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ABELIANITY CONJECTURE FOR SPECIAL COMPACT KÄHLER 3-FOLDS

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Dedicated to V. Shokurov

Abstract Using orbifold metrics of the appropriately signed Ricci curvature on orbifolds with a negative or numerically trivial canonical bundle and the two-dimensional log minimal model programme, we prove that the fundamental group of special compact Kähler 3-folds is almost abelian. This property was conjectured in all dimensions by Campana in 2004, and also for orbifolds in 2007, where the notion of specialness was introduced. We briefly recall the definition, basic properties and the role of special manifolds in classification theory.

Keywords: Special compact Kähler 3-folds; fundamental group; two-dimensional special orbifolds; klt singularities

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1. Introduction

We denote here by X an n-dimensional compact connected Kähler manifold and by $\kappa(X)$ its canonical (or Kodaira) dimension. The motivations for the following shortest, but non-transparent, definition of specialness will be explained below. See [**9**], where this notion was introduced, for more details.

Definition 1.1. A compact Kähler manifold X is said to be *special* if, for any $p > 0$, any rank-1 coherent subsheaf $\mathcal{L} \subset \Omega^p_X$ and any positive integer N, the natural meromorphic map $\Phi_{N,\mathcal{L}} \colon X \dashrightarrow \mathbb{P}(V^*_{N,\mathcal{L}})$ has an image of dimension at most $(p-1)$.† Here, $V^*_{N,\mathcal{L}}$ denotes the dual of the complex vector space of sections of $\mathrm{Sym}^N(\Omega^p_X)$ that take values in $\mathcal{L}^{\otimes N} \subset \mathrm{Sym}^N(\Omega^p_X)$ at the generic point of X.

In particular, X then has no surjective meromorphic map $f \colon X \dashrightarrow Y$ onto a manifold Y of general type and dimension $p > 0$, since otherwise $\mathcal{L} := f^*(K_Y) \subset \Omega^p_X$ would contradict the bound $(p-1)$ above. On the non-algebraic side, if X has no map

† By convention, this dimension is $-\infty$ if $V_{N,\mathcal{L}} = \{0\}$, and is always at most equal to p, by a classical result of Bogomolov.

onto a positive-dimensional projective manifold, it is obviously special too. An equivalent geometric definition of specialness of X actually requires that X satisfies the more restrictive condition of having no meromorphic map onto an orbifold of general type (see Definition 3.7 for the precise definition and the relevant notions concerning orbifolds).

Specialness is preserved by bimeromorphic maps and finite étale covers, this last assertion being surprisingly difficult to show. Special manifolds generalize in higher dimensions the rational and elliptic curves, which are obviously exactly the special curves. The next fundamental examples of special manifolds are, indeed, those that are either rationally connected or with zero canonical dimension (i.e. with $\kappa = 0$; see Example 3.8). However, the class of special manifolds is much larger than the union of these two classes, since one shows by classification that a compact Kähler surface X is special if and only if $\kappa(X) \leqslant 1$ and $\pi_1(X)$ is almost abelian. In particular, ruled elliptic surfaces are special and surfaces with $\kappa(X) = 1$ are special if and only if they do not map onto any hyperbolic curve after some finite étale cover. No such simple characterization is true when $n \geqslant 3$.

The central role of special manifolds in classification theory comes from the fact that, as shown in [**9**, 5.8], any compact Kähler manifold is canonically and functorially decomposed by its *core fibration* $c_X \colon X \to C(X)$ into its special part (the fibres of c_X) and its general-type part (the orbifold base $(C(X), \Delta(c_X))$, which is its usual base $C(X)$ together with a ramification divisor $\Delta(c_X)$ on $C(X)$ encoding the multiple fibres of c_X).

Special manifolds and general-type orbifolds are, thus, the *two* antithetical primitive pieces from which arbitrary compact Kähler manifolds are built in *one single step*. In contrast to general-type manifolds, for which no classification scheme seems to be known or even expected, special manifolds are conjectured to have many fundamental properties in common with rational and elliptic curves.

Conjecturally, indeed, an orbifold version of the $C_{n,m}$ conjecture implies that any special manifold is canonically and functorially decomposed, by means of orbifold versions of the rational quotient and of the Iitaka–Moishezon fibration, as a tower of fibrations whose orbifold fibres have either $\kappa = 0$ or $\kappa_+ = -\infty$ (a weak version of rational connectedness; see [**10**]). We stress that the orbifold considerations are essential here (as in the log minimal model programme (LMMP), but for different reasons), and that apparently there is no possibility of working in the category of varieties without additional structure. This tower decomposition allows us to lift (conditionally) to special manifolds (and even, more naturally, to special orbifolds) properties that are expected to be common to manifolds that are either rationally connected or with $\kappa = 0$, and naturally leads to the following conjectures.

Conjecture 1.2 (Campana [9, 10]).

(1) (*Abelianity conjecture.*) *A special compact Kähler manifold has an almost abelian fundamental group (i.e. $\pi_1(X)$ has an abelian subgroup of finite index).*

(2) *A compact Kähler manifold (respectively, a projective manifold defined over a number field) has an identically vanishing Kobayashi pseudometric (respectively, is potentially dense) if and only if it is special (this last statement is inspired by Lang's conjectures).*

For example, rationally connected manifolds are simply connected [**5**], and compact Kähler manifolds X with $c_1(X) = 0$ have an almost abelian fundamental group, by [**2**,**29**].

In this paper we prove the abelianity conjecture above for 3-folds (as stated above, this is known for surfaces by classification; see Proposition 3.9).

Theorem 1.3. *Let X be a compact Kähler 3-fold. If X is special†, its fundamental group is almost abelian.*

This immediately implies, among several other things, a precise solution of Shafarevich's conjecture in this case (and for all special manifolds if the abelianity conjecture holds).

Corollary 1.4. *If X is a special compact Kähler 3-fold with universal cover $\tilde{X}$, let X' be any finite étale cover of X with abelian torsion-free fundamental group and Albanese variety $\mathrm{Alb}(X')$ of dimension $d := q(X')$. Then,*

$$\tilde{X} = X' \times_{\mathrm{Alb}(X')} \widetilde{\mathrm{Alb}(X')}$$

is holomorphically convex, the universal cover $\widetilde{\mathrm{Alb}(X')}$ of $\mathrm{Alb}(X')$ being Stein, since it is isomorphic to $\mathbb{C}^d$.

2. Reduction to the two-dimensional orbifold case

We prove the theorem in this section only in the cases where no orbifold structure is needed, that is, except when either X is projective and $\kappa(X) = 2$, or $a(X) = 2$. The treatment of these two residual cases needs the consideration of two-dimensional projective special orbifolds and is the subject of the subsequent sections.

Because of the lack of a minimal model programme in the Kähler non-projective case, we need to treat it differently from the projective case. So, first assume that X is projective. We work according to the value of $\kappa(X) \leqslant 2$.

$\kappa(X) = -\infty$. By Miyaoka's theorem, X is uniruled. Let $r_X : X \to R(X)$ be its rational quotient (also known as its maximal rationally connected (MRC) fibration). Then, $R(X)$ is also special, with $\dim(R(X)) \leqslant 2$, and $\pi_1(X) \simeq \pi_1(R(X))$, since the fibres of r_X are rationally connected. Since $\pi_1(R(X))$ is almost abelian, so is $\pi_1(X)$.

$\kappa(X) = 0$. If $c_1(X) = 0$, the theorem is true, by [**2**, **29**]. One reduces to this case by the minimal model programme and [**25**] (see [**18**, (4.17.3)]). The details are as follows. There exists a terminal model X' birational to X such that $K_{X'}$ is torsion, hence trivial after finite étale (in codimension 1) cover. Then, X' has only cDV singularities. It is thus smoothable in the projective category, by [**25**]. The conclusion follows, since $\pi_1(X)$ is a quotient of the fundamental group of the generic fibre.

† This paper in fact deals with the (*a priori*) more general case of *classically special* orbifolds defined in [**10**]; see the terminological remark on [**10**, p. 5]. It is not known whether the two notions actually differ.

$\kappa(X) = 1$. Let $J_X\colon X \to Y$ be the Moishezon–Iitaka fibration. Because its base is a curve, there exists a finite étale cover of X (still written X) such that J_X has no multiple fibre.† Then, $\pi_1(X)$ is an extension of $\pi_1(Y)$ (Y being a rational or elliptic curve, since special) by a quotient of $\pi_1(X_y)$, X_y being the generic smooth fibre of J_X, which is a surface with $\kappa = 0$. Because these two groups are almost abelian, $\pi_1(X)$ is *a priori* only polycyclic. The following result, however, implies the conclusion.

