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FINITE GROUPS ADMITTING AN AUTOMORPHISM TRIVIAL ON A SYLOW 2-SUBGROUP

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In this paper we shall consider finite groups satisfying the following hypothesis.

HYPOTHESIS I. *Let G be a finite group which admits an automorphism σ of prime order p, $(p, |G|) = 1$. Assume the fixed point subgroup $B = C_G(\sigma)$ contains some Sylow 2-subgroup.*

Let $G(q)$ be a finite simple group of Lie type defined over the finite field $GF(q)$, q odd. Let p be an odd prime with $(p, |G(q)|) = 1$. With the exception of the groups ${}^3D_4(q)$, $|G(q)| = q^m \prod_i (q^{a_i} - 1) \prod_j (q^{b_j} + 1)/d$, $d = (c, q^v - 1)$ or $(c, q^v + 1)$. The integers a_i, b_j, c, v, d are independent of q and depend only on the rank and family of the group [**5**]. By matching terms it is seen that $[G(q^p) : G(q)]$ is the product of an odd integer and integer factors $((q^p)^a - 1)/(q^a - 1)$ and $((q^p)^b + 1)/(q^b + 1)$. As $((q^a)^p - 1)/(q^a - 1) = (q^a)^{p-1} + (q^a)^{p-2} + \ldots + (q^a) + 1$ is a sum of p odd integers, $((q^p)^a - 1)/(q^a - 1))$ is odd. Similarly $((q^p)^b + 1)/(q^b + 1)$ is odd and we conclude $[G(q^p) : G(q)]$ must be odd. Moreover, $(q^p)^a \pm 1 \equiv q^a \pm 1 \pmod p$ so $(p, |G(q)|) = 1$ if and only if $(p, |G(q^p)|) = 1$. Let σ be a field automorphism of $G(q^p)$ of order p. Then $C_G(\sigma) \cong G(q)$ and we conclude that $G(q^p)$ is a finite simple group satisfying Hypothesis I. A similar argument shows the groups ${}^3D_4(q^p)$ satisfy Hypothesis I.

The above remarks illustrate that the simple Lie groups $G(q^p)$, q odd, satisfy Hypothesis I. Our first step toward obtaining the converse of this statement is the following result proved in Section 1.

THEOREM 1. *Let G be a finite simple group satisfying Hypothesis I. Then $G \cong L_2(q^p)$, q odd, or G is of component type.*

The classification of groups of component type has been the object of a considerable amount of research in the last few years. (See Section 1 for definitions and notation.) Recent progress suggests that work has nearly been completed in classifying all simple groups with an involution t for which $O(C_G(t)) \neq 1$. This classification, now called the *Unbalanced Group Conjecture*, is stated in Section 1. Under the assumption of the Unbalanced Group Conjecture, simple groups with a component of type $G(q)$, q odd, have been determined. We shall show that the following conjecture can be proved from the Unbalanced Group Conjecture and the classification of groups of component type $G(q)$, q odd.

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CONJECTURE. *Let G be a finite simple group satisfying Hypothesis I. Then G is a Chevalley group over $GF(q^p)$, q odd.*

This proof as well as the proof of Theorem 1 uses several results of G. Glauberman. Because $(|G|, p) = 1$, every non-empty subset $Q \subseteq B = C_G(\sigma)$ has $N_G(Q) = N_B(Q)C_G(Q)$. This factorization is used in conjunction with the main theorem of [**2**] which states that if G contains an involution t whose centralizer is contained in B, then G has a proper normal subgroup of odd order.

In Section 2 of this paper, we shall determine the structure of finite groups which satisfy Hypothesis I with $B = C_G(\sigma)$ solvable. The following statement is the main result.

THEOREM 2. *Let G be a finite group satisfying Hypothesis I. Assume $B = C_G(\sigma)$ is solvable. Then one of the following occurs:*

i) *G is solvable with $G = O_{2'}(G)B$.*

ii) *G contains characteristic subgroups G_1, G_2 such that $G_1 \trianglelefteq G_2 \trianglelefteq G$ with G_1, G/G_2 solvable and $G_2/G_1 \cong L_1 \times \ldots \times L_n$, $L_i \cong L_2(3^p)$, $1 \leqq i \leqq n$.*

COROLLARY 2. *Let G be a finite simple group satisfying the hypothesis of Theorem 2. Then $G \cong L_2(3^p)$.*

It is seen that Corollary 2 is an immediate consequence of Theorem 2. Moreover, the argument of Section 2 shows G_1 to be the largest normal solvable subgroup of G while G_2 is the preimage in G of the largest normal semisimple subgroup of G/G_1.

