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ON ∞-COMPLEX SYMMETRIC OPERATORS

MUNEO CHŌ
Department of Mathematics, Kanagawa University,
Hiratsuka 259-1293, Japan
e-mail: chiyom01@kanagawa-u.ac.jp

EUNGIL KO
Department of Mathematics, Ewha Womans University,
Seoul 120-750, Korea
e-mail: eiko@ewha.ac.kr

and JI EUN LEE
Department of Mathematics-Applied Statistics,
Sejong University, Seoul 143-747, Korea
e-mail: jieun7@ewhain.net; jieunlee7@sejong.ac.kr
Dedicated to the memory of Professor Takayuki Furuta in deep sorrow

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Abstract. In this paper, we study spectral properties and local spectral properties of ∞-complex symmetric operators T. In particular, we prove that if T is an ∞-complex symmetric operator, then T has the decomposition property (δ) if and only if T is decomposable. Moreover, we show that if T and S are ∞-complex symmetric operators, then so is $T \otimes S$.

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1. Introduction. Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_{su}(T)$ for the spectrum, the point spectrum, the approximate point spectrum, and the surjective spectrum of T, respectively.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [**6**] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is complex symmetric with conjugation C. This concept is due to the fact that T is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [**6**]).

In 1970, J. W. Helton [**9**] initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j} = 0. \tag{1}$$

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In view of complex symmetric operators, using the identity (1), we define m-complex symmetric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C = 0$$

for some positive integer m. In this case, we say that T is m-complex symmetric with conjugation C. Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$. Then, T is an m-complex symmetric operator with conjugation C if and only if $\Delta_m(T) = 0$. Note that

$$T^*\Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T). \tag{2}$$

By (2), if T is m-complex symmetric with conjugation C, then T is n-complex symmetric with conjugation C for all $n \geq m$. It is clear that a 1-complex symmetric operator is complex symmetric. We now introduce the class of ∞-complex symmetric operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *an ∞-complex symmetric operator with conjugation C* if

$$\limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} = 0.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called a finite-complex symmetric operator if T is m-complex symmetric for some $m \geq 1$. All normal operators, algebraic operators of order 2, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, some Volterra integration operators, nilpotent operators of order k, and nilpotent perturbations of Hermitian operator are included in the class of m-complex symmetric operators. We refer the reader to [**5–8, 10, 11**], and [**2**] for more details. The class of ∞-complex symmetric operators is the large class which contains finite-complex symmetric operators.

Example 1.1. Let C be the canonical conjugation on $\mathcal{H}$ given by

$$C\left(\sum_{n=0}^{\infty} x_n e_n\right) = \sum_{n=0}^{\infty}\overline{x_n}e_n,$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. Given any $\epsilon > 0$, choose a positive integer N such that $\frac{1}{N} < \epsilon$. Fix any $m > N$. If W is the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{2^{m+n}}e_{n+1}$ $(n = 0, 1, 2, \ldots)$ for such m, then $T = I + W$ is an ∞-complex symmetric operator. Indeed, since W is a quasinilpotent operator, $\sigma(W) = \{0\}$, and $\Delta_m(T) = \Delta_m(W)$, it follows from Theorem 3.2 that

$$\begin{aligned}\|\Delta_m(T)\|^{\frac{1}{m}} &= \|\Delta_m(W)\|^{\frac{1}{m}}\\ &\leq \left(\sum_{j=0}^{m}\binom{m}{j}\|W^{*j}\|\|CW^{m-j}C\|\right)^{\frac{1}{m}}\\ &\leq \left(\sum_{j=0}^{m}\binom{m}{j}\|W^*\|^j\|W\|^{m-j}\right)^{\frac{1}{m}} \leq \left[2^m\left(\frac{1}{2^m}\right)^m\right]^{\frac{1}{m}} = \frac{1}{2^{m-1}} < \frac{1}{N} < \epsilon.\end{aligned}$$

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By taking limsup as $m \to \infty$ in the above inequality, we get that

$$\limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} \le \epsilon.$$

Since ϵ is arbitrary, it follows that T is an ∞-complex symmetric operator.

The paper is organized as follows. In Section 3, we focus on spectral properties and local spectral properties of ∞-complex symmetric operators T. In particular, we show that if T is an ∞-complex symmetric operator, then T has the decomposition property (δ) if and only if T is decomposable. In Section 4, we prove that if T and S are ∞-complex symmetric operators, then so is $T \otimes S$. As some applications, we give several examples of such operators.

2. Preliminaries. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. Given an operator $T \in \mathcal{L}(\mathcal{H})$ and a closed set $F \subseteq \mathbb{C}$, let $\mathcal{X}_T(F)$ consist of all $x \in \mathcal{H}$ such that there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{H}$ that satisfies

$$(T-\lambda)f(\lambda) = x$$

for all $\lambda \in \mathbb{C} \setminus F$. The space $\mathcal{X}_T(F)$ is called *glocal spectral subspace* of T. In particular, if T has the SVEP, then $\mathcal{X}_T(F) = H_T(F)$ holds. In general, $\mathcal{X}_T(F)$ is strictly smaller than the corresponding $H_T(F)$. We say that T has the *decomposition property* (δ) if for every open cover $\{U, V\}$ of $\mathbb{C}$, the decomposition

$$\mathcal{H} = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$$

holds. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well-known that

$$\text{Decomposable} \Rightarrow \text{Bishop's property } (\beta) \Rightarrow \text{SVEP}.$$

In general, the converse implications do not hold (see [**12**] and [**3**] for more details).

3. ∞-complex symmetric operators. In [**2**], the authors have studied spectral relations for an m-complex symmetric operator on $\mathcal{H}$. In this section, we provide

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several spectral properties of ∞-complex symmetric operators. Recall that for any $x, y \in \mathcal{H}$, two vectors x and y are C-orthogonal if $\langle Cx, y\rangle = 0$.

