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# WHAT IS THE TIME VALUE OF A STREAM OF INVESTMENTS?

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**Abstract**

The titular question is investigated for fairly general semimartingale investment and asset price processes. A discrete-time consideration suggests a stochastic differential equation and an integral expression for the time value in the continuous-time framework. It is shown that the two are equivalent if the jump part of the price process converges. The integral expression, which is the answer to the titular question, is the sum of all investments accumulated with returns on the asset (a stochastic integral) plus a term that accounts for the possible covariation between the two processes. The arbitrage-free price of the time value is the expected value of the sum (i.e. integral) of all investments discounted with the locally risk-free asset.

*Keywords:* Cash flow; compounding asset; discounting; stochastic exponential

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## 1. The problem

Consider a stream of payments – incomes less expenditures – commencing at time 0 (say) and totaling $A_t$ at time $t \geq 0$. The payments are currently and instantaneously invested in an asset with price $S_t$ per share at time $t \geq 0$. The functions $A$ and $S$ are assumed to be right-continuous with left limits (RCLL) and, moreover, $S$ is assumed to be strictly positive. The question is: what is the value $U_t$, at time $t$, of the total investments compounded with returns on $S$? There is an answer that is indisputably true in discrete-time models, and carries over to continuous time if the payments are lump sums due at isolated points in time. In more general continuous-time models, the answer is less simple and depends very much on the path of the functions $A$ and $S$: we find that $U_t$ is the sum of all investments accumulated with returns plus a term arising from the optional covariation between the two processes (if any). This statement is imprecise, and we should add that the 'sum of investments accumulated with returns' must, in general, be the (stochastic) integral of the accumulation factor with respect to the payment process. For this integral to be proper, the integrand must be predictable. However, the accumulation factor acts in arrears and is therefore not predictable in general. The optional covariance term collects what has not been accounted for in the stochastic integral.

This phenomenon has been occasionally observed in recent finance and insurance literature. The optional covariance term appears in [2] and [7], which both addressed certain situations in life insurance mathematics where payments and interest are driven by the same diffusion

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processes, and in [1], where it emerged in the equilibrium price when the risky asset and the deflator were correlated diffusion processes. Paulsen [4] carried through a systematic study of the phenomenon in a general semimartingale setting and pursued the matter in a later unpublished work [5]. In all works cited, the covariance term arose from solving the stochastic differential equation, which was taken for granted. The question remains: is the differential equation the basic relationship from which the process $U$ should be derived, or is there a way of motivating both the differential equation for $U$ and the very form of the process $U$, and then establishing that the two are consistent?

The present paper proposes a resolution: working with fairly general semimartingale processes, it first motivates both the claimed formula for $U_t$ and the stochastic differential equation for $\mathrm{d}U_t$ by inspection of their respective discrete-time counterparts, which are fully transparent, and then establishes that the former is the solution to the latter. The latter part is a special case of the general exponential equation with an exogenous driving term, which has been solved for quite general semimartingales (see [6]). The general solution is rather involved, however, and our aim is to justify the appealing formula suggested by the discrete-time heuristics. This is achieved by assuming that the jump part of the asset price process is not too unruly. A further result is that, in a financial market model, the arbitrage-free price of $U_t$ at time 0 is the expected value of the sum (i.e. integral) of all investments discounted with the locally risk-free asset.

Basic notions and results in stochastic calculus, including Itô's formula, are taken as prerequisites throughout (see [6]).

## 2. Discrete time

For a purely discrete stream of investments into an account, with $A_j - A_{j-1}$ deposited or withdrawn at time $j = 1, 2, \dots$, the statement of account at time $j$ (the end of year $j$) exhibits the calculation

$$U_j = U_{j-1} + U_{j-1}\frac{S_j - S_{j-1}}{S_{j-1}} + A_j - A_{j-1}.$$

In plain words, the balance at the end of the year is the sum of the previous balance, the interest earned on that balance (at rate $(S_j - S_{j-1})/S_{j-1}$ in year $j$), and the latest movement in the account. In difference form, the relationship reads

$$U_j - U_{j-1} = U_{j-1}\frac{S_j - S_{j-1}}{S_{j-1}} + A_j - A_{j-1}. \tag{2.1}$$

From the recursive formula

$$U_j = U_{j-1}\frac{S_j}{S_{j-1}} + A_j - A_{j-1},$$

starting from $U_0 = 0$, we obtain

$$U_j = S_j \sum_{i=1}^{j} S_i^{-1}(A_i - A_{i-1}), \tag{2.2}$$

which can be explained as follows. The payment $A_i - A_{i-1}$ at time $i$ purchases $S_i^{-1}(A_i - A_{i-1})$ units of the asset. At time $j$, the portfolio consists of

$$\sum_{i=1}^{j} S_i^{-1}(A_i - A_{i-1})$$

units of the asset, each of which has value $S_j$ at time $j$. The relationship (2.2), recast as

$$S_j^{-1}U_j = \sum_{i=1}^{j} S_i^{-1}(A_i - A_{i-1}), \tag{2.3}$$

says that the $S$-discounted time value is equal to the $S$-discounted value of all investments.

