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OPERATOR ALGEBRAS FROM THE DISCRETE HEISENBERG SEMIGROUP

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Abstract We study reflexivity and structural properties of operator algebras generated by representations of the discrete Heisenberg semigroup. We show that the left regular representation of this semigroup gives rise to a semi-simple reflexive algebra. We exhibit an example of a representation that gives rise to a non-reflexive algebra. En route, we establish reflexivity results for subspaces of $H^\infty(\mathbb{T}) \otimes \mathcal{B}(\mathcal{H})$.

Keywords: non-self-adjoint operator algebra; reflexivity; slice map; discrete Heisenberg semigroup

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1. Introduction

The theory of group representations has been a motivating force for operator algebra theory since the very beginnings of the subject. If π is a unitary representation of a group G, a much studied object is the weak-$*$ closed algebra generated by $\{\pi(g) \colon g \in G\}$. A special case of particular importance arises when π is the left regular representation $g \to L_g$ acting on $L^2(G)$; the algebra obtained in this way is the von Neumann algebra $\mathrm{VN}(G)$ of the group G.

These algebras are all self-adjoint. If $S \subseteq G$ is a semigroup, one can consider instead the non-self-adjoint algebra generated by $\{\pi(g) \colon g \in S\}$, possibly restricted to a common invariant subspace. The algebra of analytic Toeplitz operators is an instance of this construction. Such algebras have recently attracted considerable attention in the literature.

Let $\mathbb{F}_n^+$ be the free semigroup on n generators. The 'non-commutative Toeplitz algebra' is the weakly closed algebra $\mathcal{L}_n$ generated by the operators L_g, $g \in \mathbb{F}_n^+$, restricted to the invariant subspace $\ell^2(\mathbb{F}_n^+)$. It was introduced by Popescu in [**19**] and studied by him in a subsequent series of papers, and by Arias and Popescu in [**1**]. Later, Popescu [**20**] considered free products of semigroups satisfying certain additional properties, and

Davidson and and Pitts [**8**, **9**] and Davidson *et al*. [**7**] studied the algebra $\mathcal{L}_n$ within the more general framework of free semigroup algebras. On the other hand, non-self-adjoint algebras arising from representations of some Lie groups such as the Heisenberg group, the '$ax+b$ group' and $\mathrm{SL}_2(\mathbb{R})$ were considered by Katavolos and Power [**13**, **14**], by Levene [**16**] and by Levene and Power [**17**]. These authors studied problems including reflexivity and hyperreflexivity, determination of the invariant subspace lattice and semi-simplicity.

In this paper, we study operator algebras arising from representations of the discrete Heisenberg semigroup. Recall that the discrete Heisenberg group $\mathbb{H}$ consists of all matrices of the form

$$\begin{bmatrix} 1 & k & n \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}, \quad k, m, n \in \mathbb{Z}.$$

Let $\mathbb{H}^+$ be the semigroup consisting of all matrices in $\mathbb{H}$ with $k, m \in \mathbb{Z}^+$. We are interested in the weak-$*$ closed algebra $\mathcal{T}_L(\mathbb{H}^+)$ generated by the operators L_g, $g \in \mathbb{H}^+$, restricted to the invariant subspace $\ell^2(\mathbb{H}^+)$. In § 4, we show that $\mathcal{T}_L(\mathbb{H}^+)$ contains no non-trivial quasi-nilpotent or compact elements; in particular, it is semi-simple. We show that the commutant of $\mathcal{T}_L(\mathbb{H}^+)$ is the corresponding right regular representation and we identify the centre and the diagonal. In § 5 we prove that $\mathcal{T}_L(\mathbb{H}^+)$ is reflexive using a direct integral decomposition and the results of § 3.

In § 6 we study a class of representations of $\mathbb{H}^+$ that arise from representations of the irrational rotation algebra studied by Brenken [**4**]. The latter, in the multiplicity free case, are parametrized by a cocycle and a measure. When the cocycle is trivial, we show that the weak-$*$ closed algebras generated by the restriction to $\mathbb{H}^+$ are unitarily equivalent to nest algebras or equal to $\mathcal{B}(\mathcal{H})$. We also exhibit a representation (corresponding to a non-trivial cocycle) that generates a non-reflexive algebra even for the weak operator topology.

In §§ 2 and 3 we develop a technique that allows us to handle the question of reflexivity of $\mathcal{T}_L(\mathbb{H}^+)$. We introduce and study a notion of reflexivity for spaces of operators acting on tensor products of Hilbert spaces, which we think is of independent interest. Using this notion, we generalize previous results of Kraus [**15**] and Ptak [**21**], establishing reflexivity for a class of subspaces of $\mathcal{T} \otimes B(\mathcal{H})$ (where $\mathcal{T}$ is the algebra of analytic Toeplitz operators).

1.1. Preliminaries and notation

The discrete Heisenberg group $\mathbb{H}$ is generated by

$$u = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The element w is central and $uv = wvu$.

We write $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$.

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If $P \in \mathcal{B}(\mathcal{H})$ is an (orthogonal) projection, we set $P^\perp = I - P$, where I is the identity operator. We denote by $\mathcal{B}(\mathcal{H})_*$ the predual of $\mathcal{B}(\mathcal{H})$, that is, the space of all weak-$*$ continuous functionals on $\mathcal{B}(\mathcal{H})$. If $x, y \in \mathcal{H}$, we write $\omega_{x,y}$ for the vector functional in $\mathcal{B}(\mathcal{H})_*$ given by $\omega_{x,y}(A) = \langle Ax, y\rangle$, $A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{E}$ is a subset of a vector space, $[\mathcal{E}]$ will stand for the linear span of $\mathcal{E}$.

The *pre-annihilator* $\mathcal{S}_\perp$ of a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is

$$\mathcal{S}_\perp = \{\omega \in \mathcal{B}(\mathcal{H})_* \colon \omega(A) = 0 \text{ for all } A \in \mathcal{S}\}.$$

The *reflexive hull* of $\mathcal{S}$ [**18**] is

$$\operatorname{Ref} \mathcal{S} = \{A \in \mathcal{B}(\mathcal{H}) \colon \omega_{x,y}(\mathcal{S}) = \{0\} \Rightarrow \omega_{x,y}(A) = 0 \text{ for all } x, y \in \mathcal{H}\}.$$

The subspace $\mathcal{S}$ is called *reflexive* if $\mathcal{S} = \operatorname{Ref} \mathcal{S}$.

If $\mathcal{L}$ is a collection of projections on $\mathcal{H}$,

$$\operatorname{Alg} \mathcal{L} = \{A \in \mathcal{B}(\mathcal{H}) \colon AL = LAL\}$$

is the algebra of all operators leaving the ranges of the elements of $\mathcal{L}$ invariant. It is easy to see that a unital subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is reflexive if and only if $\mathcal{A} = \operatorname{Alg} \mathcal{L}$ for some collection $\mathcal{L}$ of projections on $\mathcal{H}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be their Hilbert space tensor product. If $\mathcal{S}_i \subseteq \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, we let $\mathcal{S}_1 \otimes \mathcal{S}_2$ be the weak-$*$ closed subspace of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by the operators $A_1 \otimes A_2$, where $A_i \in \mathcal{S}_i$, $i = 1, 2$. If $A \in \mathcal{B}(\mathcal{H}_1)$, we write $A \otimes \mathcal{S}_2$ for the space $\mathbb{C}A \otimes \mathcal{S}_2$. If $\omega_i \in \mathcal{B}(\mathcal{H}_i)_*$, $i = 1, 2$, we let $\omega_1 \otimes \omega_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)_*$ be the unique weak-$*$ continuous functional satisfying $(\omega_1 \otimes \omega_2)(A_1 \otimes A_2) = \omega_1(A_1)\omega_2(A_2)$, $A_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$.