1. Groups of component type. We recall some notation and terminology from [**1**] and [**3**]. A group A is *quasisimple* if A is its own commutator group and, modulo its center, A is simple. A *component* of a group is a subnormal quasisimple subgroup. The *core* of a group is its largest normal subgroup of odd order. A 2-*component* of a group is a subnormal subgroup A such that A is its own commutator subgroup and A is quasisimple modulo its core. G is of *component type* if the centralizer in G of some involution contains a 2-component. This is equivalent to requiring that the centralizer is not 2-constrained (see 2.11, [**7**]).

For any group H, we let $\tilde{E}(H)$ be the inverse image in H of the socle of $C_H(F(H))/Z(F(H))$, where $F(H)$ is the Fitting subgroup of H. We then define $E(H)$ to be the last term of the derived series of $\tilde{E}(H)$, and put $F^*(H) = E(H)F(H)$. Lemma (2.1) in [**3**] shows $E(H)$ to be the central product of uniquely determined quasisimple groups, which are called the *components* of $E(H)$ and are permuted under conjugation by H. Moreover, the components of $E(H)$ are exactly the set of all subnormal quasisimple subgroups of H. See Section 2 of [**3**] for further properties of $E(H)$ and $F^*(H)$.

The first result of this section characterizes $L_2(q^p)$, q odd, $(p, |L_2(q)|) = 1$, as the only family of simple groups satisfying Hypothesis I and not of component type.

(1.1) *Let G be a finite simple group which satisfies Hypothesis I. If the centralizer of each involution of G is 2-constrained, then* $G \cong L_2(q^p)$, *q odd.*

Proof. Let us first assume G has 2-rank at least 3. Let $S \in \mathrm{Syl}_2(B)$ and notice for any involution $t \in S$, $C_G(t)$ is σ-invariant. The coprime action of σ on $C_G(t)$ produces $T \in \mathrm{Syl}_2(C_G(t))$ with T σ-invariant and Lemma 6 of [**2**] shows T to be contained in some σ-invariant conjugate of S. Because σ-invariant Sylow 2-subgroups of G are conjugate by an element of B (Lemma 5, [**2**]), we conclude that σ acts trivially on T. By assumption G has 2-rank at least 3 so that [**4**] implies $O_{2'}(C_G(t)) = 1$ provided $\mathrm{SCN}_3(2) \neq \emptyset$. A simple group with $SCN_3(2) \neq \emptyset$, 2-rank at least 3, and 2-constrained centralizers of involutions is isomorphic to $G_2(3)$ or the sporadic group J_3 (see [**6**, Corollary A]). The group J_3 does not satisfy Hypothesis I so we may assume $O_{2'}(C(t)) = 1$ in any case. Set $X = C_G(t)$ and $Q = O_2(X)$. Lemma 5 in [**2**] and the fact that σ acts trivially on Q imply $X = C_X(Q)N_{B \cap X}(Q)$. Then, as X is 2-constrained, $C_X(Q) \subseteq Q$ and we have $X = N_{B \cap X}(Q) \subseteq B$. Theorem 1 in [**2**] shows G to have a normal subgroup N which does not contain t. This contradicts the simplicity of G.

We may now assume G has 2-rank at most 2. A result of Brauer and Suzuki implies G cannot have rank 1. Hence G is a simple group of 2-rank 2. Corollary A in [**6**] shows G is isomorphic to one of the groups $L_2(q)$, $L_3(q)$, $U_3(q)$, q odd, $U_3(4)$, A_7 or M_{11}. The last three groups do not satisfy Hypothesis I and among the groups $L_3(q)$, $U_3(q)$, only $L_3(3)$, $U_3(3)$ have 2-constrained centralizers. As $L_3(3)$, $U_3(3)$ do not admit an automorphism satisfying Hypothesis I, $G \cong L_2(r)$, r odd. The structure of $P\Gamma L(2, r)$ forces the existence of a field automorphism σ of G of order p. We conclude that $G \cong L_2(q^p)$, q odd.