Theorem 2.1 (Campana [7]). *Let $f\colon X \to Y$ be a fibration without multiple fibres in codimension 1 on Y from a compact Kähler manifold onto a manifold Y. Assume that Y and the generic fibre X_y of f both have an almost abelian fundamental group. Then, X also has an almost abelian fundamental group.*

The proof of this theorem rests on two deep results of Hodge theory: Deligne's strictness theorem for morphisms of mixed Hodge structures (MHSs), and Hain's functorial MHS on the Malčev completion of $\pi_1(X)$ when X is compact Kähler.

$\kappa(X) = 2$. When X is projective, this is the only remaining case. Note that, when $\kappa(X) = 1$, we can remove the multiple fibres of J_X by making a suitable finite étale cover of X, because Y is a curve. When $\kappa(X) = 2$ this is, in general, no longer possible. This is the reason why the notion of an orbifold base, which virtually removes the multiple fibres, is introduced in this case, and why the geometry of such orbifolds needs to be considered and defined. But once this is done, and the corresponding properties established, the proof is entirely parallel.

We now deal with the case when X is *not* projective. We work this time according to the algebraic dimension $a(X) \in \{0, 1, 2\}$ of X.

$a(X) = 0$. Recall that the Albanese map is surjective and has connected fibres when $a(X) = 0$ (see [**27**, 13.1, 13.6]). The irregularity of X can, thus, only take the values $q(X) = 0, 2$ or 3 (since $q(X) = 1$ would imply that $a(X) \geqslant 1$). The assumption that $a(X) = q(X) = 0$ leads to the finiteness of $\pi_1(X)$ according to [**6**, Corollary 5.7]. Thus, if $q(X) = 3$, X is birational to its Albanese variety, and $\pi_1(X)$ is abelian. When $q(X) = 2$, the Albanese fibration $\alpha_X\colon X \to \mathrm{Alb}(X)$ has no multiple fibre in codimension 1, by [**9**, Proposition 5.3]. Moreover, the general fibre of α_X is elliptic or rational, by [**27**, Theorem 13.8]. From Theorem 2.1, we conclude that $\pi_1(X)$ is almost abelian.

$a(X) = 1$. The algebraic reduction $a_X\colon X \to A(X)$ is then a fibration onto a curve $A(X)$ with general fibre special [**9**, Theorem 2.39]; the fundamental groups of the base and of the fibre are then almost abelian (see Theorem 3.9). As in the projective case with $\kappa(X) = 1$, we may assume that there are no multiple fibres. A final application of Theorem 2.1 then implies the result.

† Or $Y \cong \mathbb{P}^1$ and J_X has one or two multiple fibres. This case is easily treated similarly, and is easier since the orbifold base is now $\mathbb{P}^1$ minus one or two points with finite multiplicities, and, thus, has a finite cyclic orbifold fundamental group. There is not even any need to consider the orbifold structure here, since the fundamental group of $\mathbb{P}^1$ minus two points is $\mathbb{Z}$, hence abelian.

We are thus left with the following two cases:

(1) X is projective and $\kappa(X) = 2$,

(2) X is a compact Kähler 3-fold with $a(X) = 2$.

We briefly explain how the conclusion is then obtained. In both cases we have a fibration (which may be assumed to be *neat*; see Definition 3.5), after suitable modifications of X and S, $f\colon X \to S$ on a smooth projective surface S, with generic fibres elliptic curves. This defines a *smooth*, and hence Kawamata log terminal (klt) (see Example 3.3 and Definition 5.1, respectively, for these notions) orbifold base (S, Δ_f). Because X is special, so is (S, Δ_f). As above, for such a fibration, $\pi_1(X)$ is now an extension of $\pi_1(S, \Delta_f)$ by $\pi_1(X_s)$. We show the following below.

Theorem 2.2. *The fundamental group of a special geometric orbifold* (S, Δ_S) *of dimension* 2 *is almost abelian of even rank at most* 4.

The conclusion now follows from the orbifold version of Theorem 2.1, proved in [**11**, Corollary 7.6].

We stress that our proof of Theorem 2.2 depends in an essential way on the fact that *integral klt* pairs are locally uniformizable by *smooth* germs of surfaces, because we are using the existence of Ricci-flat orbifold metrics when $c_1(S, \Delta) = 0$. This is the main reason why we cannot extend Theorem 2.2 to any higher dimension. Otherwise, assuming the abundance conjecture, it seems that one could derive the abelianity conjecture in the projective case for klt orbifolds in any dimension using inductively on the dimension essentially the same arguments as the ones presented below.

3. Some basic facts on orbifolds

3.1. Notion of orbifold

The orbifolds considered here are spaces with local smooth uniformizations under the action of some finite groups. Isotropy in codimension 1 is expressed by a 'ramification' $\mathbb{Q}$-divisor.

Definition 3.1 (Ghigi and Kollár [14]). An orbifold is a pair (X, Δ) where X is a normal variety and Δ a $\mathbb{Q}$-Weil divisor of the following form:

$$\Delta = \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) \Delta_i,$$

where the $m_i \geqslant 2$ are integers, and the pair (X, Δ) is locally uniformizable in the following sense. X is covered by the domains U of finite maps

$$\varphi\colon U \to X$$

satisfying the following properties:

(i) $\varphi(U)$ is open in X,

(ii) $\varphi\colon U \to \varphi(U)$ is a Galois cover whose branching divisor is exactly $\Delta|_{\varphi(U)}$.

The support of Δ is then $|\Delta| = \sum_i \Delta_i$.

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Terminological remark

The orbifolds we consider are compatible with all situations in which this term is used: they are particular cases of the ones in [**10**], and the special manifolds defined below are actually the *classical* special ones in [**10**]. These *classical* orbifolds are particular instances of the more general versions defined in [**10**]. They are smooth Deligne–Mumford stacks too, and also klt pairs of the LMMP. Because they are locally smoothly uniformized, we can, as usual, attach to them fundamental groups and differential–geometric notions such as metrics and differential forms, which coincide with the notions introduced more generally in [**10**].

Definition 3.2. The canonical divisor of such a pair is defined as the (Weil) $\mathbb{Q}$-divisor $K_X + \Delta$.

Example 3.3. Let X be a smooth variety, and let $|\Delta| = \sum_i \Delta_i$ be a normal crossing divisor; the choice of multiplicities $m_i \geqslant 2$ on each component of the divisor defines a canonical orbifold structure on (X, Δ). Since $|\Delta|$ is locally given by the equation $z_1 \cdots z_k = 0$ in suitable coordinates $(z_1, \dots, z_n)$, the map $(z_1, \dots, z_n) \mapsto (z_1^{m_1}, \dots, z_k^{m_k}, z_{k+1}, \dots, z_n)$ gives a local uniformization. These orbifolds are said to be smooth and integral in [**10**].

In particular, an orbifold curve is simply a smooth curve with a finite set of points marked with integral multiplicities at least 2.

Since we consider only integral and finite multiplicities here, we define the following.

Definition 3.4. A geometric orbifold is an orbifold (X, Δ) with X smooth and $|\Delta|$ of normal crossings.

3.2. The orbifold base of a fibration and special manifolds

Definition 3.5 (Campana [9, Definition 1.2]). Let $f_0 \colon X_0 \to Y_0$ be a fibration (surjective morphism with connected fibres) between compact complex manifolds. A *neat* model of f consists in a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & X_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y & \xrightarrow[v]{} & Y_0 \end{array}$$

where

(1) X and Y are smooth,

(2) u and v are bimeromorphic morphisms,

(3) the locus of singular fibres of f is a normal crossing divisor of Y,

(4) every f-exceptional divisor is u-exceptional too.

Such models actually exist birationally for any fibration; this can be proven using Hironaka desingularization and Raynaud's flattening theorems [**9**, Lemma 1.3]. Note that any fibration is neat when Y is a curve.

Given a neat fibration $f\colon X \to Y$, we can naturally associate a $\mathbb{Q}$-divisor $\Delta^*(f)$ on the base Y of f. This divisor will be supported on the singular locus of f and will thus be a normal crossing: the pair $(Y, \Delta^*(f))$ will be a geometric orbifold.

The construction goes as follows (see also [**9**, 1.1.4]): if Δ_i is any component of the singular locus of f, its pullback can be written as

$$f^*(\Delta_i) = \sum_j m_j D_j + R,$$

where R is f-exceptional and D_j is mapped surjectively onto Δ_i. The multiplicity of f along Δ_i is defined by $m_i = m(f, \Delta_i) = \gcd_j(m_j)$.