THEOREM 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator with conjugation C and let λ and μ be any distinct eigenvalues of T. Then, eigenvectors of T corresponding to λ and μ are C-orthogonal. Moreover, if $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$, then $\lim_{n_k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle = 0$ where $\langle Cx_{n_k}, y_{n_k}\rangle$ is any convergent subsequence of $\langle Cx_n, y_n\rangle$.*

Proof. Let λ and μ be distinct eigenvalues of T with respect to the corresponding unit eigenvectors x and y, respectively. Since $Tx = \lambda x$ and $Ty = \mu y$, it follows that $CTC(Cx) = \overline{\lambda} Cx$ and so

$$\begin{aligned}
\langle \Delta_m(T)Cx, y\rangle &= \left\langle \left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C\right)Cx, y\right\rangle \\
&= \left\langle \left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}\overline{\lambda}^{m-j}\right)Cx, y\right\rangle = \langle (T^*-\overline{\lambda})^m Cx, y\rangle \\
&= \langle Cx, (T-\lambda)^m y\rangle = \left\langle Cx, \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^j\lambda^{m-j}y\right\rangle \\
&= \langle Cx, (\mu-\lambda)^m y\rangle = (\overline{\mu}-\overline{\lambda})^m\langle Cx, y\rangle. \qquad (3)
\end{aligned}$$

Moreover, since $\|C\| = 1$, it follows from (3) that

$$\begin{aligned}
|(\overline{\mu}-\overline{\lambda})||\langle Cx, y\rangle|^{\frac{1}{m}} &= |(\overline{\mu}-\overline{\lambda})^m\langle Cx, y\rangle|^{\frac{1}{m}} \\
&= |\langle \Delta_m(T)Cx, y\rangle|^{\frac{1}{m}} \le \|\Delta_m(T)Cx\|^{\frac{1}{m}}\|y\|^{\frac{1}{m}} \le \|\Delta_m(T)\|^{\frac{1}{m}}.
\end{aligned}$$

By taking limsup as $m \to \infty$ in the above inequality, we obtain $\langle Cx, y\rangle = 0$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$. Then, $\lim_{n\to\infty}(CTC-\overline{\lambda})Cx_n = 0$ and so $\lim_{n\to\infty}(T^l-\mu^l)y_n = 0$ and $\lim_{n\to\infty}(CT^lC-\overline{\lambda}^l)Cx_n = 0$ for every $l \in \mathbb{N}$. If $\langle Cx_{n_k}, y_{n_k}\rangle$ is any convergent subsequence of $\langle Cx_n, y_n\rangle$ such that $\lim_{k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle = a$, then it suffices to show that $a = 0$. Note that for each fix $m \ge 1$, the following relations hold:

$$\begin{aligned}
|(\overline{\mu}-\overline{\lambda})^m a| &= \lim_{n_k\to\infty}\left|(\overline{\mu}-\overline{\lambda})^m\langle Cx_{n_k}, y_{n_k}\rangle\right| \\
&= \left|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\overline{\lambda}^{m-j}\overline{\mu}^j\lim_{n_k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle\right| \\
&= \left|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\lim_{n_k\to\infty}\langle (CT^{m-j}C)Cx_{n_k}, T^j y_{n_k}\rangle\right| \\
&= \left|\lim_{n_k\to\infty}\left\langle\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C\right)Cx_{n_k}, y_{n_k}\right\rangle\right| \\
&= \lim_{n_k\to\infty}|\langle \Delta_m(T)Cx_{n_k}, y_{n_k}\rangle| \le \|\Delta_m(T)\|. \qquad (4)
\end{aligned}$$

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Since T is an ∞-complex symmetric operator, it follows from (4) that

$$|(\overline{\mu}-\overline{\lambda})| \lim_{m\to\infty} |a|^{\frac{1}{m}} = \limsup_{m\to\infty} |(\overline{\mu}-\overline{\lambda})^m a|^{\frac{1}{m}} \leq \limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} = 0.$$

Since λ and μ are distinct values, $a = 0$. Hence, $\lim_{n_k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle = 0$. □

THEOREM 3.2. *Let Q be a quasinilpotent operator. Then, $T = aI + Q$ is an ∞-complex symmetric operator for all $a \in \mathbb{C}$.*

Proof. We first show that $\Delta_k(T) = \Delta_k(Q)$ for all $k \in \mathbb{N}$. If $k = 1$, it is true clearly. Assume that it holds when $k = m$. Then, it holds

$$\begin{aligned}\Delta_{m+1}(T) &= T^*\Delta_m(T) - \Delta_m(T)(CTC)\\ &= T^*\Delta_m(Q) - \Delta_m(Q)(CTC)\\ &= (\overline{a}I + Q^*)\Delta_m(Q) - \Delta_m(Q)(C(aI+Q)C)\\ &= Q^*\Delta_m(Q) - \Delta_m(Q)(CQC) = \Delta_{m+1}(Q).\end{aligned}$$