Theorem 1. *Let G be a finite simple group satisfying Hypothesis I. Then either* $G \cong L_2(q^p)$, *q odd or G is of component type.*

Proof. By (1.1), we may assume G contains an involution t such that $C_G(t)$ is not 2-constrained. Corollary 2.11 in [**7**] implies $C_G(t)$ contains a 2-component and we conclude that G is of component type.

At this point we prove the conjecture as stated in the introduction. We first state the Unbalanced Group Conjecture and the relevant theorem for groups of component type.

Unbalanced Group Conjecture. *Let G be a finite group with* $F^*(G) = L$ *simple and* $O(C_G(t)) \neq 1$ *for some involution t in G. Then one of the following holds:*

(1) *L is a Chevalley group of odd characteristic,*
(2) *L is an alternating group of odd degree, or*
(3) *L is isomorphic to* $L_3(4)$ *or Held's group.*

Component Theorem (Aschbacher-Walter). *Let G be a finite group with* $F^*(G) = L$ *simple containing an involution t such that* $C_G(t)$ *has a component A* *with $A/Z(A)$ a Chevalley group over $GF(r^p)$ where r is an odd prime power and $p \geqq 3$. Then L is a Chevalley group over $GF(q)$ for some odd q.*

The authors wish to point out that the Unbalanced Group Conjecture has been established modulo successful completion of Harris's work on groups with an $L_2(q)$ component. The Component Theorem with certain modifications has been announced by John Walter. Aschbacher has distributed a preprint of his part of the work.

CONJECTURE. *Let G be a simple group which satisfies Hypothesis I. Then G is a Chevalley group over $GF(q^p)$, q odd.*

Proof. Let G be a minimal counterexample. Choose t to be an involution of G. If $O(C_G(t)) \neq 1$, the Unbalanced Group Conjecture implies that G is a Chevalley group over $GF(r)$, r odd, A_{2n+1}, $L_3(4)$ or Held's group. The groups A_{2n+1}, $L_3(4)$ and Held's group admit no automorphism of order p with $(p, |G|) = 1$. Hence G is a Chevalley group over $GF(r)$, r odd. Since $(p, |G|) = 1$, σ must be a field automorphism so $r = q^p$, q odd. This contradicts our choice of G.

We therefore have that $O(C_G(t)) = 1$ for every involution t of G. Let $X = C_G(t)$ and suppose X is 2-constrained for some involution t. Then $X = N_X(O_2(X)) = C_X(O_2(X))N_{B \cap X}(O_2(X)) \leqq B$, against [**2**]. Therefore $E(X) \neq 1$. Let $L_1, \ldots, L_n$ be the components of $E = E(X)$. Because σ is trivial on a Sylow 2-subgroup of G, σ leaves each L_i invariant.

Suppose $E \subseteq B$. Then $X = C_G(t) = N_X(E) = C_X(E)N_{B \cap X}(E) \subseteq O_2(X)EN_{B \cap X}(E) \subseteq B$. By [**2**], this is impossible. Hence σ is non-trivial on some L_i. Furthermore, L_i is perfect so σ is non-trivial on $L_i/Z(L_i)$. By induction, $L_i/Z(L_i)$ is a Chevalley group over $GF(r^p)$, r odd.The Component Theorem implies G is a Chevalley group over $GF(q_1)$ for some odd q_1. However, $(p, |G|) = 1$ so σ must be a field automorphism and $q_1 = q^p$, q odd. This contradicts our choice of G and the conjecture follows.

2. Groups with $B = C_G(\sigma)$ solvable. In this section we shall determine the structure of finite groups satisfying Hypothesis I with $B = C_G(\sigma)$ solvable. We prove the following main result.

THEOREM 2. *Let G be a finite group satisfying Hypothesis I. Assume $B = C_G(\sigma)$ is solvable. Then one of the following occurs:*

i) *G is solvable with $G = O_{2'}(G)B$.*

ii) *G contains characteristic subgroups G_1, G_2 such that $G_1 \trianglelefteq G_2 \trianglelefteq G$ with G_1, G/G_2 solvable and $G_2/G_1 \cong L_1 \times \ldots \times L_n$, $L_i \cong L_2(3^p)$, $1 \leqq i \leqq n$.*

Let G be a finite group satisfying the hypothesis of Theorem 2. If G is solvable, set $Q = O_{2'2}(G)$ and choose $S \in \mathrm{Syl}_2(B)$. Then $T = S \cap Q$ is a Sylow 2-subgroup of Q and $G = QN_G(T)$ by a Frattini argument. Lemma 5 in [**2**] and the fact that σ acts trivially on T imply $N_G(T) = C_G(T)N_B(T)$. As G is solvable, $C_G(T) \subseteq Q$ so $G = QN_B(T) \subseteq O_{2'}(G)B$. Hence $G = O_{2'}(G)B$ and (i) of Theorem 2 is established.