Finally, we let H^p be the Hardy space corresponding to p ($p = 2, \infty$), that is, the space consisting of all functions in $L^p(\mathbb{T})$ whose Fourier coefficients indexed by negative integers vanish. For each $\varphi \in H^\infty$, we let $T_\varphi \in \mathcal{B}(H^2)$ be the analytic Toeplitz operator with symbol φ, that is, the operator given by $T_\varphi f = \varphi f$, $f \in H^2$. We let

$$\mathcal{T} = \{T_\varphi \colon \varphi \in H^\infty\}$$

be the algebra of all analytic Toeplitz operators on H^2.

2. A reflexive hull for subspaces of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$

In this section, we introduce a reflexive hull for spaces of operators that act on the tensor product of two given Hilbert spaces. The results will be applied in § 3 to study reflexivity of subspaces of $\mathcal{T} \otimes \mathcal{B}(\mathcal{K})$ for a given Hilbert space $\mathcal{K}$.

Suppose a Hilbert space $\mathcal{H}$ decomposes as a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces. If $\omega \in \mathcal{B}(\mathcal{H}_1)_*$, then the right slice map $R_\omega \colon \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is the unique weak-$*$ continuous linear map with the property that $R_\omega(A \otimes B) = \omega(A)B$, whenever $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Similarly, one defines the left slice maps, denoted by L_τ,

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where $\tau \in \mathcal{B}(\mathcal{H}_2)_*$. We note that if $\omega = \omega_{\xi,\eta}$ for some vectors $\xi, \eta \in \mathcal{H}_1$, then for all $x, y \in \mathcal{H}_2$,

$$\langle R_\omega(T)x, y\rangle = \langle T(\xi \otimes x), \eta \otimes y\rangle, \quad T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2). \tag{2.1}$$

This equality shows that, when ω is a vector functional or, more generally, a weakly continuous functional, then R_ω is also weakly (that is, WOT–WOT) continuous (where WOT denotes weak operator topology).

If $\mathcal{S}$ is a weak-$*$ closed subspace of $\mathcal{B}(\mathcal{H}_1)$ and $T \in \mathcal{S} \otimes \mathcal{B}(\mathcal{H}_2)$, then clearly $L_\omega(T) \in \mathcal{S}$ for all $\omega \in \mathcal{B}(\mathcal{H}_2)_*$. The converse was proved in [**15**].

Lemma 2.1 (Kraus [15]). *Let $\mathcal{S}$ be a weak-$*$ closed subspace of $\mathcal{B}(\mathcal{H}_1)$ and let $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If $L_\omega(T) \in \mathcal{S}$ for all $\omega \in \mathcal{B}(\mathcal{H}_2)_*$, then $T \in \mathcal{S} \otimes \mathcal{B}(\mathcal{H}_2)$.*

Consider the set of vector functionals

$$\mathcal{E} = \{\omega_{\xi\otimes x,\eta\otimes y} : \xi, \eta \in \mathcal{H}_1,\ x, y \in \mathcal{H}_2\} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)_*.$$

The set $\mathcal{E}$ (as any subset of the dual of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ [**10**]) can be used to define a reflexive hull for subspaces of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Namely, if $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let

$$\operatorname{Ref}_e \mathcal{S} = \{T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) : \omega(\mathcal{S}) = \{0\} \Rightarrow \omega(T) = 0 \text{ for all } \omega \in \mathcal{E}\}.$$

It is clear that $\operatorname{Ref}_e(\mathcal{S})$ depends on the tensor product decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The following statements are easy consequences of the definition; we omit their proofs.

Lemma 2.2. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then*

(i) *$\operatorname{Ref}_e \mathcal{S}$ is a reflexive, hence weakly closed, subspace of operators,*

(ii) *$\operatorname{Ref} \mathcal{S} \subseteq \operatorname{Ref}_e \mathcal{S}$,*

(iii) *$\operatorname{Ref}_e \mathcal{S} = \operatorname{Ref}_e \operatorname{Ref} \mathcal{S} = \operatorname{Ref}_e \operatorname{Ref}_e \mathcal{S}$.*

It follows from Lemma 2.2 that if a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies $\operatorname{Ref}_e \mathcal{S} = \mathcal{S}$, then $\mathcal{S}$ is reflexive. Remark 2.8 shows that the converse does not hold.

Lemma 2.3. *Let $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H}_2)$ be subspaces. Then*

$$\operatorname{Ref}_e(\mathcal{U} \otimes \mathcal{V}) = (\mathcal{B}(\mathcal{H}_1) \otimes \operatorname{Ref} \mathcal{V}) \cap (\operatorname{Ref} \mathcal{U} \otimes B(\mathcal{H}_2)).$$

Proof. Note that a vector functional $\omega_{\xi\otimes x,\eta\otimes y} = \omega_{\xi,\eta} \otimes \omega_{x,y}$ annihilates $\mathcal{U} \otimes \mathcal{V}$ if and only if either $\omega_{\xi,\eta}$ annihilates $\mathcal{U}$ or $\omega_{x,y}$ annihilates $\mathcal{V}$. For if there exists $U \in \mathcal{U}$ with $\omega_{\xi,\eta}(U) \neq 0$, then for all $V \in \mathcal{V}$ we have $\omega_{\xi,\eta}(U)\omega_{x,y}(V) = 0$, and hence $\omega_{x,y}(V) = 0$.

Now let $T \in \operatorname{Ref}_e(\mathcal{U} \otimes \mathcal{V})$. Suppose that $\omega_{\xi,\eta} \in \mathcal{U}_\perp$. Then $\omega_{\xi\otimes x,\eta\otimes y}$ annihilates $\mathcal{U} \otimes \mathcal{V}$ for all $x, y \in \mathcal{H}_2$, and hence

$$\omega_{\xi,\eta}(L_{\omega_{x,y}}(T)) = (\omega_{\xi,\eta} \otimes \omega_{x,y})(T) = 0.$$

This shows that $L_{\omega_{x,y}}(T) \in \operatorname{Ref} \mathcal{U}$. Since $x, y \in \mathcal{H}_2$ are arbitrary, linearity and (norm) continuity of the map $\omega \to L_\omega$ yield $L_\omega(T) \in \operatorname{Ref} \mathcal{U}$ for all $\omega \in (\mathcal{B}(\mathcal{H}_2))_*$. By Lemma 2.1, $T \in \operatorname{Ref} \mathcal{U} \otimes \mathcal{B}(\mathcal{H}_2)$. Similarly, one obtains $T \in \mathcal{B}(\mathcal{H}_1) \otimes \operatorname{Ref} \mathcal{V}$.

Conversely, if $T \in (\mathcal{B}(\mathcal{H}_1) \otimes \operatorname{Ref} \mathcal{V}) \cap (\operatorname{Ref} \mathcal{U} \otimes \mathcal{B}(\mathcal{H}_2))$, then for each $\phi = \omega_{\xi,\eta} (\xi, \eta \in \mathcal{H}_1)$ we have $R_\phi(T) \in \operatorname{Ref} \mathcal{V}$. So, if $\omega_{x,y}$ is a vector functional annihilating $\mathcal{V}$, then it must annihilate $R_\phi(T)$, and hence

$$\phi(L_{\omega_{x,y}}(T)) = (\phi \otimes \omega_{x,y})(T) = \omega_{x,y}(R_\phi(T)) = 0.$$

Since $\phi = \omega_{\xi,\eta}$ with ξ, η arbitrary in $\mathcal{H}_1$, this implies $L_{\omega_{x,y}}(T) = 0$. Similarly, using the fact that all left slices of T must lie in $\operatorname{Ref} \mathcal{U}$, we see that

$$\omega_{\xi,\eta} \in \mathcal{U}_\perp \Rightarrow R_{\omega_{\xi,\eta}}(T) = 0.$$