Therefore, $\Delta_k(T) = \Delta_k(Q)$ for all $k \in \mathbb{N}$. We next prove $\limsup \|\Delta_m(Q)\|^{\frac{1}{m}} = 0$. Since Q is quasinilpotent, for a given ϵ with $0 < \epsilon < 1$, there exists n_0 such that $\|Q^n\| < \epsilon^n$ for all $n \geq n_0$. Let $M = \max\{\|Q\|, \|Q^2\|, \ldots, \|Q^{n_0-1}\|\}$ and m be sufficiently large. We may assume $M \geq 1$. Then, we have

$$\begin{aligned}\Delta_m(Q) = &\sum_{j=0}^{n_0-1}(-1)^{m-j}\binom{m}{j} Q^{*j}CQ^{m-j}C\\ &+\sum_{j=n_0}^{m-n_0}(-1)^{m-j}\binom{m}{j} Q^{*j}CQ^{m-j}C\\ &+\sum_{j=m-n_0+1}^{m}(-1)^{m-j}\binom{m}{j} Q^{*j}CQ^{m-j}C.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\|\Delta_m(Q)\| &\leq M\sum_{j=0}^{n_0-1}\binom{m}{j}\|Q^{m-j}\|\\ &\quad+\sum_{j=n_0}^{m-n_0}\binom{m}{j}\|Q^{*j}\|\cdot\|Q^{m-j}\| + M\sum_{j=m-n_0+1}^{m}\binom{m}{j}\|Q^{*j}\|\\ &< M\sum_{j=0}^{n_0-1}\binom{m}{j}\epsilon^{m-j} + M\sum_{j=n_0}^{m-n_0}\binom{m}{j}\epsilon^j\cdot\epsilon^{m-j} + M\sum_{j=m-n_0+1}^{m}\binom{m}{j}\epsilon^j\\ &= M\epsilon^m\left(\sum_{j=0}^{n_0-1}\binom{m}{j}\epsilon^{-j} + \sum_{j=n_0}^{m-n_0}\binom{m}{j} + \sum_{j=m-n_0+1}^{m}\binom{m}{j}\epsilon^{j-m}\right)\\ &\leq M\epsilon^m\epsilon^{1-n_0}\left(\sum_{j=0}^{n_0-1}\binom{m}{j} + \sum_{j=n_0}^{m-n_0}\binom{m}{j} + \sum_{j=m-n_0+1}^{m}\binom{m}{j}\right)\\ &= M\epsilon^m\epsilon^{1-n_0}2^m,\end{aligned}$$

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due to the fact that $\max\{1, \epsilon^{-1}, \ldots, \epsilon^{1-n_0}\} = \epsilon^{1-n_0} \geq 1$. Hence,

$$\limsup_{m\to\infty} \|\Delta_m(Q)\|^{\frac{1}{m}} \leq 2\epsilon.$$

Since ϵ is arbitrary, $\limsup_{m\to\infty} \|\Delta_m(Q)\|^{\frac{1}{m}} = 0$. This completes the proof. □

REMARK 3.3. Let T be an m-complex symmetric operator with a conjugation C. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^* (see [**2**]). However, if T is an ∞-complex symmetric operator, this does not hold. For example, let C be the conjugation on $\mathcal{H}$ given by

$$C\left(\sum_{n=0}^{\infty} x_n e_n\right) = \sum_{n=0}^{\infty}(-1)^{n+1}\overline{x_n}e_n,$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ and let W be the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{n+1}e_{n+1}$ $(n = 0, 1, 2, \ldots)$. If $T = \lambda I + W^*$, then T is an ∞-complex symmetric operator by Theorem 3.2. Moreover, $(T - \lambda I)e_0 = W^*e_0 = 0$, but $(T^* - \overline{\lambda}I)Ce_0 = WCe_0 = We_0 = e_1 \neq 0$.

THEOREM 3.4. *If* $\{T_n\}$ *is a sequence of commuting* ∞*-complex symmetric operators with conjugation C such that* $\lim_{n\to\infty} \|T_n - T\| = 0$*, then T is also* ∞*-complex symmetric with conjugation C.*

Proof. We first claim that if T and Q are in $\mathcal{L}(\mathcal{H})$ with $TQ = QT$, then

$$\|\Delta_m(T+Q)\| \leq K^m\big(\max_{l\leq n\leq m} \|\Delta_n(T)\| + \max_{l\leq n\leq m} \|Q\|^n\big),$$

where $K = \max\{K_1, K_2\}$ with $K_1 = 2(2\|Q\| + 1)$ and $K_2 = 2(2\|T\| + \|Q^*\| + 1)$. In fact, since

$$\begin{aligned}[(a+b)-(c+d)]^m &= [(a-c)+(b-d)]^m \\ &= \sum_{i=0}^{m}(-1)^i\binom{m}{i}[(a-c)+b]^{m-i}d^i \\ &= \sum_{i=0}^{m}\sum_{j=0}^{m-i}(-1)^i\binom{m}{i}\binom{m-i}{j}b^j(a-c)^{m-i-j}d^i \\ &= \sum_{m_1+m_2+m_3=m}\binom{m}{m_1, m_2, m_3}b^{m_3}(a-c)^{m_1}d^{m_2},\end{aligned}$$

it follows that

$$\Delta_m(T+Q) = \sum_{m_1+m_2+m_3=m}\binom{m}{m_1, m_2, m_3}Q^{*m_3}\Delta_{m_1}(T)CQ^{m_2}C.$$

Let $l = [\frac{m}{3}]$ be the integer part of $\frac{m}{3}$. Put

$$M_i = \sum_{m_1+m_2+m_3=m \text{ and } m_i\geq l}\binom{m}{m_1, m_2, m_3}\|Q^{*m_3}\Delta_{m_1}(T)CQ^{m_2}C\|$$

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for $i = 1, 2, 3$. Since $m_1 + m_2 + m_3 = m$, it follows that $m_j \geq l$ for some $j = 1, 2, 3$. Therefore, we get that

$$\begin{aligned} \|\Delta_m(T+Q)\| &\leq \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} \|Q^{*m_3} \Delta_{m_1}(T) C Q^{m_2} C\| \\ &\leq M_1 + M_2 + M_3. \end{aligned} \tag{5}$$

