BULL. AUSTRAL. MATH. SOC. 20H15, 52B30, 20C33, 51F15, 05E25, 57M07
VOL. 53 (1996) [361–368]

A VANISHING THEOREM FOR HYPERPLANE COHOMOLOGY

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Let $\mathcal{A}$ be a hyperplane arrangement in an arbitrary finite dimensional vector space V and let $G \leqslant GL(V)$ be an automorphism group of $\mathcal{A}$. If λ is a complex representation of G such that $(\lambda, 1)_{G_H} = 0$ for all pointwise isotropy groups G_H $(H \in \mathcal{A})$, then we prove the "local-global" result that λ does not appear in the representation of G on the Orlik-Solomon algebra of $\mathcal{A}$. The result is applied to complex reflection groups and to finite orthogonal groups. It may also be viewed as a combinatorial result concerning the homology of the lattice of intersections of $\mathcal{A}$. A more general version of the main result is also discussed.

1. INTRODUCTION

Let V be a finite dimensional vector space over a field k and suppose $\mathcal{A}$ is a finite collection ("arrangement") of hyperplanes in V. Orlik and Solomon have defined a graded exterior algebra $A(\mathcal{A})$ (in [7], see [8, Chapter 3] for an exposition), which in this work we take to be over $\mathbb{C}$, the complex numbers. In the case $k = \mathbb{C}$, they showed that

(1.1) $A(\mathcal{A}) \cong H^*(M_{\mathcal{A}}, \mathbb{C})$ as graded $\mathbb{C}$–algebras where $M_{\mathcal{A}} = V \setminus \bigcup_{H \in \mathcal{A}} H$ is the associated hyperplane complement.

If G is a finite subgroup of $\mathrm{GL}(V)$ which stabilises the set $\mathcal{A}$ then G has an induced action on $A(\mathcal{A})$ and it is this graded representation of G with which we are concerned. When $\mathcal{A}$ is the set of complexified reflecting hyperplanes of a Weyl group W, it was shown in [3] that

(1.2) $\left(H^j(M_W, \mathbb{C}), \varepsilon\right)_W = 0$ for all j where M_W is the corresponding complex hyperplane complement, ε is the alternating character of W and $(\ ,\ )_W$ denotes the standard multiplicity form (or inner product of characters).

The proof of (1.2) was by a "reduction mod p" argument, which related the Poincaré series of the multiplicity to a count of the rational regular semisimple orbits in a reductive Lie algebra via ℓ–adic cohomology (see [3] and [4]).

The purpose of the present work is to show that (see (2.3) below) for $\mathcal{A} \neq \emptyset$, we have:

Received 19th June, 1995

Theorem 1.3. *Let λ be a complex representation of G which contains no non-zero vector fixed by any group $G_H = \{g \in G \mid gv = v \text{ for } v \in H\}$ $(H \in \mathcal{A})$. Then $\left(A^j(\mathcal{A}), \lambda\right)_G = 0$ for all j.*

Clearly (1.3) may be thought of as a "local-global" result. The hypotheses relate to the restrictions of the representation λ to the "local" isotropy groups G_H $(H \in \mathcal{A})$ and the conclusion concerns the "global" representation $A^*(\mathcal{A})$.

An easy consequence of our result is

(1.4) Let G be a finite unitary reflection group acting on the $\mathbb{C}$-vector space V. Let M_G be the complement of the union of its reflecting hyperplanes. Then

$$\left(H^j(M_G, \mathbb{C}), \det{}^{\pm 1}\right)_G = 0 \quad \text{for all } j,$$

where det is the determinant character of G.

Of course (1.4) has (1.2) as a special case. Clearly our main result (2.3) has a combinatorial interpretation in terms of the lattice $L(\mathcal{A})$ (see Section 3 below and [**8**, Chapter 6]). In its combinatorial context, the result belongs to the circle of ideas discussed in [**10**] and [**2**]. Moreover in view of the connection between (1.2) and the theory of reductive groups (see [**3**]) and Lie algebras over finite fields one might expect other applications there (see [**1**]).