We may now assume G is nonsolvable and set $G_1 = S(G)$, the largest normal solvable subgroup of G. Let $\bar{G} = G/G_1$. Then σ induces an automorphism of $\bar{G}$ with $C_{\bar{G}}(\sigma) = \bar{B}$ where $\bar{B}$ denotes the image of B in $\bar{G}$. (See Lemma 3, [**2**].) If $\bar{G} = \bar{B}$, G is solvable. We conclude that $\bar{B}$ is a proper subgroup of $\bar{G}$ and $\bar{G}$ is a nonsolvable group satisfying the hypothesis of Theorem 2 with $S(\bar{G}) = 1$. Now proving Theorem 2 for $\bar{G}$ is equivalent to proving the theorem for G, so we may assume for the remainder of this section that G satisfies

HYPOTHESIS II. *Let G be a finite nonsolvable group satisfying the hypothesis of Theorem 2 with $S(G) = 1$.*

First we recall some definitions from [**3**]. Suppose $A \leqq T \leqq X$ are groups such that whenever $a \in A$, $x \in X$, and $a^x \in T$, then $a^x \in A$. In this situation we say A is *strongly closed* in T with respect to X.

The next series of propositions establish the existence of a strongly closed Abelian 2-subgroup A of G.

(2.1) *Let G satisfy Hypothesis I and choose $S \in \mathrm{Syl}_2(B)$. If $S_1 \leqq S$ is strongly closed in S with respect to B, then S_1 is strongly closed in S with respect to G.*

Proof. Suppose S_1 is strongly closed in S with respect to B and $s^g \in S$ for some $s \in S_1$, $g \in G$. Lemma 5 of [**2**] implies the existence of $b \in B$ such that $s^g = s^b$. By assumption $s^b \in S_1$ so $s^g \in S_1$ as desired.

(2.2) *Let G satisfy Hypothesis I and suppose $H \trianglelefteq B$, $S \in \mathrm{Syl}_2(B)$. Then $S \cap H$ is strongly closed in S with respect to G.*

Proof. Set $S_1 = S \cap H$ and suppose $s^b \in S$ for some $s \in S_1$, $b \in B$. Because $H \trianglelefteq B$, $s^b \in S_1$ and we conclude S_1 is strongly closed in S with respect to B. By (2.1), S_1 is strongly closed in S with respect to G.

(2.3) *Let G satisfy Hypothesis I. Then there exists $S \in \mathrm{Syl}_2(B)$ and an Abelian 2-subgroup $A \subseteq S$, $A \neq 1$, such that A is strongly closed in S with respect to G.*

Proof. By hypothesis, B is solvable so that $B/O(B)$, has a minimal normal elementary Abelian 2-subgroup. Let T be an elementary Abelian 2-group of B so that $TO(B)$ is the preimage of this subgroup in B. Then $TO(B) \triangleleft B$ and, by (2.2), $S \cap (TO(B)) = A \neq 1$ is a strongly closed Abelian 2-subgroup of S with respect to G.

Let $S \in \mathrm{Syl}_2(B)$ and $A \subseteq S$ be a strongly closed Abelian 2-group of S. By (2.3), $A \neq 1$. Set $K = \langle A^G \rangle$. Theorem A in [**6**] implies $K/O(K)$ is the central product of an Abelian 2-group and certain quasisimple groups. Because $S(G) = 1$, $O_{2',2}(K) = 1$ and $K = L_1 \times \ldots \times L_m$ where L_i is simple, $1 \leqq i \leqq m$. According to [**6**], L_i may be of Type I or II. A group of Type I is isomorphic to one of the groups $L_2(2^n)$, $n \geqq 3$, $Sz(2^{2n+1})$, $n \geqq 1$ or $U_3(2^n)$, $n \geqq 2$. Groups of