## 3. From discrete to continuous time

For a fixed time $t > 0$, let $0 = t_0 < t_1 < \cdots < t_n = t$ and consider a discrete payment stream with payments at times $t_1, \ldots, t_n$.

The relationship (2.1) carries over to

$$U_{t_j} - U_{t_{j-1}} = U_{t_{j-1}} \frac{S_{t_j} - S_{t_{j-1}}}{S_{t_{j-1}}} + A_{t_j} - A_{t_{j-1}}. \tag{3.1}$$

The continuous-time analogue is the stochastic differential equation (SDE)

$$\mathrm{d}U_t = U_{t-} \frac{\mathrm{d}S_t}{S_{t-}} + \mathrm{d}A_t, \tag{3.2}$$

with initial condition $U_0 = 0$. (Informally, think of approximating the continuous payment stream with a discrete one and refine the partition.) It is crucial that the candidate integrand $U_{t_{j-1}}/S_{t_{j-1}}$ in (3.1) is evaluated at time $t_{j-1}$ and that its increment $S_{t_j} - S_{t_{j-1}}$ is a forward difference.

Similarly, (2.3) carries over to

$$S_{t_j}^{-1}U_{t_j} = \sum_{i=1}^{j} S_{t_i}^{-1}(A_{t_i} - A_{t_{i-1}}).$$

Now, the sum on the right-hand side is not a candidate for a stochastic integral, since the integrand $S_{t_i}^{-1}$ is evaluated at the end of the interval over which the increment $A_{t_i} - A_{t_{i-1}}$ is formed. Therefore, we decompose it into the sum of a term that is a candidate for a stochastic integral and a term that is a sum of products of increments and, thus, is a candidate for an optional covariance process, as follows:

$$\begin{aligned} S_{t_j}^{-1}U_{t_j} &= \sum_{i=1}^{j} S_{t_{i-1}}^{-1}(A_{t_i} - A_{t_{i-1}}) + \sum_{i=1}^{j}(S_{t_i}^{-1} - S_{t_{i-1}}^{-1})(A_{t_i} - A_{t_{i-1}}) \\ &= \sum_{i=1}^{j} S_{t_{i-1}}^{-1}(A_{t_i} - A_{t_{i-1}}) - \sum_{i=1}^{j} \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}S_{t_i}}(A_{t_i} - A_{t_{i-1}}). \end{aligned}$$

This suggests the following continuous-time analogue:

$$S_t^{-1}U_t = \int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau + [S^{-1}, A]_t \tag{3.3}$$

$$= \int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau - \int_0^t S_{\tau-}^{-1}S_\tau^{-1}\,\mathrm{d}[S, A]_\tau. \tag{3.4}$$

## 4. Solution to the problem

In the following, we will be working in some probability space endowed with a suitable filtration satisfying the usual conditions. For an RCLL process $X$, we denote its continuous part by $X^{\mathrm{c}}$ and its jumps by $\Delta X_t = X_t - X_{t-}$.

The previous section served to motivate taking the SDE (3.2) as the defining relationship for the value process $U$ and conjecturing that this process is explicitly given by (3.3)–(3.4). In the present section, we will prove this conjecture. The problem is inextricably connected to the general exponential equation with an exogenous driving term, $U_t = A_t + \int_0^t U_{\tau-}\,\mathrm{d}Z_\tau$ (see [6, Section 9 of Chapter V] for a context-free treatment with $Z$ continuous). The solution is rather involved. The simple solution aimed at here is obtained at the expense of a slight sacrifice of generality; the most unruly asset price processes are ruled out by the assumption that

$$\sum_{0<\tau\leq t} \Delta S_\tau \quad \text{converges almost surely for each } t > 0. \tag{4.1}$$

**Theorem 4.1.** *Let $A$ and $S$ be semimartingales, the latter strictly positive and satisfying (4.1). The solution to the SDE (3.2) is*

$$U_t = S_t\biggl(\int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau + [S^{-1}, A]_t\biggr) \tag{4.2}$$

$$= S_t\biggl(\int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau - \int_0^t S_{\tau-}^{-1}S_\tau^{-1}\,\mathrm{d}[S, A]_\tau\biggr). \tag{4.3}$$

*Proof.* This is an exercise in the use of Itô's formula (see e.g. [6]). We introduce

$$V_t = \int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau + [S^{-1}, A]_t,$$

which is the expression in parentheses in (4.2).