After giving the proofs of our main statements in Section 3, we give, in Section 4, a slightly more general version of the main result. In Section 5 we give an application in the context of the finite orthogonal groups to the arrangement of "non-isotropic" hyperplanes in an orthogonal space over a finite field.

2. Notation and statement of results

Notation will be as in [**8**, Chapters 3, 6]. Given V, k and $\mathcal{A}$ as in Section 1, the algebra $A(\mathcal{A})$ is defined as follows.

(2.1) $A(\mathcal{A})$ is generated as (unital, associative, graded) $\mathbb{C}$-algebra by $\{a_H \mid H \in \mathcal{A}\}$ $(\deg a_H = 1$ for $H \in \mathcal{A})$ subject to the relations

(2.1.1) $$a_H a_K = -a_K a_H \quad (H, K \in \mathcal{A}).$$

(2.1.2) If $H_1, \ldots, H_s \in \mathcal{A}$ and $\operatorname{codim}_V (H_1 \cap \ldots \cap H_s) < s$ then

$$\sum_{i=1}^{s} (-1)^i a_{H_1} \ldots \widehat{a}_{H_i} \ldots a_{H_s} = 0.$$

Following Orlik and Solomon, one associates with $\mathcal{A}$ the lattice $L(\mathcal{A})$ of all intersections of elements of $\mathcal{A}$, ordered by the reverse of inclusion. Then $L(\mathcal{A})$ has a bottom

https://doi.org/10.1017/S0004972700017111 Published online by Cambridge University Press

element V and top element $T(L) = \bigcap_{H \in \mathcal{A}} H$. It is known that $L(\mathcal{A})$ is a geometric lattice and hence that its order complex (the simplicial complex with simplexes the chains in $L(\mathcal{A})\backslash\{V, T(L)\}$) has the homotopy type of a bouquet of spheres.

The lattice $L(\mathcal{A})$ has rank function $r(X) = \operatorname{codim}_V(X)$ $(X \in L(\mathcal{A}))$. We write $r(T(L)) = r = r(\mathcal{A})$ for the *rank* of the arrangement.

DEFINITION 2.2: For $X \in L(\mathcal{A})$ write $G_X = \{g \in G \mid gv = v \text{ for all } v \in X\}$ and $N_X = \{g \in G \mid gX = X\}$.

Clearly G_X is a normal subgroup of N_X. If $\bigcap_{g \in G_X} \operatorname{Fix} g = X$, then $N_X = N_G(G_X)$.

Since the relations (2.1) are homogeneous (in the exterior algebra), $A(\mathcal{A})$ has a natural grading. We write $A^j(\mathcal{A})$ for the j^{th} graded component.

THEOREM 2.3. *Let $\mathcal{A}$ be an arrangement in the k–vector space V (k any field) and let $G \leqslant \mathrm{GL}(V)$ be a finite group such that $G\mathcal{A} \subseteq \mathcal{A}$. Let λ be a complex representation of G satisfying*

(2.3.1) For $H \in \mathcal{A}$, we have $\left(\operatorname{Res}^G_{G_H}(\lambda), 1\right)_{G_H} = 0$.

(2.3.2) $(\lambda, 1)_G = 0$.

Then for $j = 0, 1, \ldots, r$ we have

$$\left(A^j(\mathcal{A}), \lambda\right)_G = 0.$$

where $A(\mathcal{A}) = \bigoplus_{j=0}^{r} A^j(\mathcal{A})$ is the (complex) Orlik–Solomon algebra of $\mathcal{A}$.

Note that if $\mathcal{A}$ is not empty, the condition (2.3.2) is a consequence of (2.3.1).