Type II are $L_2(q)$, $q \equiv 3, 5 \pmod 8$ or simple groups of Janko-Ree type. The automorphism σ centralizes $S \cap K \in \mathrm{Syl}_2(K)$ and thus must centralize a Sylow 2-subgroup of each L_i. Each group of Type I is a C-group so that centralizers of involutions are solvable and consequently 2-constrained. The argument (1.1) of Section 1 may then be used to show that these simple groups cannot admit an automorphism σ satisfying Hypothesis I. The solvability of B implies σ must act faithfully on L_i and consequently L_i must be a group of Type II. We are now able to prove the following:

(2.4) *Let G satisfy Hypothesis II. Then G contains a normal subgroup K such that $K = L_1 \times \ldots \times L_m$, $L_i \cong L_2(3^p)$, $1 \leqq i \leqq m$.*

Proof. The remarks preceding (2.4) show the existence of $K \trianglelefteq G$ such that $K = L_1 \times \ldots \times L_m$, L_i simple of type $L_2(q)$, $q \equiv 3, 5 \pmod 8$ or isomorphic to a simple group of Janko-Ree type.

We first show no factor L_i of K is of Janko-Ree type. Suppose L is a simple group of Janko-Ree type admitting an automorphism σ satisfying Hypothesis I. Let $T = C_L(\sigma)$ and choose $t \in T$, with t an involution. From [**3**; **9**], $C_L(t) = \langle t \rangle \times F$, $F \cong L_2(3^{2n+1})$, $n \geqq 1$. It follows that σ leaves F invariant with $C_F(\sigma)$ a solvable subgroup containing a Sylow 2-subgroup of F of order 4. The structure of $P\Gamma L(2, 3^{2n+1})$ forces σ to be a field automorphism with $GF(3)$ the fixed field of σ (see [**8**, p. 632]). Hence $F \cong L_2(3^p)$ and $C_F(\sigma) \cong A_4$, the alternating group on four letters. We conclude that $C_T(t) = \langle t \rangle \times D$, $D \cong A_4$. A Sylow 2-subgroup of L is elementary Abelian of order 8 so by a transfer argument all involutions of L are conjugate. Let $R \in \mathrm{Syl}_2(T)$. Lemma 5 in [**2**] shows T controls fusion in R and hence T has no normal subgroup of index 2. Let $\langle a, b \rangle$ be a four-group of R and set $O = O_{2'}(T)$. Then $O = C_O(a)C_O(b)C_O(ab)$. However, for $t \in \langle a, b \rangle^\#$, $C_T(t)$ has no normal subgroup of odd order so $O_{2'}(T) = 1$. The solvability of T implies $R \trianglelefteq T$ so that $[T : C_T(t)] = 7$, $|T| = 2^3 \cdot 3 \cdot 7$. Let $x \in T$ be an element of order 3 fixed by σ. Lemma 6 in [**2**] implies the existence of a σ-invariant Sylow 3-subgroup Q of L containing x. Moreover, [**9**] shows $|Q| = 3^{3p}$. Now σ must act fixed-point freely on the remaining $3^{3p} - 3$ elements of Q so $3^{3p} - 3 \equiv 0 \pmod p$. But, $3^{3p} - 3 \equiv 3^3 - 3 = 24 \pmod p$, a contradiction to our choice of p. We conclude that a group L of Janko-Ree type admits no automorphism σ satisfying Hypothesis I.

We may now conclude that $K = L_1 \times \ldots \times L_m$, $L_i \cong L_2(q)$, $q \equiv 3, 5 \pmod 8$. Because σ fixes a Sylow 2-subgroup of L_i, σ must be a field automorphism with $C_{L_i}(\sigma)$ solvable. Thus $q = 3^p$ and $L_i \cong L_2(3^p)$, $1 \leqq i \leqq m$.