Definition 3.6. The pair $(Y, \Delta^*(f))$, where $\Delta^*(f) = \sum_i (1 - 1/m_i)\Delta_i$, is called the *orbifold base* of the fibration f_0 (notation as above).

A fibration f_0 is said to be of general type if the canonical divisor of the orbifold base $(Y, \Delta^*(f))$ is *big*: $\kappa(Y, K_Y + \Delta^*(f)) = \dim(Y) > 0$.

Definition 3.7 (Campana [9, Definition 2.1]). A compact Kähler manifold X is said to be (classically) special if it does not admit any fibration of general type.

Example 3.8. The main examples of special manifolds are given by the following classes [**9**, 3.22, 5.1, 2.39].

- Rationally connected manifolds.
- Compact Kähler manifolds X with $\kappa(X) = 0$. This is a consequence of the additivity of canonical dimensions in general-type fibrations (see [**9**, Theorem 4.2], an orbifold version of Viehweg's theorem).
- The special curves are, thus, just the rational or elliptic ones.
- Compact Kähler manifolds of algebraic dimension 0 and, more generally, fibres of algebraic reductions.

Conjecturally, in an orbifold version of Iitaka's $C_{n,m}$-conjecture, special manifolds can be reconstructed as a tower (in a suitable sense) of fibrations with fibres belonging to the classes above. This reduces the abelianity conjecture, Conjecture 1.2 (1), to the case of orbifolds with either $\kappa = 0$ or $\kappa_+ = -\infty$ (see [**10**, 13.10]).

In dimension 2 only, we still have a simple topological characterization of specialness.

Proposition 3.9 (Campana [9, Proposition 3.32]). *A compact Kähler surface X is special if and only if $\kappa(X) \leqslant 1$ and if $\pi_1(X)$ is almost abelian.*

We also need the notion of the orbifold base of a fibration $f\colon X \to Y$ when X is equipped with an orbifold divisor Δ_X, at least when Y is a curve and X is a surface (for the general case, see [**10**, Definition 4.2]).

Definition 3.10. Let (X,Δ_X) be a geometric orbifold of dimension 2 and let $f\colon X\to C$ be a fibration onto a curve. We define the multiplicity of a point $y\in C$ (relative to f and Δ_X) by the following formula: $m_y(f,\Delta_X):=\gcd_i(m_i\mathrm{mult}_{\Delta_X}(F_i))$, where $f^*(y)=\sum_i m_iF_i$. The orbifold base is then the pair $(C,\Delta^*(f,\Delta_X))$, where

$$\Delta^*(f,\Delta_X)=\sum_y\left(1-\frac{1}{m_y(f,\Delta_X)}\right)\{y\}.$$

We say that f is of general type if $2g(C)-2+\deg(\Delta^*(f,\Delta_X))>0$.

The two-dimensional orbifold (X,Δ_X) is then said to be special if (X,Δ_X) does not admit any general-type fibration onto a curve, and if $\kappa(X,K_X+\Delta_X)<2$.

4. Fundamental groups and fibrations

4.1. An orbifold exact sequence

In this section, we define the fundamental groups of orbifolds and study the morphisms induced at the level of fundamental groups by classical orbifold morphisms.

Definition 4.1. Let (X,Δ) be an orbifold, and let X^* be the smooth locus of X; the fundamental group $\pi_1(X,\Delta)$ is the quotient of the group $\pi_1(X^*\setminus|\Delta|)$ by the normal subgroup generated by the loops $\gamma_j^{m_j}$, where γ_j is a small loop around the component Δ_j of multiplicity m_j.

This definition is derived from the local models. If $X=\mathbb{C}^n/G$, where G is a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ and Δ is the branching divisor of the projection $\pi\colon\mathbb{C}^n\to X$, we recover the group G as an orbifold fundamental group†:

$$\pi_1(X,\Delta)\simeq G.$$

Example 4.2. The case of orbifold curves is quite classical. The structure of the fundamental group of such an orbifold curve $(C,\Delta=(m_1,\dots,m_n))$ (we just keep in mind the deformation invariant, that is, the multiplicities, not the marked points) is determined by the sign of its canonical bundle:

$$\deg(K_C+\Delta)=2g(C)-2+\sum_{j=1}^{n}\left(1-\frac{1}{m_j}\right).$$

When this quantity is positive (respectively, 0; respectively, negative), the fundamental group of (C,Δ) is commensurable to the fundamental group of a hyperbolic curve (respectively, commensurable to $\mathbb{Z}^2$; respectively, finite and explicitly known).

When dealing with fibrations having multiple fibres, we need to consider orbifold fundamental groups. A fibration $f\colon X\to Y$ with general fibre X_y gives rise to a natural sequence of fundamental groups

$$\pi_1(X_y)\xrightarrow{i_*}\pi_1(X)\xrightarrow{f_*}\pi_1(Y)\to 1.$$

† The orbifold divisor determines the structure of the smooth Deligne–Mumford stack $\mathcal{X}$ associated with (X,Δ) and called the *root stack* by Abramovich and Vistoli.

Although f_* is surjective, since f has connected fibres, this sequence is in general not exact in the middle. Multiple fibres are responsible for this failure, remedied by the orbifold fundamental group.

We need the case where X has an orbifold structure†, and thus a slightly more general version. For the definition (of a neat fibration in the orbifold setting), we refer the reader to [**10**, Definition 4.8].

Proposition 4.3 (Campana [10, Corollary 12.10]). *Let $f\colon (X, \Delta_X) \to Y$ be a neat fibration. If X_y denotes a general fibre of f and Δ_y denotes the restriction of the orbifold structure to X_y, the sequence*

$$\pi_1(X_y, \Delta_y) \to \pi_1(X, \Delta_X) \xrightarrow{f_*} \pi_1(Y, \Delta^*(f, \Delta_X)) \to 1$$

is exact.

Remark 4.4. We omit the detailed definition of neatness required for the preceding statement because in what follows we only use it in the two following, quite simple, situations.

(a) When Δ_X is empty, this neatness assumption has already been encountered (see Definition 3.5). The content of the previous proposition is then that $\pi_1(X)$ sits in the middle of a short exact sequence (with the fundamental group of the orbifold base on the right-hand side).

(b) When Y is a curve, the fibration $f\colon (X, \Delta_X) \to Y$ is always neat.

4.2. Nilpotency class in fibrations

To complete this study of the behaviour of fundamental groups in fibrations, we prove that nilpotency conditions on the fundamental groups are preserved in fibrations between Kähler orbifolds. This remarkable fact (obviously false even for submersions between complex manifolds as the Iwasawa manifold shows) is a consequence of deep results in mixed Hodge theory.‡

Theorem 4.5 (Campana [11, Corollary 7.6]). *Let $f\colon (X, \Delta_X) \to (Y, \Delta_Y)$ be a neat fibration (see Remark 4.4) between smooth compact Kähler orbifolds. If the groups $\pi_1(X_y, \Delta_{X_y})$ and $\pi_1(Y, \Delta_Y)$ are almost abelian¶, then $\pi_1(X, \Delta_X)$ is almost abelian as well.*

Proof. This is a reduction to the case when $\Delta_X = 0$. Since f is assumed to be neat, the sequence

$$\pi_1(X_y, \Delta_{X_y}) \to \pi_1(X, \Delta_X) \xrightarrow{f_*} \pi_1(Y, \Delta_Y) \to 1$$

† Because X will not then be our initial 3-fold, but the two-dimensional supporting space of the orbifold base of either its Iitaka fibration, or of its algebraic reduction.

‡ The key ingredients are the existence of an MHS on the Malčev completion of the fundamental group of a compact Kähler manifold and the strictness of morphisms of MHSs.