We will estimate the constant M_i. Then, we have

$$\begin{aligned} M_1 &= \sum_{m_1+m_2+m_3=m \text{ and } m_1 \geq l} \binom{m}{m_1, m_2, m_3} \|Q^{*m_3} \Delta_{m_1}(T) C Q^{m_2} C\| \\ &\leq \sum_{m_1+m_2+m_3=m \text{ and } m_1 \geq l} \binom{m}{m_1, m_2, m_3} \|Q^*\|^{m_3} \|\Delta_{m_1}(T)\| \|Q\|^{m_2} \\ &\leq \max_{l \leq n \leq m} \|\Delta_n(T)\| \cdot \sum_{m_1+m_2+m_3=m \text{ and } m_1 \geq l} \binom{m}{m_1, m_2, m_3} \|Q^*\|^{m_3} \|Q\|^{m_2} \\ &= \max_{l \leq n \leq m} \|\Delta_n(T)\| \cdot (\|Q^*\| + \|Q\| + 1)^m \\ &= \max_{l \leq n \leq m} \|\Delta_n(T)\| \cdot (2\|Q\| + 1)^m \\ &= \max_{l \leq n \leq m} \|\Delta_n(T)\| \cdot \left(\frac{K_1}{2}\right)^m. \end{aligned} \tag{6}$$

Since $\|\Delta_k(T)\| \leq 2^k \|T\|^k$ for all k, it follows from a similar method of (6) that

$$\begin{aligned} M_2 &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot \sum_{m_1+m_2+m_3=m \text{ and } m_2 \geq l} \binom{m}{m_1, m_2, m_3} \|Q^*\|^{m_3} \|\Delta_{m_1}(T)\| \\ &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot \sum_{m_1+m_2+m_3=m \text{ and } m_2 \geq l} \binom{m}{m_1, m_2, m_3} \|Q^*\|^{m_3} (2\|T\|)^{m_1} \\ &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot (2\|T\| + \|Q^*\| + 1)^m \\ &= \max_{l \leq n \leq m} \|Q\|^n \cdot \left(\frac{K_2}{2}\right)^m \end{aligned}$$

and

$$\begin{aligned} M_3 &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot \sum_{m_1+m_2+m_3=m \text{ and } m_3 \geq l} \binom{m}{m_1, m_2, m_3} \|\Delta_{m_1}(T)\| \|Q\|^{m_2} \\ &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot \sum_{m_1+m_2+m_3=m \text{ and } m_3 \geq l} \binom{m}{m_1, m_2, m_3} (2\|T\|)^{m_1} \|Q\|^{m_2} \\ &\leq \max_{l \leq n \leq m} \|Q\|^n \cdot (2\|T\| + \|Q\| + 1)^m \\ &= \max_{l \leq n \leq m} \|Q\|^n \cdot \left(\frac{K_2}{2}\right)^m. \end{aligned}$$

Hence, (5) implies that

$$\begin{aligned} \|\Delta_m(T+Q)\| &\leq \left(\frac{K_1}{2}\right)^m \max_{l \leq n \leq m} \|\Delta_n(T)\| + 2\left(\frac{K_2}{2}\right)^m \max_{l \leq n \leq m} \|Q\|^n \\ &\leq K^m \big(\max_{l \leq n \leq m} \|\Delta_n(T)\| + \max_{l \leq n \leq m} \|Q\|^n \big), \end{aligned}$$

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where $K = \max\{K_1, K_2\}$ with $K_1 = 2(2\|Q\| + 1)$ and $K_2 = 2(2\|T\| + \|Q^*\| + 1)$. So this completes the proof of the claim.

If $T_nT_k = T_kT_n$ for all positive integers k, n, then $TT_n = T_nT$ for all $n \geq 1$. Given $0 < \epsilon < 1$, there exists n_0 such that $\|T - T_{n_0}\| \leq \epsilon$ and $\|\Delta_n(T_{n_0})\| \leq \epsilon^n$ for all $n \geq n_0$. By the above claim, for $m \geq 3n_0$ and $l = [\frac{m}{3}] \geq n_0$, we get that

$$\begin{aligned}\|\Delta_m(T)\|^{\frac{1}{m}} &= \|\Delta_m(T_{n_0} + T - T_{n_0})\|^{\frac{1}{m}} \\ &\leq K\big(\max_{l\leq n\leq m} \|\Delta_n(T_{n_0})\| + \max_{l\leq n\leq m} \|T - T_{n_0}\|^n\big)^{\frac{1}{m}} \\ &\leq 2^{\frac{1}{m}} K\epsilon^{\frac{l}{m}} (= 2^{\frac{1}{m}} K\epsilon^{\frac{1}{m}[\frac{m}{3}]}).\end{aligned}$$

Since ϵ is arbitrary, $\limsup_{m\to\infty} \|\Delta_m(T)\|^{\frac{1}{m}} = 0$. Hence, T is ∞-complex symmetric with conjugation C. □

Proposition 3.5. *Let R and T be in $\mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Assume that T is a complex symmetric operator with conjugation C and $RT = TR$. Then, the following statements hold:*

(i) RT is an m-complex symmetric operator with conjugation C if and only if R is an m-complex symmetric operator on $\overline{ran(T^m)}$.

(ii) If R is an ∞-complex symmetric operator with conjugation C, then RT is an ∞-complex symmetric operator with conjugation C.

Proof. (i) Since $T^* = CTC$ and $RT = TR$, it follows that

$$\begin{aligned}\Delta_m(RT) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(RT)^{*j}C(RT)^{m-j}C \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}R^{*j}T^{*j}CT^{m-j}R^{m-j}C \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}R^{*j}T^{*j}CT^{m-j}CCR^{m-j}C \\ &= T^{*m}\left[\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}R^{*m-j}CR^{m-j}C\right] = T^{*m}\Delta_m(R). \qquad (7)\end{aligned}$$

If RT is an m-complex symmetric operator with conjugation C, then from (7), we have $\langle T^{*m}\Delta_m(R)T^m x, x\rangle = 0$ and therefore $\langle \Delta_m(R)T^m x, T^m x\rangle = 0$ for all $x \in \mathcal{H}$. Hence, R is an m-complex symmetric operator on $\overline{ran(T^m)}$. If R is an m-complex symmetric operator, then from (7), we have $\Delta_m(RT) = 0$ and hence RT is an m-complex symmetric operator.