Therefore, if $\omega_{\xi,\eta} \otimes \omega_{x,y}$ annihilates $\mathcal{U} \otimes \mathcal{V}$, then either $\omega_{\xi,\eta}$ annihilates $\mathcal{U}$, in which case $R_{\omega_{\xi,\eta}}(T) = 0$, or $\omega_{x,y}$ annihilates $\mathcal{V}$, in which case $L_{\omega_{x,y}}(T) = 0$. In either case,

$$(\omega_{\xi,\eta} \otimes \omega_{x,y})(T) = \omega_{\xi,\eta}(L_{\omega_{x,y}}(T)) = \omega_{x,y}(R_{\omega_{\xi,\eta}}(T)) = 0,$$

which shows that $T \in \operatorname{Ref}_e(\mathcal{U} \otimes \mathcal{V})$. □

Remark 2.4. The intersection $(\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{V}) \cap (\mathcal{U} \otimes \mathcal{B}(\mathcal{H}_2))$ coincides with the Fubini product $F(\mathcal{U}, \mathcal{V})$ defined by Tomiyama in [**23**] for von Neumann algebras and by Kraus in [**15**] for weak-$*$ closed spaces of operators.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, and let $\mathcal{L}_1 \otimes \mathcal{L}_2$ be the smallest subspace lattice generated by $P_1 \otimes P_2$, where $P_i \in \mathcal{L}_i$, $i = 1, 2$. It follows from a result of Kraus [**15**, (3.3)] that the Fubini product $F(\operatorname{Alg} \mathcal{L}_1, \operatorname{Alg} \mathcal{L}_2)$ equals $\operatorname{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2)$. Combining this with Lemma 2.3, we obtain

$$\operatorname{Ref}_e(\operatorname{Alg} \mathcal{L}_1 \otimes \operatorname{Alg} \mathcal{L}_2) = \operatorname{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

Corollary 2.5.

(i) *If* $A \in \mathcal{B}(\mathcal{H}_1)$, *then* $\operatorname{Ref}_e(A \otimes \mathcal{V}) = A \otimes \operatorname{Ref} \mathcal{V}$.

(ii) *If* $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H}_1)$, *then* $\operatorname{Ref}_e(\mathcal{U} \otimes \mathcal{B}(\mathcal{H}_2)) = \operatorname{Ref} \mathcal{U} \otimes \mathcal{B}(\mathcal{H}_2)$.

Proof. (i) Clearly, we may assume that $A \neq 0$. If $T \in \operatorname{Ref}_e(A \otimes \mathcal{V})$, then by Lemma 2.3, $T \in (\mathcal{B}(\mathcal{H}_1) \otimes \operatorname{Ref} \mathcal{V}) \cap (\operatorname{Ref} \mathbb{C}A \otimes \mathcal{B}(\mathcal{H}_2))$. But $\operatorname{Ref} \mathbb{C}A = \mathbb{C}A$, since one-dimensional subspaces are reflexive (see, for example, [**5**, Proposition 56.5]), so $T = A \otimes B$ for some $B \in \mathcal{B}(\mathcal{H}_2)$. Thus, $A \otimes B \in \mathcal{B}(\mathcal{H}_1) \otimes \operatorname{Ref} \mathcal{V}$, which implies that $B \in \operatorname{Ref} \mathcal{V}$.

Part (ii) follows from Lemma 2.3. □

Lemma 2.6. *Let* $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ *be a subspace of operators and* $\omega \in \mathcal{B}(\mathcal{H}_1)_*$ *be a vector functional. Then* $R_\omega(\operatorname{Ref}_e \mathcal{S}) \subseteq \operatorname{Ref} R_\omega(\mathcal{S})$.

Similarly, if $\tau \in \mathcal{B}(\mathcal{H}_2)_*$ *is a vector functional, then* $L_\tau(\operatorname{Ref}_e \mathcal{S}) \subseteq \operatorname{Ref} L_\tau(\mathcal{S})$.

Proof. Let $\omega = \omega_{\xi,\eta}$, where $\xi, \eta \in \mathcal{H}_1$. Fix $T \in \operatorname{Ref}_e \mathcal{S}$ and suppose that $x, y \in \mathcal{H}_2$ are such that $\omega_{x,y}(R_\omega(\mathcal{S})) = \{0\}$. It follows from (2.1) that

$$\omega_{\xi \otimes x, \eta \otimes y}(\mathcal{S}) = \{0\}.$$

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Since $T \in \operatorname{Ref}_e \mathcal{S}$, we have that $\omega_{\xi\otimes x,\eta\otimes y}(T) = \{0\}$. By (2.1) again, $\omega_{x,y}(R_\omega(T)) = \{0\}$. We have thus shown that $R_\omega(T) \in \operatorname{Ref} R_\omega(\mathcal{S})$. The first claim is proved. The second claim follows similarly. □

Proposition 2.7. *For a projection $L \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, let $\tilde{L}$ be the projection onto the subspace $\{\xi \otimes x \colon L(\xi \otimes x) = 0\}^\perp$. Let $P, Q \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be projections. Then*

$$\operatorname{Ref}_e Q\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)P = \tilde{Q}\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)\tilde{P}.$$

In particular, there exists a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\operatorname{Ref}_e \mathcal{S}$ is strictly bigger than $\operatorname{Ref} \mathcal{S}$.

Proof. Fix projections $P, Q \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and let $\mathcal{S} = Q\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)P$. It is clear that

$$\mathcal{S}_\perp \cap \mathcal{E} = \{\omega_{\xi\otimes x,\eta\otimes y} \colon P(\xi \otimes x) = 0 \text{ or } Q(\eta \otimes y) = 0\}. \tag{2.2}$$

Hence, $T \in \operatorname{Ref}_e \mathcal{S}$ if and only if $\langle T(\xi \otimes x), \eta \otimes y\rangle = 0$ for all $\xi, \eta \in \mathcal{H}_1$ and all $x, y \in \mathcal{H}_2$ such that either $P(\xi \otimes x) = 0$ or $Q(\eta \otimes y) = 0$.

Suppose that $T \in \operatorname{Ref}_e \mathcal{S}$. If $\xi \in \mathcal{H}_1$ and $x \in \mathcal{H}_2$ are such that $P(\xi \otimes x) = 0$, then for any $\eta \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ we have $\langle T(\xi \otimes x), \eta \otimes y\rangle = 0$ and so $T(\xi \otimes x) = 0$. But $\tilde{P}^\perp(\mathcal{H}_1 \otimes \mathcal{H}_2) = \overline{[\xi \otimes x \colon P(\xi \otimes x) = 0]}$. It follows that $T\tilde{P}^\perp = 0$, or $T = T\tilde{P}$. By considering adjoints, we conclude that $T = \tilde{Q}T$, and thus $T = \tilde{Q}T\tilde{P}$. Conversely, if T is of this form, then $T \in \operatorname{Ref}_e \mathcal{S}$ by the previous paragraph.

For the last statement, it is sufficient to exhibit a projection $P \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\tilde{P}$ is strictly greater than P. It suffices to choose any $P \neq I$ that does not annihilate any non-trivial elementary tensors. For example, take $P = F^\perp$, where F is the projection onto $\{\lambda(e_1 \otimes f_1 + e_2 \otimes f_2) \colon \lambda \in \mathbb{C}\}$ and where the set $\{e_1, e_2\} \subseteq \mathcal{H}_1$ (respectively, the set $\{f_1, f_2\} \subseteq \mathcal{H}_2$) is linearly independent. Here $P \neq I$ but $\tilde{P} = I$. □

Example 2.8. Let $\mathcal{H}_1$ be infinite dimensional, let $V \in \mathcal{B}(\mathcal{H}_1)$ be an isometry and let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_2)$ be a weak-$*$ closed subspace. Then

(i) $\operatorname{Ref}(V \otimes \mathcal{S}) = V \otimes \mathcal{S}$,

(ii) if $\mathcal{S}$ is not reflexive,

$$\operatorname{Ref}(V \otimes \mathcal{S}) \subsetneqq \operatorname{Ref}_e(V \otimes \mathcal{S}).$$

Proof. The equality $\operatorname{Ref}(V \otimes \mathcal{S}) = V \otimes \mathcal{S}$ is well known when V is the identity (see, for example, [**5**, Corollary 59.7]), and the proof readily extends to the general case.

