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ON THE GROWTH OF COMPOSITIONS OF LINEAR AND MEROMORPHIC FUNCTIONS

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Let $f(z)$ be a meromorphic function; we shall investigate the asymptotic behaviour of the ratio $T(r,\, f(z+\alpha))/T(r,\, f(z))$ and $T(r,\, f(\alpha z))/T(r,\, f(z))$, and discuss the growth of the meromorphic solutions of some functional equations.

1. Introduction and main results

We shall adopt the fundamental concepts and basic notation of Nevanlinna's theory in this paper. Let $f(z)$ be a meromorphic function and $T(r,\, f(z))$ be its Nevanlinna characteristic function. We denote the order and the lower order of $f(z)$ by ρ_f and μ_f respectively in the sense of Nevanlinna. In addition, we put

$$\widehat{\rho}_f = \limsup_{r\to\infty} \log\log T(r,\, f(z))/\log r \text{ and } \widehat{\mu}_f = \liminf_{r\to\infty} \log\log T(r,\, f(z))/\log r.$$

$\widehat{\rho}_f$ and $\widehat{\mu}_f$ are said to be the hyperorder and lower hyperorder of $f(z)$ respectively.

It is obvious that $\widehat{\rho}_f > 0$ (or $\widehat{\mu}_f > 0$) implies that $\rho_f = \infty$ (or $\mu_f = \infty$). Yang [1] proposed the following open problems:

Problem A: ([1], p.168). Let $f(z)$ be a meromorphic function and

$$\lim_{r\to\infty} T(r,\, f(z+1))/T(r,\, f(z)) = \infty. \tag{1}$$

Can we conclude that $\mu_f = \infty$?

Problem B: ([1], p.251). Let f_1, f_2, g_1 and g_2 be entire functions. Suppose that

$$T(r,\, f_1) \sim T(r,\, f_2), \quad T(r,\, g_1) \sim T(r,\, g_2), \quad (r \to \infty).$$

Can we conclude that

$$T(r,\, f_1(g_1)) \sim T(r,\, f_2(g_2)), \quad (r \to \infty)?$$

We shall give the answers to these two problems in this paper. Firstly, we have the following result

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Theorem 1. *Let $f(z)$ be a meromorphic function such that $\widehat{\mu}_f < 1$, A_j $(j = 1, 2, \ldots, m)$ and $B_j\,(j = 1, 2, \ldots, n)$ be positive real numbers, and $\alpha_j\,(j = 1, 2, \ldots, m)$ and $\beta_j\,(j = 1, 2, \ldots, n)$ be complex numbers. Then*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{m} A_j T(r, f(z+\alpha_j))}{\sum_{j=1}^{n} B_j T(r, f(z+\beta_j))} \leqslant \frac{\sum_{j=1}^{m} A_j}{\sum_{j=1}^{n} B_j} \leqslant \limsup_{r\to\infty} \frac{\sum_{j=1}^{m} A_j T(r, f(z+\alpha_j))}{\sum_{j=1}^{n} B_j T(r, f(z+\beta_j))}. \tag{2}$$

Remark. (a) If (1) holds, then (2) does not hold for $m = n = 1$, $A_1 = B_1 = 1$, $\alpha_1 = 1$ and $\beta_1 = 0$. By Theorem 1 we have that $\widehat{\mu}_f \geqslant 1$; thus $\mu_f = \infty$. This gives an affirmative answer to the Problem A.

(b) For $f(z) = e^{e^z}$, we can verify that $\widehat{\mu}_f = 1$. It is easily seen that (2) does not hold for $m = n = 1$, $A_1 = B_1 = 1$, $\alpha_1 = 1$ and $\beta_1 = 0$. Therefore the conditions of Theorem 1 cannot be weakened.