COROLLARY 2.4. *With notation as in (2.3), assume $k = \mathbb{C}$ and write $M_{\mathcal{A}} = V \backslash \bigcup_{H \in \mathcal{A}} H$. Then $\left(H^j(M_{\mathcal{A}}, \mathbb{C}), \lambda\right) = 0$ for $j = 0, 1, \ldots, r$.*

COROLLARY 2.5. *With notation as in (2.3), suppose $\mathcal{A} \neq \emptyset$ and that there is a homomorphism $d : G \to k^\times$ with non–trivial restriction to G_H (each $H \in \mathcal{A}$). Then $\{d(g) \mid g \in G\}$ is a (finite) cyclic subgroup $d(G)$ of $k^\times$. Let $\iota : d(G) \to \mathbb{C}^\times$ be any monomorphism and define $\delta(g) = \iota(d(g))$ $(g \in G)$. Then $\left(A^j(\mathcal{A}), \delta\right)_G = 0$ for $j = 0, \ldots, r$.*

COROLLARY 2.6. *Let G $(\neq 1)$ be a finite unitary reflection group of rank r acting on the complex vector space V. If M_W is the complement of the reflecting hyperplanes of G in V, then $\left(H^j(M_W, \mathbb{C}), \det^{\pm 1}\right)_G = 0$ for $j = 0, 1, \ldots, r$.*

3. PROOFS

Let $X \in L(\mathcal{A})$ and write $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \leqslant X \text{ in } L(\mathcal{A})\}$. Then $\mathcal{A}_X$ is an arrangement in V which is stabilised by N_X (see (2.2). Hence we may speak of the N_X-module $A^j(\mathcal{A}_X)$ $(j = 0, 1, \ldots)$.

Lemma 3.1. *For each $j \in \{0, 1, 2 \ldots\}$, there is an isomorphism of G-modules: $A^j(\mathcal{A}) \xrightarrow{\sim} \bigoplus_{X \in (L/G)_j} \operatorname{Ind}_{N_X}^G (A^j(\mathcal{A}_X))$ where $(L/G)_j$ denotes a set of representatives of the G-orbits on $L(\mathcal{A})_j = \{Y \in L(\mathcal{A}) \mid r(Y) = j\}$.*

Proof: This is essentially proved in [**6**, (2.4) and following remarks], using results from [**7**]. Although the context in [**6**] is more specific, the arguments there yield the statement (3.1). □

(3.2) Proof of Theorem (2.3): Observe that the hypothesis (2.3.1) is equivalent to

(2.3.1)′ For $V \neq X \in L(\mathcal{A})$, we have $(\lambda, 1)_{G_X} = 0$.

This is because for $X \leqslant Y$ in $L(\mathcal{A})$ we have $G_X \leqslant G_Y$. Hence $(\lambda, 1)_{G_H} = 0$ for $H \in \mathcal{A}$ implies that $(\lambda, 1)_{G_X} = 0$ for $X \in L(\mathcal{A})$, $X \neq V$, because the elements of $\mathcal{A}$ are the atoms of $L(\mathcal{A})$.

Next, we have from (3.1), using Frobenius reciprocity,

$$(A^j(\mathcal{A}), \lambda)_G = \sum_{X \in (L/G)_j} \left(A^j(\mathcal{A}_X), \operatorname{Res}_{N_X}^G(\lambda)\right)_{N_X}. \tag{3.2.1}$$

Observe that if $j > 0$, the hypotheses (2.3.1) and (2.3.2) apply with $\mathcal{A}_X$, N_X, $\operatorname{Res}_{N_X}^G(\lambda)$ in place of $\mathcal{A}$, G, λ respectively. This is because $L(\mathcal{A}_X) = \{Y \in L(\mathcal{A}) \mid Y \leqslant X\}$; hence if (2.3.1)′ holds for $Y \in L(\mathcal{A})$, it holds also for $Y \in L(\mathcal{A}_X)$. To check (2.3.2), observe that since $\operatorname{Res}_{G_X}^G(\lambda)$ does not contain the trivial representation of G_X, we have a fortiori that $\operatorname{Res}_{N_X}^G(\lambda)$ does not contain the trivial representation of N_X. Hence $\left(\operatorname{Res}_{N_X}^G(\lambda), 1\right)_{N_X} = 0$, proving (2.3.2). If $j = 0$, the above assertion is clear from (2.3.2).

If $X \in L_j$ (that is, $r(X) = j$) then $r(\mathcal{A}_X) = j$. It follows that if (2.3) holds for $j = r = r(\mathcal{A})$, then by applying it to all triples $\left(\mathcal{A}_X, N_X, \operatorname{Res}_{N_X}^G(\lambda)\right)$ with $j = r(\mathcal{A}_X)$ and using (3.2.1), one obtains (2.3) for all j. We have therefore shown

(3.2.2) It suffices to prove (2.3) for $j = r(\mathcal{A}) = r$.