Since $\operatorname{Ref}_e(V \otimes \mathcal{S}) = V \otimes \operatorname{Ref} \mathcal{S}$ by Corollary 2.5, if $\mathcal{S}$ is not reflexive, then $\operatorname{Ref}(V \otimes \mathcal{S})$ is strictly contained in $\operatorname{Ref}_e(V \otimes \mathcal{S})$. □

3. Reflexive hulls and Fourier coefficients

We recall that for each $\varphi \in H^\infty$ we denote by T_φ the analytic Toeplitz operator on H^2 with symbol φ and by $\mathcal{T}$ the collection of all analytic Toeplitz operators on H^2. Let $\zeta_n \in H^2$ be the function given by $\zeta_n(z) = z^n$, $z \in \mathbb{T}$. We note that $\{\zeta_n : n \geqslant 0\}$ is an orthonormal basis of H^2. Let $S = T_{\zeta_1} \in \mathcal{T}$ be the unilateral shift.

For the rest of this section, we fix a Hilbert space $\mathcal{K}$. We note that $(\mathcal{T} \otimes \mathcal{B}(\mathcal{K}))' = \mathcal{T} \otimes I$. Indeed, if $T \in (\mathcal{T} \otimes \mathcal{B}(\mathcal{K}))'$, then $T \in (I \otimes \mathcal{B}(\mathcal{K}))'$ and hence $T = A \otimes I$ for some $A \in \mathcal{B}(H^2)$. It now follows that $A \in \mathcal{T}' = \mathcal{T}$ [**22**]. Thus, $(\mathcal{T} \otimes \mathcal{B}(\mathcal{K}))'' = (\mathcal{T} \otimes I)'$. Now, if $X \in (\mathcal{T} \otimes I)'$, then $X(T \otimes I) = (T \otimes I)X$ for all $T \in \mathcal{T}$. Applying left slice maps, we obtain $L_\omega(X)T = TL_\omega(X)$ for all normal functionals ω and all $T \in \mathcal{T}$. Thus, $L_\omega(X) \in \mathcal{T}' = \mathcal{T}$ for all normal functionals ω, which means by Lemma 2.1 that $X \in \mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K})$. We conclude that $(\mathcal{T} \otimes \mathcal{B}(\mathcal{K}))'' = \mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K})$ and, in particular, that $\mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K})$ is automatically weakly closed.

If $T \in \mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K})$, let $\hat{T}_n$, $n \geqslant 0$, be the operators determined by the identity

$$T(\zeta_0 \otimes x) = \sum_{n \geqslant 0} \zeta_n \otimes \hat{T}_n x, \quad x \in \mathcal{K}.$$

Alternatively, $\hat{T}_n = R_{\omega_n}(T)$, where $\omega_n = \omega_{\zeta_0, \zeta_n}$, $n \geqslant 0$.

We call $\sum_{n \geqslant 0} S^n \otimes \hat{T}_n$ the formal Fourier series of T. When $\mathcal{K}$ is one dimensional, this is the usual Fourier series of an operator $T \in \mathcal{T}$. By standard arguments, as in the scalar case, the Cesàro sums of this series converge to T in the weak-$*$ topology.

If $\mathbb{S}$ is a family $(\mathcal{S}_n)_{n \geqslant 0}$ of subspaces of $\mathcal{B}(\mathcal{K})$, we let

$$\mathcal{A}(\mathbb{S}) = \{T \in \mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K}) : \hat{T}_n \in \mathcal{S}_n, \ n \geqslant 0\}.$$

It is obvious that $\mathcal{A}(\mathbb{S})$ is a linear space; it is a subalgebra of $\mathcal{B}(H^2 \otimes \mathcal{K})$ if and only if $\mathcal{S}_n \mathcal{S}_m \subseteq \mathcal{S}_{n+m}$ for all $n, m \geqslant 0$.

Remark 3.1. If $\mathcal{S}_n$ is closed in the weak operator (respectively, the weak-$*$) topology and $\mathbb{S} = (\mathcal{S}_n)_{n \geqslant 0}$, then $\mathcal{A}(\mathbb{S})$ is closed in the weak operator (respectively, the weak-$*$) topology.

This follows from the fact that the slice maps R_{ω_n} are continuous in both the weak–weak and the weak-$*$–weak-$*$ sense.

Remark 3.2. If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{K})$ is a weak-$*$ closed space and $\mathcal{S}_n = \mathcal{S}$ for each $n \geqslant 0$, then $\mathcal{A}(\mathbb{S}) = \mathcal{T} \bar{\otimes} \mathcal{S}$.

Indeed, if $A \in \mathcal{S}$ and $k \geqslant 0$, then obviously $S^k \otimes A \in \mathcal{A}(\mathbb{S})$ and hence $\mathcal{T} \bar{\otimes} \mathcal{S} \subseteq \mathcal{A}(\mathbb{S})$, since the latter is weak-$*$ closed.

Conversely, suppose that $T \in \mathcal{T} \bar{\otimes} \mathcal{B}(\mathcal{K})$ is such that $\hat{T}_n \in \mathcal{S}$ for each $n \geqslant 0$. Then $S^n \otimes \hat{T}_n \in \mathcal{T} \bar{\otimes} \mathcal{S}$ and hence the Cesàro sums of the Fourier series of T are in $\mathcal{T} \bar{\otimes} \mathcal{S}$. But $\mathcal{T} \bar{\otimes} \mathcal{S}$ is weak-$*$ closed, and so $T \in \mathcal{T} \bar{\otimes} \mathcal{S}$.

If $\mathbb{S} = (\mathcal{S}_n)_{n \geqslant 0}$, we let $\operatorname{Ref} \mathbb{S} := (\operatorname{Ref} \mathcal{S}_n)_{n \geqslant 0}$.

Theorem 3.3. *If $\mathbb{S} = (\mathcal{S}_n)_{n\geqslant 0}$ is a sequence of subspaces of $\mathcal{B}(\mathcal{K})$, then $\operatorname{Ref}_e \mathcal{A}(\mathbb{S}) = \mathcal{A}(\operatorname{Ref}\mathbb{S})$. In particular, if $\mathcal{S}_n$ is reflexive for each $n \geqslant 0$, then $\mathcal{A}(\mathbb{S})$ is reflexive.*

Proof. First observe that $\operatorname{Ref}_e(\mathcal{T} \otimes \mathcal{B}(\mathcal{K})) = (\operatorname{Ref}\mathcal{T}) \otimes \mathcal{B}(\mathcal{K})$ by Corollary 2.5. But $\mathcal{T}$ is reflexive [**22**] and hence $\operatorname{Ref}_e(\mathcal{T} \otimes \mathcal{B}(\mathcal{K})) = \mathcal{T} \otimes \mathcal{B}(\mathcal{K})$.

Let $T \in \operatorname{Ref}_e \mathcal{A}(\mathbb{S})$. As just observed, $T \in \mathcal{T} \otimes \mathcal{B}(\mathcal{K})$. By Lemma 2.6, for each $n \geqslant 0$, writing $\omega_n = \omega_{\zeta_0,\zeta_n}$, we have

$$R_{\omega_n}(T) \in \operatorname{Ref} R_{\omega_n}(\mathcal{A}(\mathbb{S})) \subseteq \operatorname{Ref}\mathcal{S}_n,$$

since $R_{\omega_n}(\mathcal{A}(\mathbb{S})) \subseteq \mathcal{S}_n$ by the definition of $\mathcal{A}(\mathbb{S})$. In other words, $\hat{T}_n \in \operatorname{Ref}\mathcal{S}_n$ for all $n \geqslant 0$, and so $T \in \mathcal{A}(\operatorname{Ref}\mathbb{S})$.