Next we consider the asymptotic behaviour of the ratio $T(z, f(\alpha z))/T(r, f(\beta z))$, and have

Theorem 2. *Let $f(z)$ be a meromorphic function, α and β be two complex constants satisfying $|\alpha| > |\beta| > 0$. Then*

$$\liminf_{r\to\infty} \frac{T(r, f(\alpha z))}{T(r, f(\beta z))} \leqslant \left|\frac{\alpha}{\beta}\right|^{\mu_f} \leqslant \left|\frac{\alpha}{\beta}\right|^{\rho_f} \leqslant \limsup_{r\to\infty} \frac{T(r, f(\alpha z))}{T(r, f(\beta z))}. \tag{3}$$

Remark. (a) Let f_1, f_2 be meromorphic functions, $\rho_{f_1} > 0$ and

$$T(r, f_1) \sim T(r, f_2), \qquad (r \to \infty).$$

Put $g_1(z) = cz$ and $g_2(z) = z$; here c is a complex constant and $|c| > 1$. It is obvious that

$$T(r, g_1) \sim T(r, g_2), \qquad (r \to \infty).$$

But by Theorem 2 we have

$$\limsup_{r\to\infty} \frac{T(r, f_1(g_1))}{T(r, f_2(g_2))} = \limsup_{r\to\infty} \left[\frac{T(r, f_1(cz))}{T(r, f_1(z))} \cdot \frac{T(r, f_1(z))}{T(r, f_2(z))}\right] \geqslant |c|^{\rho_{f_1}} > 1.$$

Thus we give a negative answer to Problem B.

(b) Choose $f(z) = e^z$; then $\rho_f = \mu_f = 1$. It is easily to verify that

$$\lim_{r\to\infty} T(r, f(\alpha z))/T(r, f(\beta z)) = \left|\frac{\alpha}{\beta}\right|.$$

Thus the result (3) of Theorem 2 is sharp.

Considering the functional equation

$$\alpha_n f(z+n) + \alpha_{n-1} f(z+n-1) + \cdots + \alpha_1 f(z+1) = R(f(z)), \tag{4}$$

in which $R(w) = P(w)/Q(w)$, $P(w) = a_p w^p + \cdots + a_1 w + a_0$ and $Q(w) = b_q w^q + \cdots + b_1 w + b_0$ are supposed to be mutually prime, $\alpha_n, \ldots, \alpha_1$; $a_p, \ldots, a_1, a_0; b_q, \ldots, b_1, b_0$ are constants, and $\alpha_n a_p b_q \neq 0$, Yanagihara proved

Theorem A. [4] *Suppose* $\max(p, q) \geqslant n+1$. *Then any non-constant meromorphic solution* $f(z)$ *of (4) is of order* $\rho_f = \infty$.

In this paper we generalise the above Theorem A to the following Theorem 3. Here we consider the functional equation

$$\sum_{i=1}^{n} R_{i1}(z, f(z+\alpha_{i1})) R_{i2}(z, f(z+\alpha_{i2})) \ldots R_{ik}(z, f(z+\alpha_{ik})) = R_0(z, f(z)), \tag{5}$$

where α_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, k)$ are constants, $R_{ij}(z, w)$ $(i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, k)$ and $R_0(z, w)$ are rational functions of the form $R(z, w) = P(z, w)/Q(z, w)$ $P(z, w) = \sum_{j=1}^{p} a_j(z) w^j$, $Q(z, w) = \sum_{j=1}^{q} b_j(z) w^j$, in which $a_j(z)$ $(j = 0, 1, 2, \ldots, p)$ and $b_j(z)$ $(j = 0, 1, 2, \ldots, q)$ are polynomials. $P(z, w)$ and $Q(z, w)$ are supposed to be mutually prime. Denote $\partial R = \max(p, q)$. We have the following result.

Theorem 3. *Suppose* $\partial R_0 \geqslant \sum_{i=1}^{n} \sum_{j=1}^{k} \partial R_{ij} + 1$. *Then any transcendental meromorphic solution* $f(z)$ *of (5) satisfies* $\widehat{\mu}_f \geqslant 1$.