(ii) If R is an ∞-complex symmetric operator with conjugation C, then we obtain from (7) that

$$\|\Delta_m(RT)\|^{\frac{1}{m}} = \|T^{*m}\Delta_m(R)\|^{\frac{1}{m}} \leq \|T^*\|\|\Delta_m(R)\|^{\frac{1}{m}}.$$

Therefore, we have $\limsup_{m\to\infty} \|\Delta_m(RT)\|^{\frac{1}{m}} = 0$. Hence, RT is an ∞-complex symmetric operator. □

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THEOREM 3.6. *Let R and T be in $\mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. If $TS = ST$ and $S^*(CTC) = (CTC)S^*$ for a conjugation C, then*

$$\Delta_m(T+S) = \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T) \cdot \Delta_{m-j}(S), \tag{8}$$

where $\Delta_0(T) = \Delta_0(S) = I$. In particular, if T and S are m-complex symmetric and n-complex symmetric, respectively, then $T + S$ is $(m + n - 1)$-complex symmetric.

Proof. We will prove (8) by induction. If $m = 1$, then it is clear. So we consider $m = 2$. Since $TS = ST$ and $S^*(CTC) = (CTC)S^*$, it follows from (2) that

$$\begin{aligned}
\Delta_2(T+S) &= (T^* + S^*)\Delta_1(T+S) - \Delta_1(T+S)[C(T+S)C] \\
&= (T^* + S^*)(\Delta_1(T) + \Delta_1(S)) - [\Delta_1(T) + \Delta_1(S)][CTC + CSC] \\
&= \Delta_2(T) + T^*\Delta_1(S) - \Delta_1(S)CTC + S^*\Delta_1(T) - \Delta_1(T)CSC + \Delta_2(S) \\
&= \Delta_2(T) + 2\Delta_1(T)\Delta_1(S) + \Delta_2(S) \\
&= \sum_{j=0}^{2} \binom{m}{j} \Delta_j(T) \cdot \Delta_{2-j}(S),
\end{aligned}$$

where $\Delta_0(T) = \Delta_0(S) = I$. Therefore, (8) is true for $m = 2$. We assume that (8) holds for $m > 2$. Since

$$R^*\Delta_m(R) - \Delta_m(R)\,CRC = \Delta_{m+1}(R)$$

for arbitrary $R \in \mathcal{L}(\mathcal{H})$, it follows that

$$\begin{aligned}
\Delta_{m+1}(T+S) &= (T^* + S^*)\Delta_m(T+S) - \Delta_m(T+S)\,C(T+S)C \\
&= (T^* + S^*)\sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\Delta_{m-j}(S) \\
&\quad - \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\Delta_{m-j}(S)\,C(T+S)C \\
&= T^*\sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\Delta_{m-j}(S) + \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)S^*\Delta_{m-j}(S) \\
&\quad - \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)CTC\Delta_{m-j}(S) - \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\Delta_{m-j}(S)\,CSC \\
&= \sum_{j=0}^{m} \binom{m}{j} [T^*\Delta_j(T) - \Delta_j(T)CTC]\Delta_{m-j}(S) \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)[S^*\Delta_{m-j}(S) - \Delta_{m-j}(S)\,CSC] \\
&= \sum_{j=0}^{m} \binom{m}{j} \Delta_{j+1}(T)\Delta_{m-j}(S) + \sum_{j=0}^{m} \binom{m}{j} \Delta_j(T)\Delta_{m+1-j}(S) \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} \Delta_j(T)\Delta_{m+1-j}(S),
\end{aligned}$$

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where $\Delta_0(T) = \Delta_0(S) = I$. Therefore, it holds for every $m \in \mathbb{N}$. Using (8), we get the last statement. So this completes the proof. □

We next consider the decomposability of an ∞-complex symmetric operator. Put $F^* := \{\bar{z} : z \in F\}$ for any set F in $\mathbb{C}$.

THEOREM 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator with conjugation C. Then, the following statements hold:*

(i) $\mathcal{X}_{CTC}(F) \subset \mathcal{X}_{T^*}(F)$ for every closed set F in $\mathbb{C}$.

(ii) T has the decomposition property (δ) if and only if T is decomposable.

Proof. (i) Let F be a closed set in $\mathbb{C}$ and let $x \in \mathcal{X}_{CTC}(F)$. Then, there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{H}$ that satisfies $(CTC - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$.

CLAIM. The infinite series

$$g(\lambda) := \sum_{n=0}^{\infty} (-1)^n \Delta_n(T) \frac{f^{(n)}(\lambda)}{n!}$$

is uniformly convergence on all compact subset of $\mathbb{C} \setminus F$ and $\Delta_0(T) = I$.