¶ More generally, if these two groups are torsion-free nilpotent of nilpotency class at most ν, then $\pi_1(X, \Delta_X)$ is also nilpotent of nilpotency class at most ν.

is exact. Since $\pi_1(X_y, \Delta_{X_y})$ and $\pi_1(Y, \Delta_Y)$ are almost abelian groups (of finite type), $G = \pi_1(X, \Delta_X)$ is an almost polycyclic group; in particular, the group G is linear. Selberg's lemma asserts that G then has a finite index subgroup $G' \leqslant G$ that is torsion free. Consider X', the orbifold cover of (X, Δ_X) associated with G'; the latter being torsion free, X' is a normal variety with no orbifold structure (i.e. $\Delta' = 0$). Since X' has only quotient singularities, its fundamental group is isomorphic to that of $\tilde{X}$, a desingularization of X' [**17**, Theorem 7.5]. To conclude, we consider (a neat model of) the Stein factorization of

$$\tilde{X} \to X' \to X \to Y.$$

Indeed, taking a further blow-up of $\tilde{X}$, we can complete the picture:

$$\begin{array}{ccccccc} \tilde{X} & \longrightarrow & X' & \longrightarrow & X & \xrightarrow{f} & Y \\ & \searrow^{\tilde{f}} & & & & \nearrow_{g} & \\ & & & \tilde{Y} & & & \end{array}$$

with $\tilde{f}$ a neat fibration and g generically finite. It is then easy to see that we have an exact subsequence (with vertical maps having finite index images):

$$\begin{array}{ccccccc} \pi_1(\tilde{X}_y) & \longrightarrow & \pi_1(\tilde{X}) & \xrightarrow{\tilde{f}_*} & \pi_1(\tilde{Y}, \Delta^*(\tilde{f})) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_1(X_y, \Delta_{X_y}) & \longrightarrow & \pi_1(X, \Delta_X) & \xrightarrow{f_*} & \pi_1(Y, \Delta_Y) & \longrightarrow & 1 \end{array}$$

So we are reduced to the same situation with $\Delta_X = 0$, and we can apply [**11**, Corollary 7.6]. □

4.3. Fundamental groups of orbifolds with non-positive canonical bundle

Just as in the case of compact Kähler manifolds, when the orbifold first Chern class is 0 or positive, the now commonplace differential–geometric methods can then be applied to construct orbifold Kähler metrics with Ricci-curvature of the corresponding sign (using the local smooth uniformizations).

Theorem 4.6. *Let $\mathcal{X} = (X, \Delta)$ be an orbifold with X compact Kähler. If the first Chern class of $\mathcal{X}$ is non-negative, the group $\pi_1(\mathcal{X})$ is almost abelian. More precisely:*

- *if $c_1(K_X + \Delta) = 0$, then $\pi_1(X, \Delta)$ is almost abelian of even rank† bounded by $2\dim(X)$;*
- *if $c_1(K_X + \Delta) > 0$, then $\pi_1(X, \Delta)$ is finite.*

† The rank of an almost abelian group G of finite type is the maximum rank of an abelian subgroup of finite index of G.

To begin with, we recall some basic facts on differential calculus on orbifolds. The smooth functions (differential forms, hermitian metrics, etc.) on an orbifold (X,Δ) are the smooth functions on $X^*\setminus|\Delta|$ that can be smoothly extended (as the usual objects, after taking inverse images) in local uniformizations. For instance, if (X,Δ) is a geometric orbifold, a Kähler metric has the following form in coordinates charts adapted to Δ:

$$\begin{aligned}\omega_\Delta &= \omega_{\mathrm{eucl}} + \sum_{j=1}^{n} \mathrm{i}\partial\bar\partial|z_j|^{2/m_j}\\ &= \omega_{\mathrm{eucl}} + \sum_{j=1}^{n} \frac{\mathrm{i}\mathrm{d}z_j\wedge \mathrm{d}\bar z_j}{m_j^2|z_j|^{2(1-1/m_j)}}.\end{aligned}$$

Local uniformizations can also be used to compute integrals of forms of maximal degree. If $\varphi\colon U\to\varphi(U)\subset X$ is such a local cover and α an orbifold top-form on X, define

$$\int_{\varphi(U)}\alpha = \frac{1}{\deg(\varphi)}\int_U \varphi^*\alpha.$$

This local computation can then easily be globalized using partitions of unity.

The canonical divisor has been defined in Definition 3.2 as a $\mathbb{Q}$-divisor. It should be noted here that (X being compact and the uniformizations being finite) some integral multiple of this divisor defines a line bundle on X and it can be used to compute the first Chern class of X. As a by-product, the Ricci form of any orbifold volume form is a $(1,1)$-orbifold form whose cohomology class coincides with $c_1(\mathcal{X})$.

As in the manifold case, every invariant form whose cohomology class coincides with $c_1(\mathcal{X})$ is the Ricci curvature of an orbifold Kähler metric. This fact has already been noted several times in the literature (see, for instance, [**8**, **15**] when $c_1(\mathcal{X})=0$, and [**12**] when $c_1(\mathcal{X})>0$, and the references therein). We need the following statement.

Theorem 4.7 (Calabi–Yau). *Let $\mathcal{X}$ be a Kähler orbifold whose underlying space is compact and let us fix ω_0 as an orbifold Kähler metric. For any representative* (*i.e. smooth invariant* $(1,1)$ *form*) *α of $c_1(\mathcal{X})$, there exists a unique orbifold Kähler metric ω in the Kähler class* $[\omega_0]$ *such that* $\mathrm{Ricci}(\omega)=\alpha$.

Proof. We reproduce the arguments given in [**8**, Theorem 4.1]. This is a simple adaptation of the proof exposed in [**26**, pp. 85–113]. Since $\mathrm{Ricci}(\omega_0)$ and α define the same class, the orbifold $\partial\bar\partial$-lemma provides us with a smooth (orbifold) function f such that $\mathrm{Ricci}(\omega_0)=\alpha+\mathrm{i}\partial\bar\partial f$. We normalize f (by addition of a suitable constant) in such a way that

$$\int_X(\mathrm{e}^f-1)\omega_0^n=0.$$

The problem is then reduced to solving the Monge–Ampère equation

$$(\omega_0+\mathrm{i}\partial\bar\partial\varphi)^n=\mathrm{e}^f\omega_0^n. \tag{MA}$$

To do so, we apply the continuity method as in [**26**, pp. 85–113] and consider the set T of $t \in [0,1]$ for which the equation

$$(\omega_0 + \mathrm{i}\partial\bar{\partial}\varphi_t)^n = C_t \mathrm{e}^{tf}\omega_0^n \qquad (\mathrm{MA})_t$$

has a solution, where

$$C_t = \frac{\int_X \omega_0^n}{\int_X \mathrm{e}^{tf}\omega_0^n}.$$

When $t = 0$, $\varphi_0 = 0$ is an obvious solution for $(\mathrm{MA})_t$. The set T is open by linearization of the problem and the implicit functions theorem (applied in local uniformizations). To see that T is closed we need *a priori* estimates on the solution of $(\mathrm{MA})_t$, which are established using two types of argument: the maximum principle and Nash–Moser iteration (integral inequalities). These can be used in the orbifold setting as well: the maximum principle is applied in local uniformizations and integral inequalities are also valid in this formalism (see the remarks following Theorem 4.6). □

We proceed as in the manifold case to deduce Theorem 4.6 from Theorem 4.7. The positivity assumptions can be translated in the following way: in the Fano case ($c_1(\mathcal{X}) > 0$) there exists a positive $(1,1)$-form α representing $c_1(\mathcal{X})$; in the Ricci flat case ($c_1(\mathcal{X}) = 0$) we choose $\alpha = 0$. We can now apply Theorem 4.7: $\mathcal{X}$ has an orbifold Kähler metric ω with prescribed Ricci curvature $\mathrm{Ricci}(\omega) = \alpha$.

In the Fano case, by the compactness of X, we get that $\mathrm{Ricci}(\omega) \geqslant \epsilon\omega$, where $\epsilon > 0$. The following orbifold version of Myer's theorem applied to the orbifold universal cover (see [**24**]) of $\mathcal{X}$ implies the finiteness of the orbifold fundamental group.

Theorem 4.8 (Borzellino [3, Corollary 21]). *Let $(\mathcal{Y}, g)$ be a complete Riemannian orbifold (of dimension n). If its Ricci curvature satisfies the inequality*

$$\mathrm{Ricci}(g) \geqslant (n-1)k$$

for some $k > 0$, then its diameter is bounded above:

$$\mathrm{diam}(Y) \leqslant \frac{\pi}{\sqrt{k}}.$$

Moreover, the underlying space Y of $\mathcal{Y}$ is compact if $(\mathcal{Y}, g)$ is an orbifold étale cover of some compact orbifold.

When $c_1(\mathcal{X}) = 0$, the following orbifold splitting theorem, Theorem 4.9, again applied to the orbifold universal cover of $\mathcal{X}$ also implies the claim.

Theorem 4.9 (Borzellino and Zhu [4]). *Let $(\mathcal{Y}, g)$ be a compact Riemannian orbifold (of dimension n). If the Ricci curvature is everywhere non-negative, the orbifold universal cover $\tilde{\mathcal{Y}}$ admits a (metric) splitting*

$$\tilde{\mathcal{Y}} \simeq N \times \mathbb{R}^m,$$

where N is a compact orbifold. The orbifold fundamental group of $\mathcal{Y}$ is also an extension of a crystallographic group by a finite group and is, in particular, almost abelian.

