



Zeros in the character tables of symmetric groups with an ℓ -core index

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Abstract. Let $\mathcal{C}_n = [\chi_\lambda(\mu)]_{\lambda, \mu}$ be the character table for S_n , where the indices λ and μ run over the $p(n)$ many integer partitions of n . In this note, we study $Z_\ell(n)$, the number of zero entries $\chi_\lambda(\mu)$ in \mathcal{C}_n , where λ is an ℓ -core partition of n . For every prime $\ell \geq 5$, we prove an asymptotic formula of the form

$$Z_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell) p(n) \gg_\ell n^{\frac{\ell-3}{2}} e^{\pi\sqrt{2n/3}},$$

where $\sigma_\ell(n)$ is a twisted Legendre symbol divisor function, $\delta_\ell := (\ell^2 - 1)/24$, and $1/\alpha_\ell > 0$ is a normalization of the Dirichlet L -value $L\left(\left(\frac{\cdot}{\ell}\right), \frac{\ell-1}{2}\right)$. For primes ℓ and $n > \ell^6/24$, we show that $\chi_\lambda(\mu) = 0$ whenever λ and μ are both ℓ -cores. Furthermore, if $Z_\ell^*(n)$ is the number of zero entries indexed by two ℓ -cores, then, for $\ell \geq 5$, we obtain the asymptotic

$$Z_\ell^*(n) \sim \alpha_\ell^2 \cdot \sigma_\ell(n + \delta_\ell)^2 \gg_\ell n^{\ell-3}.$$

1 Introduction and statement of results

Let $\mathcal{C}_n = [\chi_\lambda(\mu)]_{\lambda, \mu}$ be the usual character table (see, for example, [6, 13, 14]) for the symmetric group S_n , where the indices λ and μ both vary over the $p(n)$ many integer partitions of n . Confirming conjectures of Miller [9], Peluse and Soundararajan [11, 12] recently proved that if ℓ is prime, then almost all of the $p(n)^2$ entries in \mathcal{C}_n , as $n \rightarrow +\infty$, are multiples of ℓ . We note that Miller conjectured that the same conclusion holds for arbitrary prime powers, a claim which remains open.

In recent papers [8, 9], Miller raised the problem of determining the limiting behavior of $Z(n)$, the number of zero entries in \mathcal{C}_n . Despite the remarkable theorem of Peluse and Soundararajan, little is known. Moreover, due to the rapid growth of $p(n)$, it is computationally infeasible to compute many values of $Z(n)$. Consequently, there are no conjectures that are supported with substantial numerics. For example, is there a limiting proportion for the zeros in \mathcal{C}_n ? Such a proportion would be given

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7	5	4	3	1
5	3	2	1	
1				

Figure 1: Hook lengths for $\lambda = (5, 4, 1)$.

by the limit

$$\lim_{n \rightarrow +\infty} \frac{Z(n)}{p(n)^2}.$$

Limited numerics suggest that such a limit might exist, and might be ≈ 0.36 (see Table 3 of [9]). However, this is a dubious guess at best. What is more, the simpler problem of determining whether $\liminf_{n \rightarrow +\infty} Z(n)/p(n)^2 > 0$ also seems to be out of reach. In view of these difficulties, McKay [5] posed a less ambitious problem; he asked for lower bounds arising from ℓ -cores that illustrate the rapid growth of $Z(n)$. Here, we answer this question, and for primes $\ell \geq 5$, we obtain asymptotic formulas for

$$(1.1) \quad Z_\ell(n) := \#\{(\lambda, \mu) : \chi_\lambda(\mu) = 0 \text{ with } \lambda \text{ an } \ell\text{-core}\}.$$

To this end, suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ is a partition of n . As is typical in the representation theory of symmetric groups, we make use of ℓ -core partitions, which are defined using *Young diagrams* of partitions, the left-justified arrays of cells where the row lengths are the parts. The *hook* for the cell in position (k, j) is the set of cells below or to the right of that cell, including the cell itself, and so its *hook length* $h_\lambda(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1$. Here, λ'_j is the number of boxes in the j th column of the diagram. We say that λ is an ℓ -core partition if none of its hook lengths are multiples of ℓ . If $c_\ell(n)$ denotes the number of ℓ -core partitions of n , then we have (see, for example, [2, 7]) the generating function

$$\sum_{n=0}^\infty c_\ell(n)q^n = \prod_{n=1}^\infty \frac{(1 - q^{\ell n})^\ell}{(1 - q^n)}.$$

Example The Young diagram of the partition $\lambda := (5, 4, 1)$, where each cell is labeled with its hook length, is given in Figure 1.