Choose any $\mu \in \mathbb{C} \setminus F$. Set $E = \{z \in \mathbb{C} : |z - \mu| < \delta\}$ where δ is the distance from μ to F. Choose a $t \in \mathbb{R}$ with $t < \delta$ such that the disc $D = \{z \in \mathbb{C} : |z - \mu| \leq t\}$ is contained in $\mathbb{C} \setminus F$. Since f is continuous on the compact set D, it follows that $K = \sup\{\|f(\xi)\| : \xi \in D\}$ is finite. For each $\lambda \in D_0 \subsetneq D$, where $D_0 = \{z \in \mathbb{C} : |z - \mu| \leq s\}$ with $s < t$ and $n \in \mathbb{N}$, Cauchy's integral formula yields that

$$\left\|\frac{f^{(n)}(\lambda)}{n!}\right\| = \left\|\frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)d\xi}{(\xi - \lambda)^n}\right\| \leq \frac{1}{2\pi}\int_{\partial D} \frac{\|f(\xi)\||d\xi|}{(|\xi - \mu| - |\mu - \lambda|)^n} \leq \frac{Kt}{(t - s)^{n+1}}.$$

Since T is an ∞-complex symmetric operator, it follows that

$$\limsup_{m\to\infty} \sup_{\lambda \in D_0} \left\|\Delta_m(T)\frac{f^{(m)}(\lambda)}{m!}\right\|^{\frac{1}{m}} \leq \limsup_{m\to\infty} \sup_{\lambda \in D_0} \|\Delta_m(T)\|^{\frac{1}{m}} \left[\frac{Kt}{(t - s)^{m+1}}\right]^{\frac{1}{m}} = 0.$$

Therefore, the series in claim converges uniformly on D_0 by the root test. Since all compact subset of $\mathbb{C} \setminus F$ can be covered by a finite number of such D_0, it follows that $g(\lambda)$ converges uniformly on compact subset of $\mathbb{C} \setminus F$.
By Claim, $g : \mathbb{C} \setminus F \to \mathcal{H}$ is an analytic function in $\mathbb{C} \setminus F$. Moreover, since $(CTC - \lambda)f(\lambda) = x$, by induction, we have

$$(CTC - \lambda)f^{(n)}(\lambda) = nf^{(n-1)}(\lambda) \tag{9}$$

for every positive integer n. Since

$$(T^* - \lambda)\Delta_m(T) = \Delta_{m+1}(T) + \Delta_m(T)(CTC - \lambda),$$

it follows from (9) that

$$(T^* - \lambda)g(\lambda) = \sum_{m=0}^{\infty} (-1)^m (T^* - \lambda)\Delta_m(T)\frac{f^{(m)}(\lambda)}{m!}$$

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$$= \sum_{m=0}^{\infty}(-1)^m[\Delta_{m+1}(T) + \Delta_m(T)(CTC - \lambda)]\frac{f^{(m)}(\lambda)}{m!}$$
$$= \sum_{m=0}^{\infty}(-1)^m \Delta_{m+1}(T)\frac{f^{(m)}(\lambda)}{m!}$$
$$+(-1)^0\Delta_0(T)(CTC - \lambda)\frac{f^{(0)}(\lambda)}{0!}$$
$$+\sum_{m=1}^{\infty}(-1)^m \Delta_m(T)(CTC - \lambda)\frac{f^{(m)}(\lambda)}{m!}$$
$$= \sum_{m=0}^{\infty}(-1)^m \Delta_{m+1}(T)\frac{f^{(m)}(\lambda)}{m!} + (CTC - \lambda)f(\lambda)$$
$$+\sum_{m=1}^{\infty}(-1)^m \Delta_m(T)\frac{f^{(m-1)}(\lambda)}{(m-1)!} = x.$$

Hence, $(T^* - \lambda)g(\lambda) = x$ on $\mathbb{C} \setminus F$ and therefore $\mathcal{X}_{CTC}(F) \subset \mathcal{X}_{T^*}(F)$.

(ii) Since T is decomposable if and only if T and T^* has the decomposition property (δ) by [**12**, Theorems 1.2.29 and 2.5.5], it suffices to show that if T has the decomposition property (δ), then T^* has the decomposition property (δ). Let $\{U, V\}$ be an arbitrary open cover of $\mathbb{C}$ and $F \subseteq U$ and $G \subseteq V$ be selected closed sets whose interiors still cover $\mathbb{C}$. Then, $F \cap \sigma(T^*)$ and $G \cap \sigma(T^*)$ are compact such that $F \cap \sigma(T^*) \subseteq U$ and $G \cap \sigma(T^*) \subseteq V$.

Claim. For a closed set F in $\mathbb{C}$, $C\mathcal{X}_T(F) = \mathcal{X}_{CTC}(F^*)$ holds.

Let F be a closed set in $\mathbb{C}$ and let $x \in \mathcal{X}_{CTC}(F)$. Then, there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{H}$ that satisfies $(CTC - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. This yields that $(T - \overline{\lambda})Cf(\lambda) = Cx$ and so $(T - \lambda)Cf(\overline{\lambda}) = Cx$ for every $\lambda \in \mathbb{C} \setminus F^*$. Since $Cf(\overline{\lambda})$ is an analytic in $\mathbb{C} \setminus F^*$, it follows that $Cx \in \mathcal{X}_T(F^*)$ and therefore $x \in C\mathcal{X}_T(F^*)$. Thus, $\mathcal{X}_{CTC}(F) \subseteq C\mathcal{X}_T(F^*)$. The converse inclusion holds by a similar method.

Moreover, since T has the decomposition property (δ), it follows that $\{U, V\}$ is an open cover of $\mathbb{C}$ such that $\mathcal{H} = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$. From the above claim, we get that

$$\mathcal{H} = C\mathcal{H} = C\mathcal{X}_T(\overline{U}) + C\mathcal{X}_T(\overline{V}) = \mathcal{X}_{CTC}(\overline{U^*}) + \mathcal{X}_{CTC}(\overline{V^*}).$$

Hence, CTC also has the decomposition property (δ). Thus by (i), we get that

$$\begin{aligned}\mathcal{H} &= \mathcal{X}_{CTC}(F) + \mathcal{X}_{CTC}(G) \subseteq \mathcal{X}_{T^*}(F) + \mathcal{X}_{T^*}(G)\\ &\subseteq \mathcal{X}_{T^*}(F \cap \sigma(T^*)) + \mathcal{X}_{T^*}(G \cap \sigma(T^*)) \subseteq \mathcal{X}_{T^*}(\overline{U}) + \mathcal{X}_{T^*}(\overline{V}).\end{aligned}$$

Thus, $\mathcal{X}_{T^*}(\overline{U}) + \mathcal{X}_{T^*}(\overline{V}) = \mathcal{H}$. Hence, T^* has the decomposition property (δ). So this completes the proof. □

Let us recall that an operator $X \in \mathcal{L}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathcal{L}(\mathcal{H})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H})$ such that $XS = TX$.