(2.5) *Let G satisfy Hypothesis II. Then $E(G) = L_1 \times \ldots \times L_n$, $L_i \cong L_2(3^p)$, $1 \leqq i \leqq n$. Moreover, $C_G(E(G)) = 1$.*

Proof. We use induction on $|G|$. Set $E = E(G)$. By (2.5), G contains a normal subgroup $K = L_1 \times \ldots \times L_m$, $L_i \cong L_2(3^p)$, $1 \leqq i \leqq m$. Then $K \leqq E$ and $E = KC_E(K)$. If $C_E(K) = 1$, $E = K$ and (2.5) holds. Because G satisfies Hypothesis II, we may assume $C_E(K)$ is a proper nonsolvable σ-invariant subgroup of E. In fact, $C_E(K)$ is the direct product of certain simple components of G so $E(C_E(K)) = C_E(K)$. By induction, $C_E(K)$ is the direct product of copies of $L_2(3^p)$. The first conclusion of (2.5) now follows.

By hypothesis, the Fitting subgroup $F(G) = 1$ so that $F^*(G) = F(G)E(G) = E(G)$. From (2.2) of [**3**], $C_G(E(G)) \subseteq E(G)$. We conclude that $C_G(E(G)) = 1$.

(2.6) *Let G satisfy Hypothesis II. Then $G/E(G)$ is solvable.*

Proof. Set $E = E(G)$. The structure of E is given in (2.5). Let $S \in \mathrm{Syl}_2(B)$. A Sylow argument shows $G = EN_G(S \cap E)$ and Lemma 5 in [**2**] implies $N_G(S \cap E) = C_G(S \cap E)N_B(S \cap E)$. Because $N_B(S \cap E)$ is solvable, it remains to show $C_G(S \cap E)$ is solvable.

Set $X_1 = C_G(S \cap E)$ and assume $X_1 \neq 1$. By (2.5), $C_G(E) = 1$ so X_1 does not centralize each factor of E. Notice X_1 must leave each factor of E invariant so, after a suitable rearrangement of the subscripts on the L_i, we may assume $X_2 = C_{X_1}(L_1)$ is a proper normal subgroup of X_1. Then X_1/X_2 is isomorphic to a group of automorphisms of L_1 which centralizes a Sylow 2-subgroup of L_1. The structure of $P\Gamma L(2, 3^p)$ forces X_1/X_2 to be solvable. A similar argument shows $X_2 = 1$ or X_2 contains a proper normal subgroup X_3 such that X_2/X_3 is solvable. Consequently, X_1 contains a subnormal series $X_1 \trianglerighteq X_2 \trianglerighteq \ldots \trianglerighteq 1$ for which X_i/X_{i+1} is solvable. We conclude $X_1 = C_G(S \cap E)$ is solvable. The result (2.6) now follows.

The proof of Theorem 2 now follows from (2.5), (2.6) and the remarks preceding (2.1). Specifically, let $G_1 = S(G)$ and choose G_2 to be the preimage in G/G_1 of $E(G/G_1)$.

Notice that the groups which satisfy the hypothesis of Theorem 2 may have $E(G) = 1$. For example, let G be isomorphic to the centralizer of a "central" element of order 3 in $PSp_4(3^p)$ where $(p, |PSp_4(3^p)|) = 1$. From [**8**], $G = UL$, $U \cap L = 1$, $|U| = 3^{3p}$, $L \cong SL_2(3^p)$ with $U = O_3(G)$. Then, if we take t to be the central involution of L, $G_1 = U\langle t\rangle$, $G_2 = G$ and because L is not subnormal in G, $E(G) = 1$.

On the other hand, consider $X = \mathrm{Sp}_8(3^p)$ where $(p, |\mathrm{Sp}_8(3)|) = 1$. It is shown in [**10**] that X contains an elementary 2-subgroup D of order 16 generated by symplectic involutions of type 2. Furthermore, $C_X(D) = L_1 \times L_2 \times L_3 \times L_4$, $L_i \cong SL_2(3^p)$ with $N_X(D)/C_X(D) \cong S_4$. Clearly the field automorphism of X of order p induces an automorphism of $N_X(D)$ which satisfies Hypothesis I. Now take H to be any finite solvable group with $(p, |H|) = 1$ and let L be a group isomorphic to $N_X(D)$. Set $G = H \times L$. The automorphism σ of order p which acts trivially on H and acts as a field automorphism of L satisfies the hypothesis of Theorem 2. In fact, if $Q = O_2(L)$, $G_1 = H \times Q$, $G_2 = HC_L(Q)$, and $G/G_2 \cong S_4$. Here $E(G) = C_L(Q) \cong L_1 \times L_2 \times L_3 \times L_4$, $L_i \cong L_2(3^p)$.

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