By inspection, we see that λ is an ℓ -core for every prime $\ell > 7$.

To state the asymptotics formulas, we let $L\left(\left(\frac{\cdot}{\ell}\right), s\right)$ be the Dirichlet L -function for the Legendre symbol $\left(\frac{\cdot}{\ell}\right)$, and let

$$(1.2) \quad \alpha_\ell := \frac{(2\pi)^{\frac{\ell-1}{2}}}{\left(\frac{\ell-3}{2}\right)! \cdot \ell^{\frac{\ell}{2}} \cdot L\left(\left(\frac{\cdot}{\ell}\right), \frac{\ell-1}{2}\right)}.$$

By the functional equations of these Dirichlet L -functions and the theory of generalized Bernoulli numbers, we have that $1/\alpha_\ell$ is always a positive integer (see page 339 of [3]). For example, we have $1/\alpha_5 = 1, 1/\alpha_7 = 8, 1/\alpha_{11} = 1, 275$, and $1/\alpha_{13} = 33, 463$. In

addition, we require the integers $\delta_\ell := (\ell^2 - 1)/24$, and the *twisted Legendre symbol divisor functions*

$$(1.3) \quad \sigma_\ell(n) := \sum_{1 \leq d|n} \left(\frac{n/d}{\ell} \right) d^{\frac{\ell-3}{2}}.$$

In terms of these quantities and functions, we obtain the following asymptotics for $Z_\ell(n)$.

Theorem 1.1 *If $\ell \geq 5$ is prime, then, as $n \rightarrow +\infty$, we have*

$$Z_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell) p(n) \gg_\ell n^{\frac{\ell-5}{2}} e^{\pi\sqrt{2n/3}}.$$

Remark Apart from a density zero subset, we have that $Z_\ell(n) = 0$ when $\ell \in \{2, 3\}$ (see [3]).

As a corollary, we find that $Z(n)/p(n)$ grows faster than any power of n .

Corollary 1.2 *If $d > 0$, then*

$$\lim_{n \rightarrow +\infty} \frac{Z(n)}{p(n) \cdot n^d} = +\infty.$$

Remark We note that Corollary 1.2 is weaker than

$$Z(n)/p(n)^2 \gg 1/\log n,$$

which can be found in the discussion after Lemma 2.3 of Peluse’s paper [11].

We turn to the problem of describing the zero entries in \mathcal{C}_n where both indices λ and μ are ℓ -core partitions. For prime ℓ and large n , the entries in \mathcal{C}_n indexed by ℓ -core pairs (λ, μ) always have $\chi_\lambda(\mu) = 0$.

Theorem 1.3 *Suppose that ℓ is prime, and let $N_\ell := (\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell)/24$. If $n > N_\ell$ and $\lambda, \mu \vdash n$ are ℓ -core partitions, then $\chi_\lambda(\mu) = 0$.*

If $Z_\ell^*(n)$ denotes the number of vanishing entries $\chi_\lambda(\mu) = 0$ indexed by ℓ -core partitions $\lambda, \mu \vdash n$, then we have the following corollary.

Corollary 1.4 *For primes ℓ , the following are true.*

- (1) *Apart from a density zero subset, we have that $Z_\ell^*(n) = 0$ when $\ell \in \{2, 3\}$.*
- (2) *If $\ell \geq 5$, then, as $n \rightarrow +\infty$, we have*

$$Z_\ell^*(n) \sim \alpha_\ell^2 \cdot \sigma_\ell(n + \delta_\ell)^2 \gg_\ell n^{\ell-3}.$$

To obtain these results, we use the well-known vanishing result that follows from the Murnaghan–Nakayama rule and says that $\chi_\lambda(\mu) = 0$ whenever μ has a part that is not the length of any hook in λ . Therefore, our goal is reduced to counting pairs of partitions (λ, μ) of large n , where μ has a part that is a multiple of ℓ , and where λ is

an ℓ -core. Theorem 1.1 is obtained by estimating these counts using asymptotics and lower bounds for various partition functions due to Hardy and Ramanujan, Hagis, and Granville and the second author.