From the arguments given in [**8**, 5.4, 6.3] we deduce that the rank of the almost abelian group $\pi_1(X, \Delta)$ is even and bounded by $2\dim(X)$.

5. Minimal model programme for klt pairs in dimension 2

5.1. Kawamata log terminal pairs as orbifolds in dimension 2

In this short section, we gather from [**19**] some well-known facts on the minimal model programme (MMP) for log pairs in dimension 2. Starting with a pair (X, Δ), where X is a smooth surface and $\Delta = \sum_j b_j \Delta_j$ is a $\mathbb{Q}$-Weil effective divisor, we can perform a sequence of divisorial contractions that ends with a birational model of the initial pair and whose geometry is simplest, according to the sign of the canonical bundle. In this process, however, $(K_X + \Delta)$-negative curves are being contracted, and the resulting surface is no longer in general smooth. The relevant preserved category of singularities is then described as follows.

Definition 5.1. A pair (X, Δ), where X is a $\mathbb{Q}$-factorial normal variety, is said to have only klt singularities if

(i) for all j, $0 < b_j < 1$;

(ii) for any (or equivalently, one) log-resolution $f: Y \to X$ of (X, Δ), in

$$K_Y + \tilde{\Delta} = f^*(K_X + \Delta) + \sum_i a_i E_i,$$

we have $a_i > -1$ for all i, $\tilde{\Delta}$ being the strict transform of Δ, and E_i the exceptional divisors of f.

A smooth (integral) orbifold is of course a klt pair. These singularities are preserved in the LMMP.

Theorem 5.2 (Kollár and Mori [19, Theorem 3.47]). *Let (X, Δ) be a klt surface. There exists a birational morphism $f: X \to S$ such that the resulting pair $(S, \Delta_S = f_*(\Delta))$ is still klt and satisfies (exactly) one of the following properties:*

(1) *$K_S + \Delta_S$ is nef;*

(2) *S admits a fibration $\pi: S \to C$ onto a (smooth) curve C, the general fibre of π being a smooth $K_S + \Delta_S$-negative rational curve;*

(3) *(S, Δ_S) is del Pezzo: $\rho(S) = 1$ and $-(K_S + \Delta_S)$ is ample.*

The crucial case in our situation is when the orbifold base (X, Δ) of either the Iitaka fibration or the algebraic reduction of our initial 3-fold is two dimensional with $\kappa(X, \Delta) = 0$. The LMMP in dimension 2 replaces it with an orbifold (S, Δ_S) with $K_S + \Delta_S \equiv 0$, to which the results of the preceding section can be applied once we prove that (S, Δ_S) is an orbifold (i.e. is locally smoothly uniformizable). This is the content of Theorem 5.3.

When $\Delta = 0$, it is well known that klt singularities coincide with quotient singularities in dimension 2 (see [**19**, Proposition 4.18]). This property still holds for $\Delta \neq 0$ with integral multiplicities (i.e. coefficients b_j of the form $1 - 1/m$).

Theorem 5.3. *Let (X,Δ) be a pair where X is a surface and Δ has integral multiplicities. The following conditions are equivalent near each point of X:*

(1) *(X,Δ) is klt,*

(2) *(X,Δ) has a finite local fundamental group,*

(3) *(X,Δ) is locally presented as a quotient: $\pi\colon \mathbb{B}^2 \to \mathbb{B}^2/G \simeq X$, where G is a finite group acting linearly (and unitarily) on $\mathbb{B}^2$ (the unit ball of $\mathbb{C}^2$) and Δ is the unique $\mathbb{Q}$-divisor on X such that $\pi^*(K_X+\Delta)=K_{\mathbb{B}^2}$.*

This statement is found and used in several places [**16**, **21**], but never with a complete accessible proof: the appendix of [**16**] consists of a list of the possible cases, and refers to the thesis [**23**] for the proof. For this reason, we give a proof of Theorem 5.3 at the end of the present text (see the appendix).

5.2. Fundamental groups and Mori contractions

We now relate the fundamental groups of a geometric orbifold (X,Δ) of dimension 2 and of its minimal model $(S, D=\Delta_S=f_*(\Delta))$ as in Theorem 5.2.

Proposition 5.4. *Let (X,Δ) be a geometric orbifold of dimension 2 and let $f\colon (X,\Delta)\to(S,D)$ be its minimal model. There exists a natural surjective morphism of groups:*

$$f^\sharp\colon \pi_1(S,D)\to\pi_1(X,\Delta).$$

In particular, if $\pi_1(S,D)$ is almost abelian, so is $\pi_1(X,\Delta)$.

Proof. We call E the union of the curves contracted by f; f being an isomorphism away from E, we have the natural maps, S^* being the smooth locus of S,

$$S^*\setminus|D| \xrightarrow{f^{-1}} X\setminus(|\Delta|\cup E)\hookrightarrow X\setminus|\Delta|$$

(note that $|\Delta|$ and E may have common components). At the level of fundamental groups, we get the natural morphisms

$$\pi_1(S^*\setminus|D|)\xrightarrow{\sim}\pi_1(X\setminus(|\Delta|\cup E))\twoheadrightarrow\pi_1(X\setminus|\Delta|)\twoheadrightarrow\pi_1(X,\Delta).$$

To get the morphism $f^\sharp$ we need only remark that the loops around the components of D are sent onto loops around corresponding components in Δ with the same multiplicities, since f is an isomorphism between S^* and $X\setminus E$. □

Example 5.5. In general, the kernel of this morphism can, however, be very big, as shown by the following example.

Let C be an elliptic curve, and let X be the blow-up of $S=C\times C$ at a point (c,c). Let $F\subset S$ be the fibre of the first projection through this point and let G be its strict transform in X. If $m>1$ is an integer, the minimal model of $(X,\Delta=(1-1/m)G)$ is $(S,D=(1-1/m)F)$ and the exceptional divisor E of the blow-up is the only $(K_X+\Delta)$-negative

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curve on X. Since the orbifold base of the first projection is $(C,(1-1/m)\{c\})$, the fundamental group of (S,D) has (a finite index subgroup that has) a surjective morphism onto a non-abelian free group. On the other hand, the orbifold base of $g\colon (X,\Delta)\to C$, g being the composition of the first projection and the blow-up, is merely C since the fibre over c has E as a component and inherits from it the multiplicity 1. It is then easy to see that (X,Δ) is special and that its fundamental group is isomorphic to the fundamental group of X, and thus abelian.

Remark 5.6. The direction of the arrow $f^\sharp$ in Proposition 5.4 is not the usual one. We explain its construction differently. The map $f\colon (X,\Delta)\to(S,D)=(X,\Delta)_{\min}$ is, in general, not a 'classical' orbifold morphism (in the sense of [**10**]) and, thus, does not induce a functorial morphism of groups, unless the multiplicities on the exceptional divisors of f are sufficiently divisible (for example, by the order of the local fundamental group of (S,D) at the point under consideration). In Example 5.5, equipping E with a multiplicity divisible by m is the right condition. This is achieved as follows.

Let Δ^+ be an orbifold divisor on X such that $f_*(\Delta^+)=f_*(\Delta)=D$, and such that $f\colon (X,\Delta^+)\to(S,\Delta)$ and $\mathrm{id}_X\colon (X,\Delta^+)\to(X,\Delta)$ are orbifold morphisms (this means, for id_X, that the multiplicity of any component of Δ divides the corresponding multiplicity in Δ^+).

We then get two functorial group morphisms: $f_+^\sharp\colon \pi_1(X,\Delta^+)\to\pi_1(S,D)$, which is an isomorphism, and $\mathrm{id}_X^\sharp\colon \pi_1(X,\Delta^+)\to\pi_1(X,\Delta)$, which is surjective (as above). Our initial $f^\sharp$ was nothing but $(\mathrm{id}_X)^\sharp\circ(f_+^\sharp)^{-1}$.

5.3. Abelianity for special klt pairs and proof of the main theorem

To conclude this section, we use the preceding proposition to prove the abelianity conjecture for geometric orbifolds of dimension 2.

Proof of Theorem 2.2. Let (S,D) be the minimal model of the pair (X,Δ_X). We argue according to the values of $\kappa=\kappa(X,\Delta)=\kappa(S,D)\in\{-\infty,0,1\}$.

If $\kappa=1$, then X admits a fibration f onto a smooth curve C whose general fibre F satisfies $F\cdot(K_X+\Delta)=0$; the general orbifold fibre (F,Δ_F) is then orbifold elliptic and its fundamental group is almost abelian (see Example 4.2). Adding to C the orbifold divisor $\Delta^*=\Delta^*(f,\Delta_X)$ of classical multiplicities, we get an exact sequence

$$\pi_1(F,\Delta_F)\to\pi_1(X,\Delta_X)\to\pi_1(C,\Delta^*)\to 1.$$

The orbifold curve (C,Δ^*) is special since so is (X,Δ), and Theorem 4.5 shows that $\pi_1(X,\Delta)$ is almost abelian.