Now the character of G on $A^r(\mathcal{A})$ has been calculated by Orlik and Solomon in terms of the lattice $L = L(\mathcal{A})$ (see [**8**, (6.1.14)]):

(3.2.3) For $g \in G$, trace $(g, A^r(\mathcal{A})) = (-1)^r \mu_g(T(L))$ where μ_g is the Möbius function of $L^g = \{Y \in L \mid gY = Y\}$.

It follows that (abusing notation by identifying λ with its character)

$$(3.2.4) \qquad (A^r(\mathcal{A}),\lambda) = (-1)^r\,|G|^{-1}\sum_{g\in G}\mu_g(T(L))\lambda(g) = (-1)^r\,|G|^{-1}\sum_{g\in G}(1+\mu_g(T(L)))\lambda(g),$$

since (2.3.2) implies that $\sum_{g\in G}\lambda(g)=0$. But by a result of Rota [9], $1+\mu_g(T(L)) = \chi(L^g)$, where χ denotes Euler characteristic. Thus

$$1+\mu_g(T(L)) = \sum_{j=0}^{r-2}(-1)^j n_j(L^g),$$

where $n_j(L^g)$ is the number of chains $\sigma = X_0 < \ldots < X_j$ with $X_i \in L^g\backslash\{T(L),V\}$. Hence

$$(3.2.5) \qquad \sum_{g\in G}(1+\mu_g(T(L)))\lambda(g) = \sum_{g\in G}\sum_{j=0}^{r-2}(-1)^j n_j(L^g)\lambda(g) = \sum_{j=0}^{r-2}(-1)^j\sum_{\sigma\in C_j(L)}\left(\sum_{g\in G_\sigma}\lambda(g)\right),$$

where $C_j(L)$ is the set of chains $X_0 < \ldots < X_j = \sigma$ with $X_i \in L\backslash\{T(L),V\}$ and G_σ is the stabiliser of σ in G, that is, $\{g\in G \mid gX_i = X_i \text{ for } i=0,1,\ldots,j\}$ (the last equality is obtained by reversing the order of summation in the previous one). But G_σ contains G_{X_0}, since if $g\in G_{X_0}$, g fixes X_0 pointwise and hence fixes any subspace of X_0. (Recall $X_0 < X_i$ means that X_i is a subspace of X_0.) It follows that since $X_0 \neq V$, the restriction of λ to G_σ does not contain 1_{G_σ}, whence the inner sum $\sum_{g\in G_\sigma}\lambda(g)=0$.

Thus from (3.2.4), we obtain $(A^r(\mathcal{A}),\lambda)=0$ and by (3.2.2) the proof is complete. ∎

(3.3) Deduction of Corollaries. Corollary (2.4) follows from the statement [**8**, (5.4.14)].

(3.3.1) Suppose $\mathcal{A}$ is a complex arrangement in (2.3). There is a G–equivariant isomorphism of graded $\mathbb{C}$–algebras: $A(\mathcal{A})\to H^*(M_{\mathcal{A}},\mathbb{C})$.

The G–equivariance is not pointed out in [**8**, 6.1.14], but is obvious from the isomorphism, which is explicit.

Suppose now that we have the situation of (2.5). The character δ clearly satisfies the conditions of (2.3) and (2.5) follows immediately.

In the situation of (2.6), for each $X\in L$, $X\neq V$, G_X contains a non–trivial reflection. Hence $\mathrm{Res}^G_{G_X}(\det^{\pm 1})$ is non–trivial and the result follows from (2.4).

4. A generalisation

Essentially the same proof as (3.2) yields the following slightly more general result.