Theorem 1.3 concerns the cases where (λ, μ) are both ℓ -cores, and is a consequence of the fact (see Theorem 3.1) that every large ℓ -core has a part that is a multiple of ℓ . This fact is proved using the “abacus theory” of ℓ -cores, and is a generalization of Section 3 of [10] by Sze and the second author in the case of four-core partitions. Corollary 1.4 then follows from the asymptotics for $c_\ell(n)$ due to Granville and the second author.

This paper is organized as follows. Section 2 recalls the well-known vanishing result and bounds, as well as the asymptotics and estimates for the relevant partition functions. Section 3 gives the abacus theory of ℓ -cores and the statement and proof of Theorem 3.1. In Section 4, we employ these results to prove Theorems 1.1 and 1.3 and Corollaries 1.2 and 1.4.

2 Nuts and bolts

In this section, we recall essential facts that we require for the proofs of our results. We first state a criterion that guarantees the vanishing of character values, and then we give estimates for the relevant partition functions.

2.1 Criterion for the vanishing of $\chi_\lambda(\mu)$

Here, we recall a standard partition theoretic criterion that guarantees the vanishing of a character value $\chi_\lambda(\mu)$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_s)$ and $\mu = (\mu_1, \dots, \mu_t)$ are partitions of size n , and let $\{h_\lambda(i, j)\}$ be the multiset of hook lengths for λ . Thanks to the Murnaghan–Nakayama formula (see, for example, Theorem 2.4.7 of [6]), we have that $\chi_\lambda(\mu) = 0$ when $\{\mu_i\}$ is not a subset of $\{h_\lambda(i, j)\}$.

Given a prime ℓ , this immediately gives natural families of vanishing character table entries indexed by pairs of partitions (λ, μ) of n , where μ has a part that is a multiple of ℓ , and λ is an ℓ -core partition. To make use of this observation, we recall that a partition μ is A -regular if none of its parts μ_i are multiples of A . If $p_A(n)$ denotes the number of A -regular partitions of n , then one easily confirms the generating function

$$\sum_{n=0}^{\infty} p_A(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{An})}{(1 - q^n)} = \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \dots + q^{(A-1)n}),$$

which shows that $p_A(n)$ is also the number of partitions of n where the parts appear at most $A - 1$ times. In terms of $p(n)$, $p_\ell(n)$, and $c_\ell(n)$, we have the following lower bounds for $Z(n)$.

Lemma 2.1 *If ℓ is prime, then the following are true.*

(1) *If $\mu \vdash n$ is not an ℓ -regular partition and $\lambda \vdash n$ is an ℓ -core partition, then $\chi_\lambda(\mu) = 0$.*

(2) *If n is a positive integer, then we have*

$$Z_\ell(n) \geq (p(n) - p_\ell(n)) \cdot c_\ell(n).$$

Proof (1) By hypothesis, μ is not ℓ -regular, meaning that it has a part that is a multiple of ℓ . As λ is an ℓ -core, none of its hook lengths are multiples of ℓ . Therefore, $\chi_\lambda(\mu) = 0$ by Murnaghan–Nakayama.

(2) The number of partitions of n that are not ℓ -regular is $p(n) - p_\ell(n)$. Therefore, (1) gives the conclusion that

$$Z_\ell(n) \geq (p(n) - p_\ell(n))c_\ell(n). \quad \blacksquare$$

2.2 Estimates for some partition functions

Here, we recall asymptotics and lower bounds for the partition functions we require to prove Theorem 1.1 and Corollary 1.4. First, we have the celebrated Hardy–Ramanujan asymptotic for $p(n)$.

Theorem 2.2 As $n \rightarrow +\infty$, we have

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot \exp(\pi\sqrt{2n/3}).$$

Hagis obtained asymptotics for $p_A(n)$, the number of A -regular partitions of n . Letting $t = A - 1$ in Corollary 4.2 of [4], we have the following asymptotic formula.