Conversely, suppose that $T \in \mathcal{A}(\operatorname{Ref}\mathbb{S})$, that is, $\hat{T}_n \in \operatorname{Ref}\mathcal{S}_n$ for each $n \geqslant 0$. By Corollary 2.5, $S^n \otimes \hat{T}_n \in \operatorname{Ref}_e(S^n \otimes \mathcal{S}_n)$, $n \geqslant 0$. Since $S^n \otimes \mathcal{S}_n \subseteq \mathcal{A}(\mathbb{S})$, we conclude that $S^n \otimes \hat{T}_n \in \operatorname{Ref}_e \mathcal{A}(\mathbb{S})$, $n \geqslant 0$. By Lemma 2.2 (i) and the fact that T is in the weak-$*$ closed linear hull of $\{S^n \otimes \hat{T}_n \colon n \geqslant 0\}$ we have that $T \in \operatorname{Ref}_e \mathcal{A}(\mathbb{S})$.

Suppose that $\mathcal{S}_n$ is reflexive for each $n \geqslant 0$. By Lemma 2.2 (ii) and the first part of the proof,

$$\mathcal{A}(\mathbb{S}) \subseteq \operatorname{Ref}\mathcal{A}(\mathbb{S}) \subseteq \operatorname{Ref}_e \mathcal{A}(\mathbb{S}) = \mathcal{A}(\mathbb{S})$$

and hence $\mathcal{A}(\mathbb{S})$ is reflexive. □

As an immediate corollary of Theorem 3.3 we obtain the following result, proved for reflexive algebras by Kraus [**15**] and Ptak [**21**].

Corollary 3.4. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{K})$ be a reflexive subspace. Then $\mathcal{T} \otimes \mathcal{S}$ is reflexive.*

Remark 3.5. We note that $\operatorname{Ref}_e \mathcal{A}(\mathbb{S})$ is in general strictly larger than $\operatorname{Ref}\mathcal{A}(\mathbb{S})$. Indeed, let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{K})$ be a non-reflexive weak-$*$ closed subspace and $\mathbb{S} = (\mathcal{S}_n)_{n\geqslant 0}$ be the family with $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_n = \{0\}$ if $n \neq 1$. Then $\mathcal{A}(\mathbb{S}) = S \otimes \mathcal{S}$ is reflexive (Example 2.8 (i)). However, by Theorem 3.3, $\operatorname{Ref}_e \mathcal{A}(\mathbb{S}) = S \otimes \operatorname{Ref}\mathcal{S}$, which strictly contains $\mathcal{A}(\mathbb{S})$.

The following corollary will be used in Theorem 5.2.

Corollary 3.6. *Let $U, V \in \mathcal{B}(\mathcal{K})$ satisfy $UV = \lambda VU$ for some $\lambda \in \mathbb{C}$. Suppose that V is invertible and that the weak-$*$ closure $\mathcal{W}_0$ of the polynomials in U is reflexive. Then the weak-$*$ closed unital operator algebra $\mathcal{W} \subseteq \mathcal{B}(H^2 \otimes \mathcal{K})$ generated by $I \otimes U$ and $S \otimes V$ is reflexive.*

Proof. The commutation relation $UV = \lambda VU$ implies that $\mathcal{W}$ is the weak-$*$ closed linear hull of the set $\{S^k \otimes V^k U^m \colon k, m \geqslant 0\}$.

Let $\mathbb{S} = (V^n \mathcal{W}_0)_{n\geqslant 0}$. We claim that $\mathcal{W} = \mathcal{A}(\mathbb{S})$. Suppose that $T \in \mathcal{T} \otimes \mathcal{B}(\mathcal{K})$ and that $\hat{T}_n \in V^n \mathcal{W}_0$, $n \geqslant 0$. Then

$$S^n \otimes \hat{T}_n \in S^n \otimes V^n \mathcal{W}_0 = (S^n \otimes V^n)(I \otimes \mathcal{W}_0) \subseteq \mathcal{W}.$$

It follows by approximation (in the w^*-topology) that $T \in \mathcal{W}$. Thus, $\mathcal{A}(\mathbb{S}) \subseteq \mathcal{W}$.

To show that $\mathcal{W} \subseteq \mathcal{A}(\mathbb{S})$, it suffices to prove that $S^k \otimes V^k U^m \in \mathcal{A}(\mathbb{S})$, for each $k, m \geqslant 0$. So, fix such k and m and note that if $x, y \in \mathcal{K}$, then

$$\begin{aligned}\langle R_{\omega_{\zeta_0,\zeta_n}}(S^k \otimes V^k U^m)x, y\rangle &= \langle (S^k \otimes V^k U^m)(\zeta_0 \otimes x), \zeta_n \otimes y\rangle \\ &= \langle \zeta_k \otimes V^k U^m x, \zeta_n \otimes y\rangle \\ &= \delta_{k,n}\langle V^k U^m x, y\rangle.\end{aligned}$$

Thus, $R_{\omega_{\zeta_0,\zeta_n}}(S^k \otimes V^k U^m) = \delta_{k,n} V^k U^m \in V^n \mathcal{W}_0$ for all n and hence $S^k \otimes V^k U^m \in \mathcal{A}(\mathbb{S})$ as required.

Now observe that, since V is invertible and $\mathcal{W}_0$ is reflexive, each $\mathcal{S}_n = V^n \mathcal{W}_0$ is reflexive. It therefore follows from Theorem 3.3 that $\mathcal{W} = \mathcal{A}(\mathbb{S})$ is reflexive. □

Remark 3.7. Both a special case of Theorem 3.3 and Corollary 3.4 were obtained independently by Kakariadis in [**12**, Theorem 2.8].

4. The structure of $\mathcal{T}_L(\mathbb{H}^+)$

In this section we study the weak-$*$ closed operator algebra $\mathcal{T}_L(\mathbb{H}^+)$ generated by the image of the left regular representation of $\mathbb{H}^+$ restricted to the invariant subspace $\mathcal{H} = \ell^2(\mathbb{H}^+)$. We identify $\mathcal{H}$ with $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, where the element of the canonical orthonormal basis of $\mathcal{H}$ corresponding to $w^n u^k v^m \in \mathbb{H}^+$ is identified with the elementary tensor $w^n \otimes u^k \otimes v^m$. Then $\mathcal{T}_L(\mathbb{H}^+)$ is generated by the operators L_u, L_v and L_w on $\mathcal{H}$, which act as follows:

$$\left.\begin{aligned} L_u(w^n \otimes u^k \otimes v^m) &= w^n \otimes u^{k+1} \otimes v^m, \\ L_v(w^n \otimes u^k \otimes v^m) &= w^{n-k} \otimes u^k \otimes v^{m+1}, \\ L_w(w^n \otimes u^k \otimes v^m) &= w^{n+1} \otimes u^k \otimes v^m, \end{aligned}\right\} \quad (n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+.$$

By the commutation relations, $\mathcal{T}_L(\mathbb{H}_+)$ coincides with the weak-$*$ closed linear span of the set

$$\{L_w^n L_u^k L_v^m \colon (n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+\}.$$

Throughout this section we will identify $\ell^2(\mathbb{Z})$ with $L^2(\mathbb{T})$ via Fourier transform in the first coordinate w. In this way, the identity function ζ_1 on $\mathbb{T}$ is identified with w and $\mathcal{T}_L(\mathbb{H}^+)$ is identified with an operator algebra acting on $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z}_+ \times \mathbb{Z}_+)$. Let $\mathcal{C}$ be the weak-$*$ closed linear span of $\{L_w^n \colon n \in \mathbb{Z}\}$. This is an abelian von Neumann algebra; it consists of all operators $\{L_f \colon f \in L^\infty(\mathbb{T})\}$, where

$$L_f(w^n \otimes u^k \otimes v^m) = (f w^n) \otimes u^k \otimes v^m.$$

Thus, $\mathcal{C} = \mathcal{M} \otimes 1 \otimes 1$, where $\mathcal{M} \subseteq \mathcal{B}(L^2(\mathbb{T}))$ is the multiplication maximal abelian self-adjoint subalgebra (MASA) of $L^\infty(\mathbb{T})$.