Considering the equation of Schröder,

$$f(cz) = Q(f(z)), \tag{6}$$

in which $Q(z)$ is a polynomial of degree n and c is a constant satisfying $|c| > 1$, Shimomura proved

Theorem B. [3] *Suppose* $f(z)$ *is a non-constant entire solution of (6); then the order* $\rho_f = \log n / \log |c|$.

Theorem C. [3] *If* $|c| < 1$, *then (6) has no non-constant entire solution.*

In this paper we generalise the above theorems to the following Theorem 4. Here we consider the functional equation

$$R_1(z, f(cz)) = R_2(z, f(z)), \tag{7}$$

in which $R_j(z, w)$ $(j = 1, 2)$ have the same form as the above $R(z, w)$, and c is a constant satisfying $|c| > 1$. We have the following result:

Theorem 4. *(a) If $\partial R_2 \geqslant \partial R_1$, and $f(z)$ is a transcendental meromorphic solution of (7), then*

$$\rho_f = \mu_f = \log \frac{\partial R_2}{\partial R_1} \Big/ \log|c|.$$

(b) If $\partial R_2 < \partial H_1$, then (7) has no transcendental meromorphic solution.

2. Proof of Theorem 1

Firstly, without loss of generality, we suppose $|\alpha_1| = \max(|\alpha_1|, |\alpha_2|, \dots |\alpha_m|)$, $|\beta_1| = \max(|\beta_1|, |\beta_q|, \dots, |\beta_n|)$ and put $t = |\alpha_1| + |\beta_1|$. If $t = 0$, then (2) obviously holds.

Below, we suppose $t > 0$ and put

$$\Omega = \liminf_{r\to\infty} \sum_{j=1}^{m} A_j T(r, f(z+\alpha_j)) \Big/ \sum_{j=1}^{n} B_j T(r, f(z+\beta_j)),$$

(Ω is finite or infinite). If $\Omega = 0$, then $\Omega < \sum_{j=1}^{m} A_j \Big/ \sum_{j=1}^{n} B_j$ holds. Next, we suppose $\Omega > 0$. Thus for any positive number $\sigma < \Omega$, there exists $r_1 > 0$ such that

$$\sum_{j=1}^{m} A_j T(r, f(z+\alpha_j)) > \sigma \sum_{j=1}^{n} B_j T(r, f(z+\beta_j)), \tag{8}$$

when $r \geqslant r_1$. We choose a number a which is not a Valiron deficient value of $f(z+\alpha_j)$ $(j = 1, 2, \dots, m)$, $f(z+\beta_j)$ $(j = 1, 2, \dots, n)$ and $f(z)$. Therefore for any $\varepsilon > 0$, there exists $r_2 > 0$ such that the following four inequalities hold when $r \geqslant r_2$.

$$T(r, f(z+\alpha_j)) \leqslant (1+\varepsilon) N(r, f(z+\alpha_j) = a), \ (j = 1, 2, \dots, m); \tag{9}$$

$$T(r, f(z+\beta_j)) \geqslant (1-\varepsilon) N(r, f(z+\beta_j) = a), \ (j = 1, 2, \dots, n); \tag{10}$$

$$N(r+|\alpha_1|, f(z) = a) \leqslant (1+\varepsilon) T(r+|\alpha_1|, f(z)); \tag{11}$$

$$N(r-|\beta_1|, f(z) = a) \geqslant (1-\varepsilon) T(r-|\beta_1|, f(z)). \tag{12}$$

It is obvious that

$$N(r, f(z+\alpha_j) = a) \leqslant N(r+|\alpha_j|, f(z) = a) \leqslant N(r+|\alpha_1|, f(z) = a).$$