When (S,D) admits a structure of Mori fibre space over a (special) curve (in particular, $\kappa=-\infty$), we still have a fibration on X whose fibres are special (in fact rational) and we can proceed as above.

There are two cases left to consider:

- $\kappa(S,D)=0$,
- (S,D) is log del Pezzo.

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In the first case, $K_S + D$ being *nef* it is semi-ample (due to log abundance for surfaces; see [**13**]) and then torsion. In particular, the orbifold (S, D) is Ricci flat. In the second case, (S, D) is Fano. In both cases, we can apply Theorem 4.6 to conclude that the fundamental group of (S, D) is almost abelian. Finally, Proposition 5.4 can be applied (it is only here we need it) to show that $\pi_1(X, \Delta_X)$ is almost abelian as well. □

Recall that (X, Δ_X) was the two-dimensional orbifold base of either the Iitaka fibration, or of the algebraic reduction of our initial special Kähler 3-fold, in the two cases left open (see the end of § 2) in the proof of Theorem 1.3. Since the smooth fibres are then elliptic curves, and the orbifold base is also special, it then has an abelian fundamental group, and, thus, has the 3-fold under study, by Theorem 4.5.

Appendix A. Classification of klt singularities of surfaces

In this appendix we give a proof of the classification of klt singularities for integral pairs (see Theorem 5.3). The method is first to treat the singular case (using the method of [**19**, Theorem 4.7]) and then to reduce the smooth case to the preceding one using orbifold étale covers. Then let (X, Δ), $\Delta = \sum_j (1 - 1/m_j) D_j$, be a germ of a klt pair.

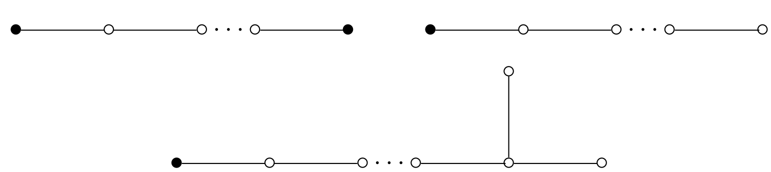
A.1. The singular case

To begin with we quote a useful (negativity) lemma on connected quadratic forms.

Lemma A 1 (Kollár and Mori [19, Corollary 4.2]). *Let $E := \bigcup_j E_j$ be a connected exceptional curve on a smooth complex surface. Let $A := \sum_j a_j E_j$ and let $B := \sum_j b_j E_j$, with $a_j, b_j \in \mathbb{R}$. Assume that $A \cdot E_j \geqslant B \cdot E_j$ for any j. Then either $A = B$ or $a_j < b_j$ for any j.*

We first consider the case in which the germ X is singular.

Lemma A 2. *Assume that X is singular, and that $\Delta \neq 0$. Let $f \colon X' \to X$ be the minimal resolution of the germ X, $\Delta' = \sum_k (1 - 1/m_k) D'_k$ being the strict transform of Δ in X' and $E = \bigcup_j E_j$ the exceptional divisor. The extended dual graph of $f^*(\Delta)$ is then one of those below (which are the dual graphs when $\Delta \neq 0$), in which the (non-compact) components of Δ' are indicated by black dots. Moreover, all intersections are transversal and all white dots are smooth rational curves. These are (-2)-curves, except possibly in the first case (see Remark A 3 below for additional constraints in the first case):*



Proof. We assume knowledge of the classification of Duval (i.e. canonical) and of klt germs of two-dimensional singularities, and, more precisely, the fact that the dual graphs of their minimal resolutions are given either by Dynkin diagrams of (-2)-curves of type

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A_n, D_n, E_k (for $k = 6, 7, 8$) or by Hirzebruch–Jung chains with transversal intersections (see [**19**, Theorem 4.7] for the method of the proof we now adapt).

From negativity, rational numbers a_i are uniquely determined by the numerical equalities

$$\forall j, \quad d_j := (K_{X'} + \Delta') \cdot E_j = A \cdot E_j, \tag{$\star$}$$

with $A := \sum_i a_i E_i$. Write also that $e_i := -E_i^2$.

Assume first that there exist (at least) two components D_1 and D_2 in Δ', of multiplicities m_1 and m_2, respectively. Consider a shortest chain $E_1, \dots, E_n$ of components of E joining them, such that D_1 (respectively, D_2) meets E_1 (respectively, E_n), but no other E_ℓ. An easy computation gives that

$$d_j = \begin{cases} (e_1 - 2) + \left(1 - \dfrac{1}{m_1}\right)(D_1 \cdot E_1) & \text{if } j = 1, \\ (e_n - 2) + \left(1 - \dfrac{1}{m_n}\right)(D_2 \cdot E_n) & \text{if } j = n, \\ e_j - 2 & \text{otherwise.} \end{cases}$$

Consider now the curve $B := -\beta(\sum_1^n E_\ell)$, for some $\beta > 0$ to be chosen later. We assume first that $n \geqslant 2$. The intersection numbers are then given by

$$B \cdot E_j = \begin{cases} \beta(e_j - 1), & j = 1, n, \\ \beta(e_j - 2), & 1 < j < n. \end{cases}$$

We now choose the value of β according to the self-intersection of the E_j:

(1) if $e_i \neq 2$, for some $i \neq 1, n$, we set

$$\begin{aligned} \beta &= \inf\left\{1, \frac{d_j}{e_j - 1},\ j = 1, n\right\} \\ &= \inf\left\{1, 1 + \frac{(1 - 1/m_1)(E_1 \cdot D_1) - 1}{e_1 - 1}, 1 + \frac{(1 - 1/m_2)(E_n \cdot D_2) - 1}{e_n - 1}\right\}, \end{aligned}$$

(2) if $e_i = 2$, for all $i \neq 1, n$, we set

$$\beta = \inf\left\{\frac{d_j}{e_j - 1},\ j = 1, n\right\}.$$

We thus have that $A \cdot E_j \geqslant B \cdot E_j$ for any j, and equality for some j. From Lemma A 1 we get that $A = B$ and $a_j = -\beta > -1$, since (X, Δ) is klt. Using ($\star$) easily implies that

$$\left(1 - \frac{1}{m_1}\right)(D_1 \cdot E_1) < 1 \quad \text{and} \quad \left(1 - \frac{1}{m_2}\right)(D_2 \cdot E_n) < 1.$$

Because $(1 - 1/m) \geqslant \frac{1}{2}$, we get that $D_1 \cdot E_1 = D_2 \cdot E_n = 1$, and we are thus in the first case of the lemma.

Assume now that $n = 1$ and note that then D_i meets E_1 for $i = 1, 2$. We then have that

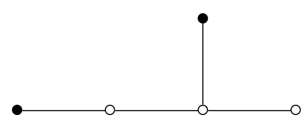
$$(e_1 - 2) + \left(1 - \frac{1}{m_1}\right)(D_1 \cdot E_1) + \left(1 - \frac{1}{m_2}\right)(D_2 \cdot E_1) = \beta e_1^2$$

for some $\beta < 1$ (by the klt condition). The solutions are easily determined to be either

(i) $E_1 \cdot D_1 = E_1 \cdot D_2 = 1$ and m_1, m_2 are arbitrary or

(ii) (up to permutation) $E_1 \cdot D_1 = m_1 = 2$ and $E_1 \cdot D_2 = 1$, with m_2 arbitrary.

This last case (which is not a normal crossing) is excluded by two blow-ups that make the total transform of Δ a normal crossings divisor. We indeed get the following extended dual graph (in which the E'_i, $i = 1, 2, 3$, appear as white circles in the order E'_1, E'_3, E'_2, while the left-hand (respectively, upper) black dot is the strict transform of D_1 (respectively, D_2)):



(which is a non-klt case). We also have that $e'_1 = e_1 + 2$, $e'_3 = 1$ and $e'_2 = 2$. A direct resolution of the linear system ($\star$) shows that this pair is not klt.