If $(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}t}) \in \mathbb{T} \times \mathbb{T}$ $(s, t \in [0, 2\pi))$, let $W_{s,t} \in \mathcal{B}(\mathcal{H})$ be the unitary operator given by

$$W_{s,t}(w^n \otimes u^k \otimes v^m) = w^n \otimes \mathrm{e}^{\mathrm{i}sk} u^k \otimes \mathrm{e}^{\mathrm{i}tm} v^m, \quad (n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+.$$

We define an action of the 2-torus $\mathbb{T}\times\mathbb{T}$ on $\mathcal{B}(\mathcal{H})$ by

$$\rho_{s,t}(A) = W_{s,t}AW_{s,t}^*, \quad A \in \mathcal{B}(\mathcal{H}).$$

Observe that

$$\rho_{s,t}(L_u) = \mathrm{e}^{\mathrm{i}s}L_u, \qquad \rho_{s,t}(L_v) = \mathrm{e}^{\mathrm{i}t}L_v, \qquad \rho_{s,t}(L_w) = L_w.$$

Hence, $\rho_{s,t}$ leaves $\mathcal{T}_L(\mathbb{H}^+)$ invariant. Since $\rho_{s,t}$ is unitarily implemented, it also leaves $\operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$ invariant.

If $A\in\mathcal{T}_L(\mathbb{H}^+)$ is a 'trigonometric polynomial', namely a sum

$$A = \sum_{(k,m)\in\Omega} L_{f_{k,m}}L_u^kL_v^m,$$

where $\Omega\subseteq\mathbb{Z}_+\times\mathbb{Z}_+$ is finite and $f_{k,m}\in L^\infty(\mathbb{T})((k,m)\in\Omega)$, then it is easy to observe that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\rho_{s,t}(A)\mathrm{e}^{-\mathrm{i}sk}\mathrm{e}^{-\mathrm{i}tm}\,\mathrm{d}t\,\mathrm{d}s = L_{f_{k,m}}L_u^kL_v^m.$$

We will need the following proposition, which is a version of well-known facts adapted to our setting.

Proposition 4.1. *For $k,m\in\mathbb{Z}_+$, let $Q_{k,m}\in\mathcal{B}(\mathcal{H})$ be the orthogonal projection onto the subspace $L^2(\mathbb{T})\otimes[u^k]\otimes[v^m]$ spanned by the vectors of the form $f\otimes u^k\otimes v^m$, $f\in L^2(\mathbb{T})$. If $A\in\mathcal{B}(\mathcal{H})$ and $p,q\in\mathbb{Z}$, set*

$$\Phi_{p,q}(A) = \sum_{k,m} Q_{k+p,m+q}AQ_{k,m},$$

where the sum is taken over all $k,m\in\mathbb{Z}_+$ such that $k+p,m+q\in\mathbb{Z}_+$. The following statements hold.

(i) $\Phi_{p,q}(A) = \dfrac{1}{4\pi^2}\displaystyle\int_0^{2\pi}\int_0^{2\pi}\rho_{s,t}(A)\mathrm{e}^{-\mathrm{i}sp}\mathrm{e}^{-\mathrm{i}tq}\,\mathrm{d}t\,\mathrm{d}s.$

(ii) *If $0<r<1$, then the series*

$$\sum_{p,q\in\mathbb{Z}}\Phi_{p,q}(A)r^{|p|+|q|}$$

converges absolutely in norm to an operator A_r; moreover, $\|A_r\|\leqslant\|A\|$ and $w^*\text{-}\lim_{r\nearrow 1}A_r = A$.

(iii) *If $\Phi_{p,q}(A)=0$ for all $p,q\in\mathbb{Z}$, then $A=0$.*

(iv) *If $A\in\mathcal{T}_L(\mathbb{H}^+)$ and $B\in\operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$, then $\Phi_{p,q}(A), A_r\in\mathcal{T}_L(\mathbb{H}^+)$ and $\Phi_{p,q}(B), B_r\in\operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$, for all $p,q\in\mathbb{Z}$.*

Proof. (i) Let $x = Q_{k_1,m_1}x$ and $y = Q_{k_2,m_2}y$. We have

$$\langle \Phi_{p,q}(A)x, y\rangle = \Big\langle \sum Q_{k+p,m+q}AQ_{k,m}x, y\Big\rangle = \delta_{k_1+p,k_2}\delta_{m_1+q,m_2}\langle Ax, y\rangle,$$

where the summation takes place over all $k, m \in \mathbb{Z}_+$ with $k + p, m + q \in \mathbb{Z}_+$. On the other hand, we have

$$\langle \rho_{s,t}(A)x, y\rangle = \langle W_{s,t}AW^*_{s,t}x, y\rangle = \mathrm{e}^{-\mathrm{i}sk_1-\mathrm{i}tm_1}\mathrm{e}^{\mathrm{i}sk_2+\mathrm{i}tm_2}\langle Ax, y\rangle$$

and hence

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}&\langle \rho_{s,t}(A)\mathrm{e}^{-\mathrm{i}sp}\mathrm{e}^{-\mathrm{i}tq}x, y\rangle \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\langle Ax, y\rangle \mathrm{e}^{\mathrm{i}s(k_2-k_1-p)}\mathrm{e}^{\mathrm{i}t(m_2-m_1-q)}\,\mathrm{d}t\,\mathrm{d}s \\ &= \delta_{k_1+p,k_2}\delta_{m_1+q,m_2}\langle Ax, y\rangle.\end{aligned}$$

(ii) Let F be the operator-valued function defined on $\mathbb{T}\times\mathbb{T}$ by $F(s,t) = \rho_{s,t}(A)$, and let $\hat{F}$ be its Fourier transform. By (i), $\hat{F}(p,q) = \Phi_{p,q}(A)$. If $P_r(s,t)$ denotes the two-dimensional Poisson kernel, then one readily sees that $A_r = (F * P_r)(0,0)$.

The claim therefore follows from the well-known properties of the Poisson kernel.

Part (iii) is an immediate consequence of (ii).

(iv) It follows from (i) that $\Phi_{p,q}(A) \in \mathcal{T}_L(\mathbb{H}^+)$ and $\Phi_{p,q}(B) \in \operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$, since $\rho_{s,t}$ leaves $\mathcal{T}_L(\mathbb{H}^+)$ and $\operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$ invariant. Now (ii) implies that $A_r \in \mathcal{T}_L(\mathbb{H}^+)$ and $B_r \in \operatorname{Ref}\mathcal{T}_L(\mathbb{H}^+)$. □

We isolate some consequences of Proposition 4.1, as follows.

Corollary 4.2. *If $A \in \mathcal{T}_L(\mathbb{H}^+)$, then we have the following.*

(i) $\Phi_{k,m}(A) = 0$ *unless* $k \geqslant 0$ *and* $m \geqslant 0$.

(ii) *For each* $k, m \geqslant 0$, *the operator* $L_{k,m} \equiv (L_v^m)^*(L_u^k)^*\Phi_{k,m}(A)$ *is in* $\mathcal{C}$. *Hence, there exists* $f_{k,m}(A) \in L^\infty(\mathbb{T})$ *such that* $L_{k,m} = L_{f_{k,m}(A)}$. *We have* $\Phi_{k,m}(A) = L_{f_{k,m}}(A)L_u^kL_v^m$.