Hence it follows from (9), (11) and the above inequality that

$$T(r, f(z+\alpha_j)) \leqslant (1+\varepsilon)^2 T(r+|\alpha_1|, f(z)), \quad (j = 1, 2, \dots, m) \tag{13}$$

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when $r \geqslant r_2$. We can also have

$$N(r, f(z+\beta_j)=a) \geqslant N(r-|\beta_j|, f(z)=a) \geqslant N(r-|\beta_1|, f(z)=a).$$

Hence it follows from (10), (12) and the above inequality that

$$T(r, f(z+\beta_j)) \geqslant (1-\varepsilon)^2 T(r-|\beta_1|, f(z)), \quad (j=1, 2, \dots, n) \tag{14}$$

when $r \geqslant r_2$. So (8), (13) and (14) yield that

$$T(r+|\alpha_1|, f(z)) > A\, T(r-|\beta_1|, f(z)),$$

when $r \geqslant \max(r_1, r_2)$; here $A = \sigma((1-\varepsilon)/(1+\varepsilon))^2 \sum_{j=1}^{n} B_j / \sum_{j=1}^{m} A_j$. Put $r_0 = \max(r_1, r_2) + |\beta_1|$. It follows that

$$T(r+t, f(z)) > A\, T(r, f(z)), \tag{15}$$

when $r \geqslant r_0$.

Suppose $\Omega > \sum_{j=1}^{m} A_j \Big/ \sum_{j=1}^{m} B_j$ Then we can choose suitable σ and ε such that $A > 1$. From (15) we easily deduce

$$T(r_0+kt, f(z)) > A^k T(r_0, f(z)),$$

in which k is any natural number. For an arbitrarily real number $r \geqslant r_0$, we assume $r \in [r_0+kt, r_0+(k+1)t)$ and obtain that

$$T(r, f(z)) \geqslant T(r_0+kt, f(z)) > A^k T(r_0, f(z)) > A^{(r-r_0-t)/t} T(r_0, f(z)).$$

It follows that $\widehat{\mu}_f \geqslant 1$. This is a contradiction. Therefore $\Omega \leqslant \sum_{j=1}^{m} A_j \Big/ \sum_{j=1}^{n} B_j$. The following inequality is thus proved.

$$\liminf_{r\to\infty} \sum_{j=1}^{m} A_j T(r, f(z+\alpha_j)) \Big/ \sum_{j=1}^{n} B_j T(r, f(z+\beta_j)) \leqslant \sum_{j=1}^{m} A_j \Big/ \sum_{j=1}^{n} B_j. \tag{16}$$

By the same method we can also prove

$$\liminf_{r\to\infty} \sum_{j=1}^{n} B_j T(r, f(z+\beta_j)) \Big/ \sum_{j=1}^{m} A_j T(r, f(z+\alpha_j)) \leqslant \sum_{j=1}^{n} B_j \Big/ \sum_{j=1}^{m} A_j.$$

This implies that

$$\limsup_{r\to\infty} \sum_{j=1}^{m} A_j T(r, f(z+\alpha_j)) \Big/ \sum_{j=1}^{n} B_j T(r, f(z+\beta_j)) \geqslant \sum_{j=1}^{m} A_j \Big/ \sum_{j=1}^{n} B_j.$$

which, together with (16), proves (2). The proof of Theorem 1 is complete □

3. Proof of Theorem 2

At first we put

$$\Omega = \liminf_{r\to\infty} T(r,\, f(\alpha z))/T(r,\, f(\beta z));$$

(Ω is finite or infinite). If $\Omega \leqslant 1$, it is obvious that $\Omega < \left|\frac{\alpha}{\beta}\right|^{u_f}$. Below, we suppose $\Omega > 1$. Then for any positive number $\sigma < \Omega$, there exists $r_1 > 0$ such that

$$T(r,\, f(\alpha z)) > \sigma T(r,\, f(\beta z)), \tag{17}$$

when $r \geqslant r_1$. We choose a number a which is not a Valiron deficient value of $f(\alpha z)$ and $f(\beta z)$. Thus for any $\varepsilon > 0$, there exists $r_2 > 0$ such that the following inequality holds when $r \geqslant r_2$.