We will need the dynamics of the processes $A$, $S^{-1}$, $[S^{-1}, A]$, and $V$. Straightforwardly,

$$\mathrm{d}A_t = \mathrm{d}A_t^{\mathrm{c}} + \Delta A_t.$$

For the remaining processes, we apply Itŏ's formula in differential form, using the devices $X_t\,\mathrm{d}Y_t^{\mathrm{c}} = X_{t-}\,\mathrm{d}Y_t^{\mathrm{c}}$ and $\Delta X_t\,\mathrm{d}Y_t^{\mathrm{c}} = 0$ and the facts that $[X, Y]$ is of bounded variation, $[X, Y]^{\mathrm{c}} = [X^{\mathrm{c}}, Y^{\mathrm{c}}] = 0$ if $X$ (or $Y$) is of bounded variation, and $\Delta[X, Y]_t = \Delta X_t \Delta Y_t$. Under assumption (4.1) the following simple version of Itô's formula applies to $S$:

$$\mathrm{d}f(S_t) = f'(S_t)\,\mathrm{d}S_t^{\mathrm{c}} + \tfrac{1}{2}f''(S_t)\,\mathrm{d}[S, S]_t^{\mathrm{c}} + f(S_t) - f(S_{t-}).$$

For $f(x) = x^{-1}$, we obtain

$$\begin{aligned}\mathrm{d}S_t^{-1} &= -S_t^{-2}\,\mathrm{d}S_t^{\mathrm{c}} + \tfrac{1}{2}2S_t^{-3}\,\mathrm{d}[S, S]_t^{\mathrm{c}} + S_t^{-1} - S_{t-}^{-1} \\ &= -S_t^{-2}\,\mathrm{d}S_t^{\mathrm{c}} + S_t^{-3}\,\mathrm{d}[S, S]_t^{\mathrm{c}} - S_t^{-1}S_{t-}^{-1}\Delta S_t.\end{aligned}$$

Using this, we derive

$$\begin{aligned}\mathrm{d}[S^{-1}, A]_t &= -S_t^{-2}\,\mathrm{d}[S, A]_t^{\mathrm{c}} - S_t^{-1}S_{t-}^{-1}\Delta S_t\Delta A_t \\ &= -S_t^{-1}S_{t-}^{-1}(\mathrm{d}[S, A]_t^{\mathrm{c}} + \Delta S_t\Delta A_t) \\ &= -S_t^{-1}S_{t-}^{-1}\,\mathrm{d}[S, A]_t.\end{aligned} \tag{4.4}$$

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Straightforwardly,

$$\mathrm{d}V_t = S_{t-}^{-1}\,\mathrm{d}A_t + \mathrm{d}[S^{-1}, A]_t.$$

Drawing on these preliminaries, we integrate $U_t = S_t V_t$ by parts and obtain

$$\begin{aligned}
\mathrm{d}U_t &= \mathrm{d}S_t V_{t-} + S_{t-}\,\mathrm{d}V_t + \mathrm{d}[S, V]_t \\
&= \frac{\mathrm{d}S_t}{S_{t-}} U_{t-} + S_{t-}(S_{t-}^{-1}\,\mathrm{d}A_t + \mathrm{d}[S^{-1}, A]_t) + S_{t-}^{-1}\,\mathrm{d}[S, A]_t + \mathrm{d}[S, [S^{-1}, A]]_t \\
&= \frac{\mathrm{d}S_t}{S_{t-}} U_{t-} + \mathrm{d}A_t + S_{t-}\,\mathrm{d}[S^{-1}, A]_t + S_{t-}^{-1}\,\mathrm{d}[S, A]_t + \mathrm{d}[S, [S^{-1}, A]]_t. \qquad (4.5)
\end{aligned}$$

We now only need to establish that the three last terms in (4.5) sum to zero. Indeed, by virtue of (4.4), they are

$$\begin{aligned}
&S_{t-}(-S_t^{-1}S_{t-}^{-1}\,\mathrm{d}[S, A]_t) + S_{t-}^{-1}\,\mathrm{d}[S, A]_t + \Delta S_t \Delta S_t^{-1} \Delta A_t \\
&\quad = (-S_t^{-1} + S_{t-}^{-1})\,\mathrm{d}[S, A]_t + \Delta S_t \Delta S_t^{-1} \Delta A_t \\
&\quad = -\Delta S_t^{-1} \Delta S_t \Delta A_t + \Delta S_t \Delta S_t^{-1} \Delta A_t \\
&\quad = 0.
\end{aligned}$$

This proves that $U$ defined by (4.2) is the solution to (3.2). The form (4.3) follows from (4.4). Since the optional covariance process $[S, A]$ is RCLL and of bounded variation, the integral with respect to it is well defined path-wise in the Stieltjes sense.