Theorem 4.1. *Let $\mathcal{A}$ be an arrangement in the k-vector space V (k any field) and let $G \leqslant \mathrm{GL}(V)$ be a finite group such that $G\mathcal{A} \subseteq \mathcal{A}$. Let λ be a complex representation of G satisfying*

(4.1.1) For any chain $\sigma = X_0 < X_1 < \ldots < X_j$ in $L(\mathcal{A})$ (including the empty chain), we have $\left(Res^G_{G_\sigma}(\lambda), 1\right)_{N_\sigma} = 0$, where N_σ is the isotropy group of σ in G.

Then for $j = 0, 1, \ldots, r$ we have

$$\left(A^j(\mathcal{A}), \lambda\right)_G = 0.$$

where $A(\mathcal{A}) = \bigoplus_{i=0}^{j} A^j(\mathcal{A})$ is the (complex) Orlik-Solomon algebra of $\mathcal{A}$.

Remark 4.2. An immediate consequence of (4.1) is that if $\mathcal{A}$ is the arrangement of all hyperplanes in an n-dimensional vector space over the finite field $\mathbb{F}_q$ and G=GL(n, $\mathbb{F}_q$), then only pricipal series representations of G may appear in $A(\mathcal{A})$. This result is of course not new.

5. An application – finite orthogonal groups

Let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$ (q odd) and let $G = O^{\pm}(n, q)$ be the isometry group of a non-degenerate symmetric bilinear form $\beta(\ ,\ )$ on V. It is well known (see [**8**, 6.32] or [**11**]) that G is generated by reflections (isometries fixing a hyperplane pointwise) in a set of hyperplanes of V. Any such reflection r has the form

$$r(v) = v - 2\frac{\beta(v, v_0)}{\beta(v_0, v_0)}v_0 \quad (v \in V) \tag{5.1}$$

for some $v_0 \in V$ such that $\beta(v_0, v_0) \neq 0$, that is, for some non-isotropic v_0.

It follows easily that the set $\mathcal{A}$ of hyperplanes corresponding to reflections in G is

$$\mathcal{A} = \{u^\perp \mid u \text{ is a non-isotropic vector in } V\}. \tag{5.2}$$

We refer to the hyperplanes of (5.2) as *non-isotropic*. Note that in [**8**, 6.32] Orlik and Terao erroneously state that $\mathcal{A}$ is the set of all hyperplanes of V.

Our result (2.5) may now be applied as follows.

Theorem 5.3. *Let $\beta(\ ,\ )$ be a symmetric, non-degenerate bilinear form on the finite dimensional vector space V over $\mathbb{F}_q$ (q odd). Assume $\dim V \geqslant 2$. Let G be the isometry group of (V, β) and let δ be the "sign" character of G ($\delta(g) = \det g = \pm 1 \in \mathbb{C}$*

for $g \in G$). If $\mathcal{A}$ is the arrangement of non-isotropic hyperplanes of V, then G acts on $(V, \mathcal{A})$ and we have

$$\left(A^j(\mathcal{A}), \delta\right)_G = 0 \quad \textit{for} \quad j = 0, 1, \ldots, \dim V - 2.$$

Using the fact that $A^n(\mathcal{A}) \cong H_{n-2}(L(\mathcal{A}))$ where $L(\mathcal{A})$ is the lattice of the arrangement $\mathcal{A}$ and $n = \dim V$, we deduce immediately

Corollary 5.4. *Let L be the lattice of intersections of the non-isotropic hyperplanes of (V, β) (notation as in (5.3)). Then $(H_{n-2}(L), \delta)_G = 0$, where H_{n-2} denotes homology with complex coefficients of the order complex of L and δ and G are as in (5.3).*

Example 5.5. Take $n = 2$ in (5.4) and let $\beta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1 y_1 + x_2 y_2$.

It is then easily verified that

$$G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in \mathbb{F}_q,\ a^2 + b^2 = 1 \right\}.$$

The representation $H_0(L)$ is then the permutation representation of G on the non-isotropic lines of $\mathbb{F}_q^2$ and the formula (5.4) reads as follows.

(5.5.1) Let $n_0 = \#\{(a, b) \in \mathbb{F}_q^2 \mid a^2 + b^2 = 1\}$. Then $n_0 = q + 1$ (respectively, $q - 1$) if -1 is a non-square (respectively, square) in $\mathbb{F}_q$.

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