Theorem 2.3 If $A \geq 2$, then we have

$$p_A(n) = C_A(24n - 1 + A)^{-\frac{3}{4}} \exp\left(C\sqrt{\frac{A-1}{A}\left(n + \frac{A-1}{24}\right)}\right) \left(1 + O(n^{-\frac{1}{2}})\right),$$

where $C := \pi\sqrt{2/3}$ and $C_A := \sqrt{12}A^{-\frac{3}{4}}(A-1)^{\frac{1}{4}}$.

Finally, we recall facts about $c_t(n)$, the number of t -core partitions of n that were obtained by Granville and the second author in [3]. In terms of α_ℓ defined in (1.2), and the twisted Legendre symbol divisor functions $\sigma_\ell(n)$ defined in (1.3), we have the following theorem.

Theorem 2.4 The following are true.

(1) We have that

$$c_2(n) = \begin{cases} 1, & \text{if } n \text{ is a triangular number,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If n is a nonnegative integer, then

$$c_3(n) = \sum_{d|(3n+1)} \left(\frac{d}{3}\right).$$

In particular, $c_3(n) = 0$ for almost all n .

(3) If $t \geq 4$ and n is a nonnegative integer, then $c_t(n) > 0$.

(4) If n is a nonnegative integer, then $c_5(n) = \sigma_5(n + 1)$.

(5) If $\ell \geq 7$ is prime, then, as $n \rightarrow +\infty$, we have

$$c_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell).$$

(6) If $\ell \geq 11$ is prime and n is sufficiently large, then we have

$$c_\ell(n) > \frac{2\alpha_\ell}{5} \cdot n^{\frac{\ell-3}{2}}.$$

Proof Claim (1) is a straightforward observation. Claims (2), (4), and (5) are proved on pages 339 and 340 of [3]. Claim (3) is Theorem 1 of [3], whereas (6) is Theorem 4 of [3]. ■

3 Abaci and large ℓ -core partitions

Throughout this section, suppose that ℓ is prime. The main result here is the following theorem, which shows that every sufficiently large ℓ -core partition has a part that is a multiple of ℓ .

Theorem 3.1 *Suppose that ℓ is prime, and let $N_\ell := (\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell)/24$. If $n > N_\ell$, then every ℓ -core partition of size n has a part that is a multiple of ℓ .*

Remark We note that $N_\ell < \ell^6/24$ is not optimal. Indeed, if we let N_ℓ^{\max} be the largest n admitting an ℓ -regular ℓ -core partition, then it turns out that $N_3^{\max} = 10$ and $N_5 = 16$.

3.1 Abaci theory

We make use of the theory of *abaci* for partitions (see, for example, [1, 6]). In particular, let $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ be a partition of n . For each $1 \leq i \leq s$, define the *i th structure number* $B_i := \lambda_i - i + s$, so that $B_i = h_\lambda(i, 1)$, the hook length of cell $(i, 1)$.

Using these structure numbers, we represent the partition λ as an ℓ -abacus \mathfrak{A}_λ , consisting of beads placed on rods numbered $0, 1, \dots, \ell - 1$. For each B_i , there is a unique pair of integers (r_i, c_i) for which $B_i = \ell(r_i - 1) + c_i$ and $0 \leq c_i \leq \ell - 1$. The abacus \mathfrak{A}_λ then consists of s beads, where for each i , one places a bead in position (r_i, c_i) .

Lemma 3.2 [6, Lemma 2.7.13] *Assuming the notation above, λ is an ℓ -core if and only if all of the beads in \mathfrak{A}_λ lie at the top of their respective rods without gaps.*

In view of this lemma, we may represent an abacus of an ℓ -core partition by ℓ -tuples of nonnegative integers, say $(b_0, \dots, b_{\ell-1})$, where b_i denotes the number of beads in column i . However, such representations are not unique as they generally allow for parts of size zero. We have the following elementary lemma.

Lemma 3.3 [10, Lemma 1] *The following abaci both represent the same ℓ -core partition:*

$$(b_0, b_1, \dots, b_{\ell-1}) \quad \text{and} \quad (b_{\ell-1} + 1, b_0, b_1, \dots, b_{\ell-2}).$$

By repeatedly applying this lemma, we may canonically define the unique abacus representation for an ℓ -core to be the one with zero beads in the first column. Thus, when we talk about *the* abacus representation of an ℓ -core λ , we will always mean the abacus of the form $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$.¹ Using these abaci, we offer the following lemma that will allow us to rule out the existence of partitions that are simultaneously ℓ -core and ℓ -regular for all but finitely many n .