Assume next that our dual graph contains a fork (of white vertices). We know that $K_{X'} \equiv 0$ in this case ($e_i = 2$, for all i). Let $D_1, E_1, \dots, E_{n-2}, E_{n-1}, E_n$ be a shortest subgraph connecting the component D_1 to the fork E_{n-2}, E_{n-1}, E_n, the last two vertices being the end points of the fork. The sequence is so labelled that D_1 meets E_1 only, and that $E_1, \dots, E_{n-2}$ is a chain. Consider the curve

$$B := -\left(1 - \frac{1}{m_1}\right)(D_1 \cdot E_1)\left(\sum_{\ell=1}^{n-2} E_\ell + \tfrac{1}{2}(E_{n-1} + E_n)\right).$$

A computation similar to the preceding one (but simpler, since now $K_{X'} \equiv 0$) shows that $A \cdot E_j = B \cdot E_j$ for all j. Thus, $A = B$ and we are in the second case.

The only case left is no fork and only one component in Δ'. This is the last diagram (with transversal intersection by the klt and integral conditions, as above) since the degenerate case where $n = 2$ in the preceding case would lead to the existence of a fork by the connectedness of E. □

Remark A 3. In the first case, the computation additionally gives that $(e_1 - 1)m_1 = (e_n - 1)m_2$ and also $e_i = 2$, for all $i \neq 1, n$. The arguments of [**1**, III (5.1)] show that the singularity of X is cyclic of type $A_{N,q}$ with $N = (n-1)(e_1-1)(e_n-1) + (e_1-1) + (e_r-1)$ and $q = (n-1)(e_1-1)+1$, with $N/q = (e_n - 1) + (e_1 - 1)/((n-1)(e_1-1)+1)$.

Lemma A 4. *Let (X, Δ) be a two-dimensional germ of a pair as above. This germ is klt if and only if has a finite local fundamental group.*

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Proof. If $\Delta = 0$, this proof is that of [**19**, Proposition 4.18]. Otherwise, if the singularity is of type A_n, an easy computation shows (as in [**20**, Chapter IV, § 13–14]) that the local fundamental group has order nm_1m_2. If the singularity is of type D_n, we reduce it to the preceding case by a double cover. □

To deal with the smooth case, we need preliminary observations (see Lemma A 5, Example A 8 and Lemma A 10) on orbifold étale covers.

A.2. Orbifold étale covers

This first lemma ensures that orbifold étale covers of klt pairs will remain klt.

Lemma A 5 (Kollár and Mori [19, Proposition 5.20]). *Let $g\colon (X', \Delta') \to (X, \Delta)$ be a finite map between normal germs of varieties. Assume that $K_{X'} + \Delta'$ and $K_X + \Delta$ are $\mathbb{Q}$-Cartier†, and that $K_{X'} + \Delta' = g^*(K_X + \Delta)$. Then, $K_{X'} + \Delta'$ is klt if and only if $K_X + \Delta$ is klt.*

We return to dimension 2 with integral pairs.

Definition A 6. We say that a finite map $g\colon (X', \Delta') \to (X, \Delta)$ of degree d is *orbifold étale* if

(i) the map g ramifies only above the support of $\Delta = \sum_k (1 - 1/m_k) D_k$,

(ii) g has order of ramification r_k dividing m_k along D_k,

(iii) $\Delta' = g^*(\sum_k (1 - r_k/m_k) D_k)$.

Remark A 7. In this preceding case the local fundamental group of (X', Δ') has index d in the local fundamental group of (X, Δ). Moreover, from the ramification formula we get that

$$K'_X + \Delta' = g^*\left(K_X + \sum_{k=1}^{r}\left(1 - \frac{1}{r_k}\right)D_k\right) + \sum_{k=1}^{r}\left(1 - \frac{r_k}{m_k}\right)g^*\left(\frac{D_k}{r_k}\right) = g^*(K_X + \Delta)$$

since $(1 - 1/r_k) + (1 - r_k/m_k)/r_k = 1 - 1/m_k$. It also follows that (X, Δ) is klt if and only if (X', Δ') is too.

We apply this remark only when $r_1 = m_1$ and $r_k = 1$ for $r \geqslant 2$. In this case, $\Delta' = g^*(\sum_{k=2}^{k=r}(1 - 1/m_k)D_k)$ and the component D_1 'disappears'.

Example A 8. We now always assume that X is a smooth germ.

(1) Assume that D_1 is a cusp of the equation $x^p = y^q$ (p, q coprime) and of multiplicity m_1, which we write symbolically as $\Delta_1 = (p, q; m_1)$. Then, X' is the singularity of the equation $z^{m_1} = y^q - x^p$. Thus, if (X, Δ) is klt, we have that $1/p + 1/q + 1/m_1 > 1$. And so (p, q, m_1) is, up to order, either $(2, 2, m)$ or $(2, 3, m)$, $m = 3, 4, 5$. Moreover, if the support of Δ is reducible (if $r \geqslant 2$), then X' must be an A_n singularity and the support of Δ' must have at most two components. Thus, $(p, q; m_1)$ is, up to order, $(2, 2; m)$ and $r \leqslant 3$.

† In dimension 2 the $\mathbb{Q}$-Cartier assumption is superfluous.

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If D_1 is not *a priori* assumed to be a cusp, but just to be given by a parametrization $x(t) = t^q$, $y(t) = t^p$, with p, q coprime, then since the cover of degree m_1 ramified exactly over D_1 will be a klt singularity, and so will have the usual normal form in suitable coordinates, we see *a posteriori* that D_1 was indeed a cusp in suitable coordinates.

(2) Assume that the support of Δ has two smooth components D_1, D_2, tangent at order $p \geqslant 2$ at the origin. They have (in this order), in suitable coordinates, the equations $y = 0$ and $y = x^p$. Consider the map $g\colon \mathbb{C}^2 \to X$ given by $g(u, v) = (x, y) := (u, v^{m_1})$. Thus, using Remark A 7, we see that making this (orbifold étale) cover g we are led to the case (again klt) where

$$\Delta' = g^*\biggl(\sum_{k=2}^{k=r}\biggl(1 - \frac{1}{m_k}\biggr)D_k\biggr),$$

with D_2' having the equation $v^{m_1} = u^p$. In this case, letting d be the gcd of (m_1, p), the germ D_2' splits into d irreducible components that are cusps of type (p', m_1') and multiplicity m_2, with $p' := p/d$, $m_1' := m_1/d$, and are thus smooth if and only if either $p = m_1 = d$ or $p = d \neq m_1$ or $m_1 = d \neq p$.

A.3. The smooth case

We next consider the remaining case in which the germ X is smooth. Write $\Delta = \sum_{k=1}^{r}(1 - 1/m_k)D_k$. For $k = 1 \dots r$, let $t_k \geqslant 1$ be the multiplicity of the germ D_k at the origin. Thus, D_k is smooth if and only if $t_k = 1$. We say that an irreducible germ of curve in $X \simeq \mathbb{C}^2$ has a (p, q)-cusp at the origin if its equation in suitable coordinates is $y^q - x^p = 0$ (with p and q coprime); in this case the multiplicity is $\inf(p, q)$.

Lemma A 9. *If (X, Δ) is klt, the only possibilities for the data r, t_k, m_k are the following.*

(1) *$t_k = 1$ for all k. Then $r \leqslant 3$. If $r = 3$, then $1/m_1 + 1/m_2 + 1/m_3 > 1$ (i.e. either $(m_1, m_2, m_3) = (2, 2, m_3)$ with $m_3 \geqslant 2$ arbitrary, or $(m_1, m_2, m_3) = (2, 3, m_3)$ with $2 \leqslant m_3 \leqslant 5$).*

(2) *$t_k = 2$ for some k. Then either $r = 1$ and D_k has a $(2, q)$-cusp at the origin with $1/2 + 1/q + 1/m_k > 1$, or $r = 2$ (and we assume that $k = 1$). There are two subcases:*

   (a) *D_1 has a $(2, q)$-cusp with $m_1 = 2$ and D_2 is smooth and has intersection multiplicity 2 with D_1; m_2 and q (odd) are arbitrary.*

   (b) *D_1 has a $(2, 3)$-cusp with $m_1 = 2$, D_2 (smooth) has intersection multiplicity 3 with D_1 and $m_2 = 2$.*

(3) *$t_k = 3$ for some k. Then $r = 1$ and D_1 is a $(3, 5)$-cusp or $(3, 4)$-cusp with $m_1 = 2$.*

Proof. We make a blow-up $f\colon X_1 \to X$ at the origin with exceptional divisor E_1 and define $K_1 := K_{X_1}$ and $K = K_X$. Let Δ_1 be the strict transform of Δ in X_1. Then $K_1 + \Delta_1 = f^*(K + \Delta) + cE_1$, where $c = 1 - \sum_k t_k(1 - 1/m_k)$. The klt condition implies that $c > -1$, that is, $\sum_k t_k(1 - 1/m_k) < 2$. Since $(1 - 1/m_k) \geqslant 1$, this implies that

$\sum_k t_k < 4$. Thus, $r \leqslant 3$ and we get the following list of possible values:

(1) if $t_k \geqslant 3$ for some k, then $r = 1$ and $t_1 = 3$;

(2) if $t_k = 2$ for some k, then $r = 1, 2$, and $(t_1, t_2) = (2, 1)$ if $r = 2$;

(3) if $t_k = 1$ for all k, then $r = 1, 2$ or 3.