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Furthermore, two operators S and T are *quasisimilar* if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$. A closed subspace $\mathcal{M}$ is *hyperinvariant* for T if it is invariant for every operator in $\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : TS = ST\}$ of T. Next, we give various useful results from Theorem 3.7 and [**12**].

COROLLARY 3.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator. If T has the decomposition property (δ), then the following statements hold:*

(i) *If $F \subset \mathbb{C}$ is closed, then the operator $S =: T/_{H_T(F)}$, induced by T, on the quotient space $\mathcal{H}/H_T(F)$ satisfies $\sigma(S) \subset \overline{\sigma(T) \setminus F}$.*
(ii) *If $\mathcal{M}$ is a spectral maximal space of T, then $\mathcal{M} = H_T(\sigma(T|_{\mathcal{M}}))$.*
(iii) *$f(T)$ is decomposable where f is any analytic function on some open neighbourhood of $\sigma(T)$.*
(iv) *If T has real spectrum on $\mathcal{H}$, then $\exp(iT)$ is decomposable.*
(v) *If $\sigma(T)$ is not singleton and $S \in \mathcal{L}(\mathcal{H})$ is quasisimilar to T, then S has a non-trivial hyperinvariant subspace.*
(vi) *$\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathcal{H}\}$.*

4. Tensor products of ∞-complex symmetric operators. Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ denote the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable complex Hilbert spaces. For operators $T \in \mathcal{L}(\mathcal{H}_1)$ and $S \in \mathcal{L}(\mathcal{H}_2)$, we define the *tensor product* operator $T \otimes S$ on $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by

$$(T \otimes S)\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j\right) = \sum_{j=1}^{n} \alpha_j T x_j \otimes S y_j.$$

Then, it is well known that $T \otimes S \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Since $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$ and $T \otimes I = \oplus_n^{\infty} T$, it is clear that an operator T is an m-complex symmetric operator with conjugation C if and only if $T \otimes I$ and $I \otimes T$ are m-complex symmetric operators with conjugation C. We replace the notation $\Delta_m(T; C)$ with $\Delta_m(T)$ as follows if necessary:

$$\Delta_m(T; C) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C.$$

Similarly, for conjugations C and D on $\mathcal{H}$, we define $C \otimes D$ on $\mathcal{H} \otimes \mathcal{H}$ by

$$(C \otimes D)\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j\right) = \sum_{j=1}^{n} \overline{\alpha_j} C x_j \otimes D y_j.$$

Then, $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$ (see Lemma 4.6 or [**6**, Lemma 6]). In this section, we prove the following results.

THEOREM 4.1. *Let T and S be an m-complex symmetric operator and n-complex symmetric operator with conjugations C and D, respectively. Then, $T \otimes S$ is an $(m + n - 1)$-complex symmetric operator with conjugation $C \otimes D$.*

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THEOREM 4.2. *Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. Then, $T \otimes S$ is an ∞-complex symmetric operator with conjugation $C \otimes D$.*

COROLLARY 4.3. *Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. Then, $(T \otimes S)^*$ has the property* (β) if and only if $T \otimes S$ is decomposable.

Proof. The proof follows from Theorem 4.2 and [**12**]. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called a 2-*normal* operator if T is unitarily equivalent to an operator matrix of the form $\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where N_i are mutually commuting normal operators for $i = 1, 2, 3, 4$.

COROLLARY 4.4. *If T is an m-complex symmetric operator with a conjugation C and S is a 2-normal operator, then $T \otimes U^*NU$ is an m-complex symmetric operator where $S = U^*NU$ with $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$ and a unitary U.*

Proof. If S is a 2-normal operator, then there exists a unitary operator U such that $S = U^*NU$ where $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$. Thus, S is a complex symmetric operator from [**8**, Theorem 1]. Hence, $T \otimes U^*NU$ is an m-complex symmetric operator from Theorem 4.1. □

EXAMPLE 4.5. Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$. Assume that N is normal and $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$. Then, T is a 5-complex symmetric operator with conjugation C from [**2**, Example 3.2]. Hence, $T \otimes N = \begin{pmatrix} 0 & N & 0 \\ 0 & 0 & 2N \\ 0 & 0 & 0 \end{pmatrix}$ is 5-complex symmetric from Theorem 4.1.

Before the proof of Theorems 4.1 and 4.2, we first recapture the following lemma from [**1**].

LEMMA 4.6 [**1**]. *If C and D be conjugations on $\mathcal{H}$, then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.*

Assume that operators $T, S \in \mathcal{L}(\mathcal{H})$ satisfy $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Since $S^{*j}(CT^kC) = (CT^kC)S^{*j}$ holds for all $j, k \in \mathbb{N}$ and

$$(ab - cd)^m = [(a - c)b + c(b - d)]^m = \sum_{j=0}^{m} \binom{m}{j} (a - c)^{m-j} b^{m-j} c^j (b - d)^j,$$

it follows that

$$\begin{aligned} \Delta_m(TS) &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (TS)^{*j} C (TS)^{m-j} C \\ &= [(T^* - CTC)S^* + CTC(S^* - CSC)]^m \end{aligned}$$

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$$= \sum_{j=0}^{m} \binom{m}{j} (T^* - CTC)^{m-j} S^{*m-j} CT^j C (S^* - CSC)^j$$
$$= \sum_{j=0}^{m} \binom{m}{j} \Delta_{m-j}(T) S^{*m-j} CT^j C \Delta_j(S), \tag{10}$$

where $\Delta_m(T) = (T^* - CTC)^m$.