Proof. Since $\mathcal{T}_L(\mathbb{H}^+)$ is the weak-$*$ closed hull of its trigonometric polynomials and the map $\Phi_{k,m}$ is weak-$*$ continuous, it suffices to assume that A is of the form $A = \sum_{(k,m)\in\Omega} L_{f_{k,m}}L_u^kL_v^m$, where $\Omega \subseteq \mathbb{Z}_+\times\mathbb{Z}_+$ is finite. Now (i) is obvious. For (ii), we have

$$\Phi_{k,m}(A) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\rho_{s,t}(A)\mathrm{e}^{-\mathrm{i}sk}\mathrm{e}^{-\mathrm{i}tm}\,\mathrm{d}t\,\mathrm{d}s = L_{f_{k,m}}L_u^kL_v^m = L_u^kL_v^mL_{f_{k,m}};$$

hence $(L_v^m)^*(L_u^k)^*\Phi_{k,m}(A) = L_{f_{k,m}}$, which is in $\mathcal{C}$. □

We can now identify the diagonal and the centre of $\mathcal{T}_L(\mathbb{H}^+)$.

Corollary 4.3. *The diagonal and the centre of $\mathcal{T}_L(\mathbb{H}^+)$ both coincide with $\mathcal{C}$.*

Proof. The maps $\rho_{s,t}$ are automorphisms of $\mathcal{T}_L(\mathbb{H}^+)$ and hence leave its centre $\mathcal{Z}$ invariant. By Proposition 4.1 (i), if $A \in \mathcal{Z}$, then $\Phi_{k,m}(A) \in \mathcal{Z}$. By Corollary 4.2 (ii), $L_{f_{k,m}(A)} L_u^k L_v^m \in \mathcal{Z}$ for each $k, m \geqslant 0$. It is now immediate that if such an operator commutes with all L_u and L_v, then $L_{f_{k,m}(A)} = 0$ unless $k = m = 0$. Thus, $A = L_{f_{0,0}(A)} \in \mathcal{C}$.

It follows from Proposition 4.1 (i) that $\Phi_{k,m}(A)^* = \Phi_{-k,-m}(A^*)$. Hence, by Corollary 4.2 (i), if A and A^* are both in $\mathcal{T}_L(\mathbb{H}^+)$, then $\Phi_{k,m}(A) = 0$ unless $k = m = 0$. Thus, each A_r is in $\mathcal{C}$ and hence so is A.

We have shown that the centre and the diagonal are contained in $\mathcal{C}$. The opposite inclusions are obvious. □

In some of the results that follow we adapt techniques used by Davidson and Pitts in [**9**]. Along with the left regular representation L of $\mathbb{H}^+$ defined above, we consider the restriction of its right regular representation to $\mathcal{H} = \ell^2(\mathbb{H}^+)$. This is generated by the operators

$$\left.\begin{aligned} R_u(w^n \otimes u^k \otimes v^m) &= w^{n-m} \otimes u^{k+1} \otimes v^m, \\ R_v(w^n \otimes u^k \otimes v^m) &= w^n \otimes u^k \otimes v^{m+1}, \\ R_w(w^n \otimes u^k \otimes v^m) &= w^{n+1} \otimes u^k \otimes v^m, \end{aligned}\right\} \quad (n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+.$$

We denote by $\mathcal{T}_R(\mathbb{H}^+)$ the weak-$*$ closed subalgebra of $\mathcal{B}(\ell^2(\mathbb{H}^+))$ generated by

$$\{R_w^n, R_u^k, R_v^m : (n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+\}.$$

It is trivial to verify that $\mathcal{T}_L(\mathbb{H}^+)$ and $\mathcal{T}_R(\mathbb{H}^+)$ commute.

Lemma 4.4. *Suppose that the operator $A \in \mathcal{B}(\mathcal{H})$ commutes with $\mathcal{T}_R(\mathbb{H}^+)$ and that $A(w^0 \otimes u^0 \otimes v^0) = 0$. Then $A = 0$.*

Proof. For each $(n,k,m) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+$ we have

$$\begin{aligned} A(w^n \otimes u^k \otimes v^m) &= A R_{v^m} R_{u^k} R_{w^n}(w^0 \otimes u^0 \otimes v^0) \\ &= R_{v^m} R_{u^k} R_{w^n} A(w^0 \otimes u^0 \otimes v^0) \\ &= 0. \end{aligned}$$

Hence, $A = 0$. □

The argument below is standard; for the case of the unilateral shift, see [**6**, Proposition V.1.1]. We include a proof for the convenience of the reader.

Proposition 4.5. *If $A \in \mathcal{B}(\mathcal{H})$ commutes with R_u or R_v, then $\|A\|$ equals the essential norm $\|A\|_e \equiv \inf\{\|A + K\| : K$ compact$\}$. In particular, the algebra $\mathcal{T}_L(\mathbb{H}^+)$ does not contain non-zero compact operators.*

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Proof. Assume that A commutes with R_v (the other case is similar). It is easy to see that $(R_v^n)_n$ tends to 0 weakly. Indeed, if x, y are in $\mathcal{H}$ and we write $x = \sum_m x_m \otimes v^m$, $y = \sum_m y_m \otimes v^m$, where x_m, y_m are in $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z}_+)$, then

$$\langle R_v^n x, y\rangle = \sum_m \langle x_m, y_{m+n}\rangle \to 0,$$

since $(\|x_m\|)$ and $(\|y_m\|)$ are square integrable.

Suppose, by way of contradiction, that there is a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that $\|A+K\| < \|A\|$. Then there is a unit vector $x \in \mathcal{H}$ which satisfies $\|Ax\| > \|A+K\|$. But $\|(A+K)R_v^n x\| \leqslant \|A+K\|$, since R_v^n is an isometry. On the other hand, since R_v^n tends to 0 weakly, we have $\lim_n \|KR_v^n x\| = 0$. Thus,

$$\lim_n \|(A+K)R_v^n x\| = \lim_n \|AR_v^n x\| = \lim_n \|R_v^n Ax\| = \|Ax\|,$$

a contradiction. □

Theorem 4.6. *The algebra $\mathcal{T}_L(\mathbb{H}^+)$ does not contain quasi-nilpotent operators. In particular, $\mathcal{T}_L(\mathbb{H}^+)$ is semi-simple.*

Proof. Let $A \in \mathcal{T}_L(\mathbb{H}^+)$ be non-zero and define $f_{k,m} = f_{k,m}(A) \in L^\infty(\mathbb{T})$ as in Corollary 4.2. Recall that for $r \in (0,1)$ we have set

$$A_r = \sum_{k,m\geqslant 0} r^{k+m} L_{f_{k,m}} L_u^k L_v^m.$$

Let

$$\begin{aligned} E &= \{(k,m)\colon f_{k,m} \neq 0\}, \\ \rho &= \inf\{k+m\colon (k,m) \in E\}, \\ k_0 &= \inf\{k\colon (k,m) \in E, k+m = \rho\}, \\ m_0 &= \rho - k_0. \end{aligned}$$

If $g, h \in L^2(\mathbb{T})$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} &\langle A_r^n(g \otimes u^0 \otimes v^0), (h \otimes u^{nk_0} \otimes v^{nm_0})\rangle \\ &\qquad = \sum_\gamma r^{\sum k_i} r^{\sum m_i} \langle (f_{k_1,m_1} \cdots f_{k_n,m_n})\phi_\gamma g \otimes u^{\sum k_i} \otimes v^{\sum m_i}, (h \otimes u^{nk_0} \otimes v^{nm_0})\rangle, \end{aligned}$$

where the summation is over all $\gamma = ((k_1,m_1),(k_2,m_2),\dots,(k_n,m_n))$ with $(k_i,m_i) \in E$ and ϕ_γ is a function of modulus 1 such that

$$L_u^{k_1} L_v^{m_1} \cdots L_u^{k_n} L_v^{m_n} = L_{\phi_\gamma} L_u^{\sum k_i} L_v^{\sum m_i}.$$

For a term in the above sum to be non-zero, we must have $\sum k_i = nk_0$ and $\sum m_i = nm_0$. Thus, since $k_i + m_i \geqslant \rho = k_0 + m_0$ for each i and $\sum(k_i + m_i) = n(k_0 + m_0)$, we obtain

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$k_i + m_i = k_0 + m_0$ for all $i = 1, \ldots, n$. But $k_i \geqslant k_0$ for all i; hence, the condition $\sum k_i = nk_0$ gives $k_i = k_0$ for all i and so $m_i = m_0$ for all i.