We now examine the distribution of possible multiplicities.

If $t_1 \geqslant 2$ and $r = 1$, the étale orbifold cover (see Example A 8) of degree m_1 ramified along the (p, q)-cusp D_1 leads to the singularity $z^{m_1} = y^q - x^p$ with the zero orbifold divisor, which is klt if and only if the claimed inequality $1/p + 1/q + 1/m_1 > 1$ holds. If $t_1 = 2, 3$, we thus get the cases described in Lemma A 9 (2) and (3). We are left with the case $t_1 = 2$, $t_2 = 1$, $r = 2$. Thus, D_1 is a $(2, q)$-cusp of multiplicity m_1 and D_2 is smooth of multiplicity m_2 with q odd. We distinguish two cases according to whether the intersection multiplicity of D_1 and D_2 is 2 or more.

In the first case, the orbifold étale cover of (X, Δ) ramified to order m_2 along D_2 leads to the orbifold divisor Δ' supported on the locus of the equation $x^2 = y^{m_2 q}$ and with multiplicity m_1. Since (X', Δ') is still klt, we derive that $1/2 + 1/qm_2 + 1/m_1 > 1$, and it yields $m_1 = 2$ since $qm_2 \geqslant 6$.

If the intersection multiplicity of D_1 and D_2 is 3 or more, the orbifold étale cover of (X, Δ) ramified to order m_2 along D_2 leads to the orbifold divisor Δ' supported on the locus of the equation $x^{2m_2} = y^q$ and with multiplicity m_1 (still klt). If q and m_2 are coprime, we get the inequality $1/q + 1/2m_2 + 1/m_1 > 1$, where the only possible solution is $m_1 = m_2 = 2$ and $q = 3$.

We now show that q and m_2 are coprime, which will complete the proof. Otherwise, let $d > 1$ be their gcd. Then, Δ' consists of d components of multiplicity m_1 and the equations $x^t = \varepsilon y^s$ with $t = q/d$ and $s = 2m_2/d$. These components need to be smooth, since $d > 1$. Thus, either $t = 1$ or $s = 1$. Since q is odd it is not divisible by $2m_2$, and so $q = d$ divides m_2. Because Δ' is supported by $d = q \leqslant 3$ smooth components, we finally get $d = q = 3$ components having pairwise tangency of order $2m_2/q \geqslant 2$ at the origin. This contradicts Lemma A 10. □

Lemma A 10. *Let (X, Δ) be a two-dimensional smooth germ, the support of Δ consisting of $s \geqslant 2$ smooth germs having two-by-two contact order $t \geqslant 1$ at the origin and each of multiplicity m_k $(k = 1, \dots, r)$. Then, (X, Δ) is klt if and only if*

$$\sum_k \left(1 - \frac{1}{m_k}\right) < 1 + \frac{1}{t}.$$

The solutions $(t, r, m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r)$ are

- *$t = 1$, $r = 2$ and (m_1, m_2);*
- *$t = 1$, $r = 3$ and $(2, 2, m_3)$ or $(2, 3, 3 \leqslant m_3 \leqslant 5)$;*
- *$t = 2$, $r = 2$ and $(2, m_2)$, $(3, m_2 \leqslant 5)$;*

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- *$t = 3$, $r = 2$ and $(2, m_2 \leqslant 5)$;*
- *$t \geqslant 4$, $r = 2$ and $(2, 2)$.*

Proof. We get a log-resolution of this pair by performing t successive suitable point blow-ups with successive exceptional divisors $E_1, \dots, E_t$. If $b\colon X' \to X$ is the resulting composition, one checks that, in this process,

$$K_{X'} = b^*(K_X) + \sum_{h=1}^{t} hE_h,$$

while

$$b^*(\Delta) = \Delta' + \left(\sum_k \left(1 - \frac{1}{m_k}\right)\right)\left(\sum_{h=1}^{t} hE_h\right),$$

where Δ' is the strict transform of Δ. We thus see that the initial pair is klt if and only if the coefficient of E_t in $K_{X'} + \Delta' - b^*(K_X + \Delta)$ is strictly greater than -1, i.e. if

$$\sum_k \left(1 - \frac{1}{m_k}\right) < 1 + \frac{1}{t}.$$

The solutions are then easily determined. □

Proposition A 11. *The examples found in Figure 1 show the full range of integral pairs that are klt with a smooth ambient surface. We use the notation of [**28**].*

Proof. To prove Proposition A 11 we are, thus, left with the case where X is smooth and $\Delta = \sum_{k=1}^{k=r}(1 - 1/m_k)D_k$, the D_k being smooth at the origin. We know that $r \leqslant 3$. When $r = 1$, any m_1 leads to the klt situation. When $r = 2$, the possible situations are described in Lemma A 10. We thus assume that $r = 3$, and that not all three components are normal crossings.

We then show that (after reordering) D_1 and D_2 are tangent to order $t \geqslant 2$, while D_3 is transversal to them, and that $1/m_1 + 1/m_2 + 1/tm_3 > 1$. So one has that either $m_1 = m_2 = 2$, m_3 and t arbitrary, or $2 = t = m_1 = m_3$ and $m_2 = 3$.

Assume indeed that D_1 and D_2 have contact order $t \geqslant 2$. We first remark that D_3 is transversal to them. If we assume this is not so, then after possible reordering of the components, we may assume that the order of contact of D_3 with D_1 and D_2 is at least t. In this situation, the proof of Lemma A 10 still applies to show that $\sum_{k=1}^{3}(1 - 1/m_k) < 1 + 1/t \leqslant 1 + 1/2$, which is impossible since $m_k \geqslant 2$ for all k.

The conclusion now simply follows from considering the orbifold étale cover ramified to order m_3 along D_3, which replaces Δ by Δ', supported by two smooth components of multiplicities m_1 and m_2 and having contact order $t' = tm_3$. This implies, by Lemma A 10, that $1/m_1 + 1/m_2 + 1/tm_3 > 1$ since the inequality $\sum_k(1 - 1/m_k) < 1 + 1/t'$ is equivalent to $\sum_k 1/m_k + 1/t' > 1$ when $r = 2$. The solutions are the classical ones. □

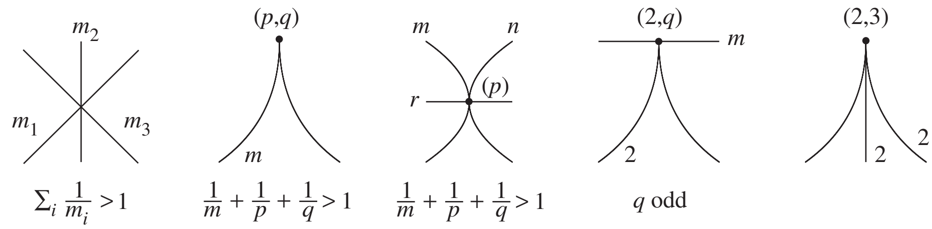

Figure 1. Germs of klt pairs with smooth base (and conditions on the coefficients).

A.4. The local fundamental groups

Proposition A 12. *Let (X, Δ) be a two-dimensional germ of integral pair. If this germ is klt, it has a finite local fundamental group.*

Proof. If X is singular, this is Lemma A 4. When X is smooth, this is a direct consequence of the construction of suitable orbifold étale covers, as explained above. □

To conclude, we briefly sum up the following proof.

Proof of Theorem 5.3. The previous proposition deals with (1) ⇒ (2), whereas (3) ⇒ (1) is just Lemma A 5. Assume finally that (X, Δ) is a (two-dimensional) pair with finite fundamental group. We consider the étale Galois cover

$$p\colon S^* \to X^* \setminus |\Delta|$$

of $X^* \setminus |\Delta|$ corresponding to the finite group $\pi_1(X, \Delta)$; the map p is easily seen to extend as a finite map (still denoted by p)

$$p\colon S \to X,$$

whose branch locus is exactly Δ (with multiplicities). It remains to check that S is smooth: it is a consequence of a classical result of Mumford [**22**], since the (local) fundamental group of $S \setminus \{s\}$ is now trivial at any point $s \in S$. The action of the Galois group can finally be linearized locally around each fixed point. □

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