Lemma 3.4 *Suppose that $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$ is the abacus corresponding to an ℓ -core λ , and suppose that there is an integer $k \geq 0$ such that, for each $1 \leq i \leq \ell - 1$, we have either $b_i \leq k$ or $b_i \geq k + \ell$. If there is at least one j for which $b_j \geq k + \ell$, then λ is not an ℓ -regular partition.*

Remark Let $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$ be the abacus of an ℓ -core λ . If $\min(b_1, \dots, b_{\ell-1}) \geq \ell$, then the proof of the lemma will show that λ has a part of exact size ℓ . These are the cases where one can choose $k = 0$ in the lemma.

Proof By our hypothesis, we may fix j for which $b_j \geq k + \ell$. Let δ denote the total number of columns with length at least $k + \ell$. Note that if B_i and $B_{i'}$ are structure numbers corresponding to consecutive beads in column j between rows $k + 1$ and $k + \ell$, then $|i - i'| = \delta$. Furthermore, we have $|B_i - B_{i'}| = \ell$. Generalizing the observation that $B_{i-1} - B_i = \lambda_{i-1} - \lambda_i + 1$, we have

$$|\lambda_i - \lambda_{i'}| = |B_i - B_{i'}| - \delta = \ell - \delta.$$

In particular, the difference between parts corresponding to consecutive beads in column j between rows $k + 1$ and $k + \ell$ is fixed and coprime to ℓ . As a consequence, these parts form a modulus $\ell - \delta$ arithmetic progression consisting of ℓ values. Thus, the parts cover all residue classes modulo ℓ , and so includes a part that is a multiple of ℓ . ■

Example Let $\ell = 3$, and consider the three-core abacus $(0, 4, 1)$ as shown below.

1	·	○	○
2	·	○	·
3	·	○	·
4	·	○	·

We illustrate Lemma 3.4 with $k = 1$. Since $b_1 = 3 + 1 = 4$ and $b_2 = 1$, the lemma asserts that λ has a part that is a multiple of 3. The structure numbers are found to be $B_1 = 10, B_2 = 7, B_3 = 4, B_4 = 2$, and $B_5 = 1$, and we compute that $\lambda_1 = 10 + 1 - 5 = 6, \lambda_2 = 4, \lambda_3 = 2$, and $\lambda_4 = \lambda_5 = 1$. In particular, λ_1 is a multiple of 3.

Finally, consider the abacus with the bead in row 4 removed. One easily checks that the corresponding partition is 3-regular, demonstrating that the condition on the size of the gap in column lengths cannot be relaxed.

¹These abaci correspond to those representations of λ without parts of size 0.

With two more observations, we will be able to construct an abacus which gives an upper bound for the size of an ℓ -regular ℓ -core partition.

First, suppose that λ and λ' are ℓ -cores with $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$ and $\mathfrak{A}_{\lambda'} = (0, b'_1, \dots, b'_{\ell-1})$. Then we say $\mathfrak{A}_\lambda \leq \mathfrak{A}_{\lambda'}$ if $b_i \leq b'_i$ for all $1 \leq i \leq \ell - 1$. This relation endows the set of ℓ -core abaci with the structure of a directed partially ordered set. It is not hard to show that $\mathfrak{A}_\lambda \leq \mathfrak{A}_{\lambda'}$ implies $|\lambda| \leq |\lambda'|$.

For the purpose of obtaining N_ℓ , the following lemma allows us to restrict our attention to those abaci where the b_i are weakly increasing.