From (10), we have the following result.

LEMMA 4.7. *Let T and S be m-complex symmetric and n-complex symmetric with conjugation C, respectively. If T commutes with S and $S^*(CTC) = (CTC)S^*$, then TS is $(m+n-1)$-complex symmetric with conjugation C.*

Proof. From (10), it holds

$$\Delta_{m+n-1}(TS) = \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} \Delta_{m+n-1-j}(T) \cdot S^{*m+n-1-j} \cdot CT^j C \cdot \Delta_j(S).$$

(i) If $0 \le j \le n-1$, then $m+n-1-j \ge m$ and hence $\Delta_{m+n-1-j}(T) = 0$.
(ii) If $n \le j$, then $\Delta_j(S) = 0$.

Therefore, $\Delta_{m+n-1}(TS) = 0$. This completes the proof. □

Proof of Theorem 4.1. By Lemma 4.6, $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$. It is clear that $T \otimes I$ and $I \otimes S$ are m-complex symmetric and n-complex symmetric with conjugation $C \otimes D$, respectively. Since operators $T \otimes I$ and $I \otimes S$ satisfy

$$(T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I) \text{ and}$$

$$(I \otimes S)^* \big((C \otimes D)(T \otimes I)(C \otimes D) \big) = \big((C \otimes D)(T \otimes I)(C \otimes D) \big) (I \otimes S)^*,$$

it follows from Lemma 4.7 that $(T \otimes I)(I \otimes S) = T \otimes S$ is $(m+n-1)$-complex symmetric with conjugation $C \otimes D$. This completes the proof. □

LEMMA 4.8. *Let T and S be ∞-complex symmetric operators with conjugation C. Assume that $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Then, TS is an ∞-complex symmetric operator with conjugation C.*

Proof. Suppose that T and S are ∞-complex symmetric operators. Then, for a given $0 < \epsilon < 1$, there exist N_1 and N_2 such that $\|\Delta_{n_1}(T)\| \le \epsilon^n$ and $\|\Delta_{n_2}(S)\| \le \epsilon^n$ for $n_1 \ge N_1$ and $n_2 \ge N_2$. Put $N = max\{N_1, N_2\}$. Then, it suffices to show that there is a constant $K > 0$ such that for $m \ge 2N$,

$$\|\Delta_m(TS)\| \le K^m \epsilon^{\frac{m}{2}}.$$

Put $l = [\frac{m}{2}]$ denote the integer part of $\frac{m}{2}$. Then by Equation (10), we have

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$$\Delta_m(TS;C) = \sum_{j=0}^{l} \binom{m}{j} \Delta_{m-j}(T;C) S^{*m-j} C T^j C \Delta_j(S;C) + \sum_{j=l+1}^{m} \binom{m}{j} \Delta_{m-j}(T;C) S^{*m-j} C T^j C \Delta_j(S;C). \tag{11}$$

For $j \leq l = [\frac{m}{2}]$, $m - j \geq [\frac{m}{2}] = l \geq N$, $\|\Delta_{m-j}(T)\| \leq \epsilon^{m-j} \leq \epsilon^l$ holds. Since $\|C\| = 1$, $\|\Delta_j(S)\| \leq 2^j \|S\|^j$ for all $j \geq 1$. Thus by (11), we obtain

$$\begin{aligned}
&\|\sum_{j=0}^{l} \binom{m}{j} \Delta_{m-j}(T;C) S^{*m-j} C T^j C \Delta_j(S;C)\| \\
&\leq \sum_{j=0}^{l} \binom{m}{j} \|\Delta_{m-j}(T;C)\| \|S^{*m-j}\| \|C T^j C\| \|\Delta_j(S;C)\| \\
&\leq \sum_{j=0}^{l} \binom{m}{j} \epsilon^{m-j} \|S\|^{m-j} \|T^j\| (2^j \|S\|^j) \\
&\leq \epsilon^l \|S\|^m \sum_{j=0}^{m} \binom{m}{j} \|T\|^j 2^j = \epsilon^l \|S\|^m (1 + 2\|T\|)^m.
\end{aligned} \tag{12}$$

Similarly, for $j \geq l + 1 \geq N$, $\|\Delta_j(S)\| \leq \epsilon^l$, we get

$$\|\sum_{j=l+1}^{m} \binom{m}{j} \Delta_{m-j}(T;C) S^{*m-j} C T^j C \Delta_j(S;C)\| \leq \epsilon^l \|T\|^m (1 + 2\|S\|)^m. \tag{13}$$

From (12) and (13), we know that for $n \geq 2N$

$$\|\Delta_m(TS;C)\| \leq \epsilon^{[\frac{m}{2}]} \big(\|S\|^m (1 + 2\|T\|)^m + \|T\|^m (1 + 2\|S\|)^m \big).$$

Thus, $\limsup_{m \to \infty} \|\Delta_m(TS;C)\|^{\frac{1}{m}} = 0$. Hence, TS is an ∞-complex symmetric operator with conjugation C. □

Proof of Theorem 4.2. It is clear that $T \otimes I$ and $I \otimes S$ are ∞-complex symmetric operators on $\mathcal{H} \otimes \mathcal{H}$, respectively. Since $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$ by Lemma 4.6 and $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies

$$(I \otimes S)^* \big((C \otimes D)(T \otimes I)(C \otimes D) \big) = \big((C \otimes D)(T \otimes I)(C \otimes D) \big) (I \otimes S)^*,$$

it follows from Lemma 4.8 that $(T \otimes I)(I \otimes S) = T \otimes S$ is an ∞-complex symmetric operator with conjugation $C \otimes D$. □

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