0, \qquad \mu > 0, \qquad \lambda > 0$

Figure 3

5. Rupture of separatrices

To obtain the global picture we have to study the rupture of the saddle connections of $X_{\varepsilon,0,0}$.

We use the technique of the Melnikov's Integral [**1**].

First, we consider a general situation for $X(x, y) = (P(x, y), Q(x, y))$ and $Z(x, y) = (Z_1(x, y), Z_2(x, y))$.

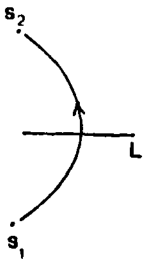


Figure 4

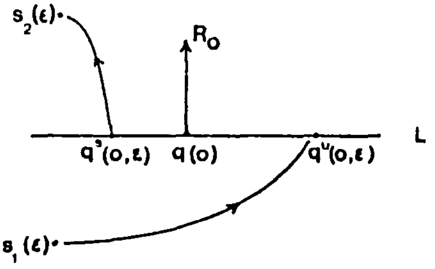


Figure 5

Let us assume that there is a saddle connection γ for the vector field X, such that $\alpha(\gamma) = s_1$ and $\omega(\gamma) = s_2$, (Figure 4).

We study the rupture of γ when X is perturbed to $X + \varepsilon Z$.

Let us consider $p \in \gamma$, and $q(t)$ the orbit of X such that $q(0) = p$ and $R_0 = X(p)$.

Let L be a transversal segment in $q(0)$ to R_0. Let us denote by $q^s(t, \varepsilon)$ and $q^u(t, \varepsilon)$ the orbits of $X + \varepsilon Z$ that parametrise the stable and unstable manifold of the hyperbolic saddles $s_2(\varepsilon)$ and $s_1(\varepsilon)$ respectively (see Figure 5).

The ε-deviation of Melnikov in $q(0)$ is the number

$$d(\varepsilon) = \det(R_0, q^u(0, \varepsilon) - q^s(0, \varepsilon)).$$

We also have that

$$d'(0) = \int_{-\infty}^{\infty} \exp\left[-\int_0^t \operatorname{div} X(q(\tau))d\tau\right] \det[X(q(t)), Z(q(t))]dt$$

(see [1]).

In our case, X is a Hamiltonian vector field, so

$$\begin{aligned} d'(0) &= \int_{-\infty}^{\infty} \det[X(q(t)), Z(q(t))]dt \\ &= \int_\gamma Z_2(x, y)dx - Z_1(x, y)dy. \end{aligned}$$

Let $x = \varphi(s)$, $y = \phi(s)$ with $a < s < b$ be another parametrisation of γ.

Then

$$d'(0) = \int_a^b [Z_2(x(s), y(s))\varphi'(s) - Z_1(x(s), y(s))\phi'(s)]ds.$$

Let γ_i, $i = 1, 2, 3, 4$ be the saddle connection in the phase diagram of X (see Figure 6).

For $i = 1, 2$ we parametrise γ_i by

$$x = (-1)^{i+1}\frac{\cos\theta}{\sqrt{a}} - \sqrt{\frac{b-a}{a(a+b)}}; \qquad y = \frac{\sin\theta}{\sqrt{b}} \quad \text{with} \quad -\alpha < \theta < \alpha$$

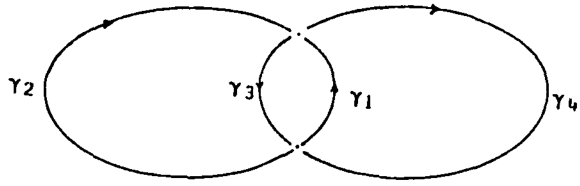


Figure 6

where α is such that $\cos\alpha = (-1)^{i+1}\sqrt{(b-a)/(b+a)}$ and $\sin\alpha = \sqrt{2a/(a+b)}$.