## 5. Discussion of the result

If the covariance process $[S, A]$ vanishes, then (4.3) reduces to

$$U_t = S_t \int_0^t S_{\tau-}^{-1}\,\mathrm{d}A_\tau, \qquad (5.1)$$

which might be felt to be the natural answer. The reason why it is, in general, not correct became clear in the discrete-time analysis in Section 2; money earns interest only after it has been deposited (see (2.2)) and, therefore, discounting operates in arrears (see (2.3)). Thus, in (5.1), the term $S_{\tau-}^{-1}$ should be replaced by $S_\tau^{-1}$. This statement is fine for bounded variation processes $A$ and $S$, since such processes inherit the essential features of their discrete-time rudiments and the integral

$$\int_0^t S_\tau^{-1}\,\mathrm{d}A_\tau$$

is well defined path by path. The statement is not fine for diffusion processes $A$ and $S$ (it is indeed void since $S^{-1}$ is continuous). For such processes, it is admittedly a matter of choice to take the SDE (3.2) as the defining relationship. However, as we have seen, this choice leads to a coherent answer for all processes, whether of bounded variation or of diffusion type. We observe that $U$ defined by (4.2)–(4.3) is linear in $A$, as it should be.

The form (5.1) appeared, for example, in [3], where it was correct since $A$ and $S$ were independent Lévy processes. Stochastic independence is, of course, not sufficient to get rid of the covariance term in (4.3); to see this, just take $A$ and $S$ to be purely deterministic with some common jumps, for example $A_t = \lfloor t \rfloor$ and $S_t = \mathrm{e}^{r\lfloor t \rfloor}$, where $\lfloor t \rfloor$ denotes the integer part of $t$.

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## 6. The price of the accumulated investments

Suppose that $S$ is a traded asset in a market with some locally risk-free asset with price process $B_t = \exp(\int_0^t r_s \, \mathrm{d}s)$, and suppose that prices of contingent claims are expected $B$-discounted values under some martingale measure. The following result states that the price of the accumulated investments in $S$ is the sum of the prices of the payments, as it ought to be.

**Theorem 6.1.** *Let* Q *be a probability measure under which the discounted price process* $B_t^{-1} S_t$ *is a martingale, and assume that* $B_t^{-1} U_t$ *is integrable with respect to* Q*. Then*

$$\mathrm{E}^{\mathrm{Q}}[B_t^{-1} U_t] = \mathrm{E}^{\mathrm{Q}}\bigg[\int_0^t B_\tau^{-1} \, \mathrm{d}A_\tau\bigg]. \tag{6.1}$$

*Proof.* Introduce the discounted values $\tilde{S}_t = B_t^{-1} S_t$ and $\tilde{U}_t = B_t^{-1} U_t$. Use $\mathrm{d}B_t = B_t r_t \, \mathrm{d}t$ to write

$$\mathrm{d}S_t = B_t \, \mathrm{d}\tilde{S}_t + B_t r_t \, \mathrm{d}t \tilde{S}_t, \qquad \mathrm{d}U_t = B_t \, \mathrm{d}\tilde{U}_t + B_t r_t \, \mathrm{d}t \tilde{U}_t.$$

Inserting these expressions into (3.2) gives

$$B_t \, \mathrm{d}\tilde{U}_t + B_t r_t \, \mathrm{d}t \tilde{U}_t = B_t \tilde{U}_{t-} \frac{B_t \, \mathrm{d}\tilde{S}_t + B_t r_t \, \mathrm{d}t \tilde{S}_t}{B_t \tilde{S}_{t-}} + \mathrm{d}A_t.$$

Upon multiplying by $B_t^{-1}$ and cancelling terms (recall that $X_{t-} \, \mathrm{d}t = X_t \, \mathrm{d}t$), we obtain

$$\mathrm{d}\tilde{U}_t = \tilde{U}_{t-} \frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_{t-}} + B_t^{-1} \, \mathrm{d}A_t.$$

Integrating this equation gives

$$\tilde{U}_t = \int_0^t \tilde{U}_{\tau-} \frac{\mathrm{d}\tilde{S}_\tau}{\tilde{S}_{\tau-}} + \int_0^t B_\tau^{-1} \, \mathrm{d}A_\tau,$$

which is interesting in its own right. By forming the expected value under Q and using the fact that $\tilde{S}$ is a Q-martingale, we arrive at (6.1).

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