$$\begin{aligned} T(t,\, f(\alpha z)) &\leqslant (1+\varepsilon)N(r,\, f(\alpha z) = a) = (1+\varepsilon)N\left(\left|\frac{\alpha}{\beta}\right| r,\, f(\beta z) = a\right) \\ &\leqslant (1+\varepsilon)^2 T\left(\left|\frac{\alpha}{\beta}\right| r,\, f(\beta z)\right). \end{aligned} \tag{18}$$

Put $t = |\alpha/\beta| > 1$ and $r_0 = \max(r_1, r_2)$. It follows from (17) and (18) that

$$T(tr,\, f(\beta z)) > A\,T(r,\, f(\beta z)), \tag{19}$$

when $r \geqslant r_0$; here $A = \sigma/(1+\varepsilon)^2$. Since $\Omega > 1$, we can choose suitable σ and ε such that $A > 1$. Hence (19) implies

$$T\left(r_0 t^k,\, f(\beta z)\right) > A^k T(r_0,\, f(\beta z)),$$

in which k is any natural number. For an arbitrarily real number $r \geqslant r_0$, we assume $r \in \left[t^k r_0,\, t^{k+1} r_0\right)$. By the same method as in the proof of Theorem 1 we can deduce $\mu_f \geqslant \log A/\log t$. Making $\varepsilon \to 0$ and $\sigma \to \Omega$, we obtain $\mu_f \geqslant \log \Omega/\log t$, that is,

$$\liminf_{r\to\infty} T(r,\, f(\alpha z))/T(r,\, f(\beta z)) \leqslant \left|\frac{\alpha}{\beta}\right|^{\mu_f}.$$

By a similar method we can also prove that

$$\limsup_{r\to\infty} T(r,\, f(\alpha z))/T(r,\, f(\beta z)) \geqslant \left|\frac{\alpha}{\beta}\right|^{\rho_f}.$$

Theorem 2 is thus proved. □

4. Proof of Theorem 3 and Theorem 4

In order to prove Theorem 3 and Theorem 4, we need the following

Lemma. [2] *Suppose $R(z, w)$ is defined as before, and $f(z)$ is a meromorphic function. Then*

$$T(r, R(z, f(z))) = \partial R \cdot T(r, f(z)) + O(\log r).$$

Proof of Theorem 3: Firstly, the following inequality follows from (5) and the above lemma.

$$\partial R_0 T(r, f(z)) \leqslant \sum_{i=1}^{n} \sum_{j=1}^{k} \partial R_{ij} T(r, f(z + \alpha_{ij})) + O(\log r). \tag{20}$$

If $\widehat{\mu}_f < 1$, since $f(z)$ is transcendental, then we deduce from (20) and Theorem 1 that $\partial R_0 \leqslant \sum_{i=1}^{n} \sum_{j=1}^{k} \partial R_{ij}$. This is a contradiction. Theorem 3 is thus proved. □

Proof of Theorem 4: (a) By the above lemma and (7) we have

$$\partial R_1 T(r, f(cz)) = \partial R_2 T(r, f(z)) + O(\log r). \tag{21}$$

It follows from Theorem 2 and (21) that $|c|^{\rho_f} = |c|^{\mu_f} = \partial R_2 / \partial R_1$, that is, $\rho_f = \mu_f = \log(\partial R_2 / \partial R_1) / \log|c|$.

(b) Suppose (7) has a transcendental meromorphic solution $f(z)$; then by the above lemma and (7) we can deduce (21). By Theorem 2 we have $\partial R_2 \geqslant |c|^{\rho_f} \partial R_1 \geqslant \partial R_1$. This is a contradiction. Theorem 4 is thus proved. □

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