For $i = 3, 4$, γ_i is parametrised as follows:

$$x = (-1)^i \frac{\cos\theta}{\sqrt{a}} + \sqrt{\frac{b-a}{a(a+b)}}; \qquad y = -\frac{\sin\theta}{\sqrt{b}} \quad \text{with} \quad -\alpha < \theta < \alpha$$

where α is such that $\cos\alpha = (-1)^{i+1}\sqrt{(b-a)/(a+b)}$ and $\sin\alpha = \sqrt{2a/(a+b)}$.

Let $d_i(\varepsilon)$ be the ε-deviation associated with γ_i, $i = 1, \dots, 4$.

Then

$$d_i'(0) = -\frac{2}{3}\left(\frac{2a}{b(a+b)}\right)^{3/2} \cdot G \qquad i = 1, 2$$

and

$$d_i'(0) = \frac{2}{3}\left(\frac{2a}{b(a+b)}\right)^{3/2} \cdot G \qquad i = 3, 4.$$

If we take $G > 0$ the ruptures of the saddle separatrices are as shown in Figure 7.

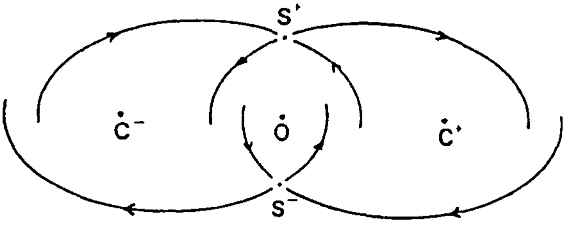


Figure 7

If we apply the method of Melnikov to study the generation of limit cycles by the rupture of the graphic, we find that the criterion fails in this case. So it remains as an open problem to decide the existence of more limit cycles in the global phase portrait of the family.

However, the possible distribution of limit cycles, besides $C_5^1 \supset 3C_1^1$, using Jibin notation [8], are $C_5^1 \supset 2C_1^1 \cup C_1^2$, $C_5^1 \supset 2C_1^2 \cup C_1^1$ and $C_5^1 \supset 3C_1^2$.

https://doi.org/10.1017/S0004972700029531 Published online by Cambridge University Press

References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, ***Theory of bifurcations of dynamic systems on a plane*** (Israel Program for Scientific Translations, Jerusalem, 1971).

[2] R. Bamón, 'Quadratic vector fields in the plane have a finite number of limit cycles', *I.H.E.S.* **64** (1987), 111–142.

[3] T.R. Blows and N.G. Lloyd, 'The number of limit cycles of certain polynomial differential equations', *Proc. Roy. Soc. Edinburgh Sect.* **A 98** (1984), 215–239.

[4] W.A. Coppel, 'A survey of quadratic systems', *J. Differential Equations* **2** (1966), 293–304.

[5] C. Chicone and Tian Jinghuang, 'On general properties of quadratic systems', *Amer. Math. Monthly* **89** (1982), 167–178.

[6] P. Holmes and D. Rand, 'Phase portraits and bifurcations of the nonlinear oscillator $\ddot{x} + \left(x + \gamma x^2\right)\dot{x} + \beta x + \delta x^3 = 0$', *Internat. J. Non-Linear Mech.* **15** (1980), 449–458.

[7] Li Jibin, 'Distribution of limit cycles of the planar cubic system.', *Sci. Sinica Ser. A* **28** (1985), 36–46.

[8] Li Jibin and Q-M Huang, 'Bifurcations of limit cycles forming compound eyes in the cubic system (Hilbert number $H_3 \geqslant 11$)', *J. Yunnan University* **1** (1985), 7–16.

[9] N.G. Lloyd, 'Limit cycles of polynomial systems, some recent developments', in *New direction in dynamical systems*, Editors T. Bedford and T. Swift (London Math. Soc. Lecture Note Ser. 127, 1988).

[10] N.G. Lloyd, T.R. Blows and M.C. Kalenge, 'Some cubic systems with several limit cycles', *Nonlinearity* **1** (1988), 653–669.

[11] C. Rousseau, 'Bifurcations of limit cycles at infinity in polynomial vector fields', (Preprint Université de Montréal 1986).

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