Lemma 3.5 *Suppose that $\lambda = (\lambda_1, \dots, \lambda_s)$ is an ℓ -core partition of n with abacus $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$. If there exist $1 \leq i < j \leq \ell - 1$ for which $b_j < b_i$, then the abacus \mathfrak{A}' obtained by swapping b_i and b_j represents an ℓ -core partition λ' with $\lambda' \vdash n' > n$.*

Proof We may write

$$n = \sum_{k=1}^s \lambda_k = \sum_{k=1}^s (B_k + k - s) = \sum_{k=1}^s B_k + \sum_{k=1}^s (k - s),$$

and likewise $n' = \sum_{k=1}^s B'_k + \sum_{k=1}^s (k - s)$, where s remains the same because we have not changed the total number of beads. Since the second sum is the same in both expressions, it suffices to prove that $\sum_{i=1}^s B_k < \sum_{i=1}^s B'_k$. Computing columnwise, we have

$$\begin{aligned} \sum_{k=1}^s B_k &= \sum_{m=1}^{b_i} (3(m-1) + i) + \sum_{m=1}^{b_j} (3(m-1) + j) + \sum_{\substack{k=1 \\ k \neq i, j}}^{\ell-1} \sum_{m=1}^{b_k} (3(m-1) + k) \\ &< \sum_{m=1}^{b_j} (3(m-1) + j) + \sum_{m=1}^{b_i} (3(m-1) + i) + \sum_{\substack{k=1 \\ k \neq i, j}}^{\ell-1} \sum_{m=1}^{b_k} (3(m-1) + k) \\ &= \sum_{k=1}^s B'_k, \end{aligned}$$

as desired, where the inequality holds since $i < j$ and $b_i > b_j$. ■

3.2 Proof of Theorem 3.1

We aim to find an upper bound on n such that $\lambda \vdash n$ can be an ℓ -regular ℓ -core partition. To do this, we will construct a partition Λ such that λ being an ℓ -regular ℓ -core implies $|\lambda| \leq |\Lambda|$. By Lemma 3.5, it suffices to restrict our attention to those ℓ -cores whose abaci have weakly increasing column lengths. Suppose that λ is a weakly increasing ℓ -regular ℓ -core with abacus $\mathfrak{A}_\lambda = (0, b_1, \dots, b_{\ell-1})$. By Lemma 3.4, we must have $\min(b_1, \dots, b_{\ell-1}) = b_1 \leq \ell - 1$. By the same logic, we must have $b_i \leq i(\ell - 1)$, for all $1 \leq i \leq \ell - 1$.

Then, if Λ is the partition with abacus $\mathfrak{A}_\Lambda = (0, \ell - 1, 2(\ell - 1), \dots, (\ell - 1)^2)$, we immediately have $\mathfrak{A}_\lambda \leq \mathfrak{A}_\Lambda$, which implies that $|\lambda| \leq |\Lambda|$. Then $N_\ell := |\Lambda|$ gives an upper

bound on n . By direct calculation, we find that

$$|\Lambda| = \sum_{i=1}^s (B_i + i - s) = \sum_{i=1}^{\ell-1} \sum_{j=1}^{i(\ell-1)} (\ell(j-1) + i) + \sum_{i=1}^s (i - s) = \frac{\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell}{24},$$

where $s = \ell(\ell - 1)^2/2$, giving the desired conclusion.

4 Proofs of our results

We are now in a position to prove Theorems 1.1 and 1.3 and Corollaries 1.2 and 1.4.

Proof of Theorem 1.1 We note that Theorems 2.2 and 2.3 imply that

$$\lim_{n \rightarrow +\infty} \frac{p(n) - p_\ell(n)}{p(n)} = 1.$$

The claim now follows by combining Lemma 2.1(2) and Theorems 2.2 and 2.4 (4–6). ■

Proof of Corollary 1.2 This claim follows from Theorem 1.1 by choosing primes $\ell \rightarrow +\infty$. ■

Proof of Theorem 1.3 By Theorem 3.1, every ℓ -core partition of size $n > N_\ell$ has a part that is a multiple of ℓ . Since every hook length of an ℓ -core is not a multiple of ℓ , it follows from Murnaghan–Nakayama that whenever $\lambda, \mu \vdash n$ are ℓ -cores with $n > N_\ell$, we have $\chi_\lambda(\mu) = 0$. ■

Proof of Corollary 1.4 Thanks to Theorem 1.3, we have that $Z_\ell^*(n) = c_\ell(n)^2$ for a sufficiently large n . The claimed asymptotics and inequalities follow from Theorem 2.4 (4–6). ■

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