Hence, there is only one non-zero term in the above sum and we obtain

$$\langle A_r^n(g \otimes u^0 \otimes v^0), (h \otimes u^{nk_0} \otimes v^{nm_0})\rangle = r^{n(k_0+m_0)}\langle (f^n_{k_0,m_0}\phi_{\gamma_0} g), h\rangle,$$

where $\gamma_0 = ((k_0, m_0), (k_0, m_0), \ldots, (k_0, m_0))$ (and the term (k_0, m_0) appears n times). Now, since $\|A_r\| \leqslant \|A\|$ for each r and $A_r \to A$ in the weak-$*$ topology,

$$\begin{aligned}|\langle A^n(g \otimes u^0 \otimes v^0), (h \otimes u^{nk_0} \otimes v^{nm_0})\rangle| &= \lim_{r \nearrow 1} |\langle A_r^n(g \otimes u^0 \otimes v^0), (h \otimes u^{nk_0} \otimes v^{nm_0})\rangle| \\ &= \lim_{r \nearrow 1} r^{n(k_0+m_0)} |\langle (f^n_{k_0,m_0}\phi_{\gamma_0} g), h\rangle| \\ &= |\langle (f^n_{k_0,m_0}\phi_{\gamma_0} g), h\rangle|.\end{aligned}$$

Since ϕ_{γ_0} is unimodular,

$$\begin{aligned}\|A^n\| &\geqslant \sup\{|\langle A^n(g \otimes u^0 \otimes v^0), h \otimes u^{nk_0} \otimes v^{nm_0}\rangle| \colon \|g\|_2 \leqslant 1,\ \|h\|_2 \leqslant 1\} \\ &= \sup\{|\langle f^n_{k_0,m_0}\phi_{\gamma_0} g, h\rangle| \colon \|g\|_2 \leqslant 1,\ \|h\|_2 \leqslant 1\} \\ &= \|f^n_{k_0,m_0}\|_\infty.\end{aligned}$$

Thus,

$$\|A^n\|^{1/n} \geqslant \|f_{k_0,m_0}\|_\infty$$

for all n, and hence the spectral radius of A is non-zero. □

Theorem 4.7. *The commutant of $\mathcal{T}_R(\mathbb{H}^+)$ is $\mathcal{T}_L(\mathbb{H}^+)$.*

Proof. Let A be in the commutant of $\mathcal{T}_R(\mathbb{H}^+)$. Then

$$A(w^0 \otimes u^0 \otimes v^0) = \sum_{k,m \geqslant 0} \phi_{k,m} \otimes u^k \otimes v^m$$

for some $\phi_{k,m} \in L^2(\mathbb{T})$.

We show that $\phi_{k,m} \in L^\infty(\mathbb{T})$. Let $g \in L^\infty(\mathbb{T})$. Since $L_g A = A L_g$ (note that $L_g \in \mathcal{Z} \subseteq \mathcal{T}_R(\mathbb{H}^+)$), we have

$$\begin{aligned}A(g \otimes u^0 \otimes v^0) &= L_g A(w^0 \otimes u^0 \otimes v^0) \\ &= \sum_{k,m \geqslant 0} L_g(\phi_{k,m} \otimes u^k \otimes v^m) \\ &= \sum_{k,m \geqslant 0} (g\phi_{k,m} \otimes u^k \otimes v^m)\end{aligned}$$

and so

$$\langle A(g \otimes u^0 \otimes v^0), (g \otimes u^k \otimes v^m)\rangle = \langle g\phi_{k,m}, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \phi_{k,m}(t)|g(t)|^2 \,\mathrm{d}t.$$

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Therefore,

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\phi_{k,m}(t)|g(t)|^2\,\mathrm{d}t\right| \leqslant \|A\|\|g\|_2^2.$$

Using this inequality for characteristic functions in the place of g, one sees that $\phi_{k,m}$ induces a linear functional on $L^1(\mathbb{T})$ of norm not larger than $\|A\|$; thus, $\phi_{k,m} \in L^\infty(\mathbb{T})$.

We show that if $r \in (0,1)$, the operator

$$A_r = \sum_{k,m\in\mathbb{Z}} \Phi_{k,m}(A)r^{|k|+|m|}$$

defined in Proposition 4.1 is in the commutant of $\mathcal{T}_R(\mathbb{H}^+)$. It suffices to show that

$$\Phi_{k,m}(A) = \sum_{i,j} Q_{k+i,m+j}AQ_{i,j}$$

is in the commutant of $\mathcal{T}_R(\mathbb{H}^+)$ for all $k,m \in \mathbb{Z}$. We have $R_uQ_{k,m} = Q_{k+1,m}R_u$ and hence

$$\begin{aligned}\sum_{i,j} Q_{k+i,m+j}AQ_{i,j}R_u &= \sum_{i,j} Q_{k+i,m+j}AR_uQ_{i-1,j}\\ &= \sum_{i,j} Q_{k+i,m+j}R_uAQ_{i-1,j}\\ &= R_u\sum_{i,j} Q_{k-1+i,m+j}AQ_{i-1,j}.\end{aligned}$$

Similarly, $R_vQ_{k,m} = Q_{k,m+1}R_v$ and hence

$$\begin{aligned}\sum_{i,j} Q_{k+i,m+j}AQ_{i,j}R_v &= \sum_{i,j} Q_{k+i,m+j}AR_vQ_{i,j-1}\\ &= \sum_{i,j} Q_{k+i,m+j}R_vAQ_{i,j-1}\\ &= R_v\sum_{i,j} Q_{k+i,m-1+j}AQ_{i,j-1}.\end{aligned}$$

Now set

$$B_r = \sum_{k,m\geqslant 0} r^{k+m}L_{\phi_{k,m}}L_{u^k}L_{v^m}.$$

Since $\phi_{k,m} \in L^\infty(\mathbb{T})$, the series converges absolutely to an operator in $\mathcal{T}_L(\mathbb{H}^+)$.

Clearly, $\Phi_{k,m}(A)(w^0\otimes u^0\otimes v^0) = \phi_{k,m}\otimes u^k\otimes v^m$ and so $A_r(w^0\otimes u^0\otimes v^0) = B_r(w^0\otimes u^0\otimes v^0)$. Since both A_r and B_r are in the commutant of $\mathcal{T}_R(\mathbb{H}^+)$, Lemma 4.4 implies that $A_r = B_r$. Hence, $A_r \in \mathcal{T}_L(\mathbb{H}^+)$. Since $\mathcal{T}_L(\mathbb{H}^+)$ is weak-$*$ closed, Proposition 4.1 (ii) implies that $A \in \mathcal{T}_L(\mathbb{H}^+)$. □

The following properties of $\mathcal{T}_L(\mathbb{H}^+)$ follow from Theorem 4.7.

Corollary 4.8.

(i) *The algebra* $\mathcal{T}_L(\mathbb{H}^+)$ *has the bicommutant property* $\mathcal{T}_L(\mathbb{H}^+)'' = \mathcal{T}_L(\mathbb{H}^+)$.

(ii) $\mathcal{T}_L(\mathbb{H}^+)$ *is an inverse closed algebra.*

(iii) $\mathcal{T}_L(\mathbb{H}^+)$ *is closed in the weak operator topology.*

5. Reflexivity of $\mathcal{T}_L(\mathbb{H}^+)$

In this section we establish the reflexivity of the algebra $\mathcal{T}_L(\mathbb{H}^+)$. Let $F\colon L^2(\mathbb{T})\otimes L^2(\mathbb{T})\otimes L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{Z})$ be the tensor product of three copies of the Fourier transform. Let $\mathcal{K} = H^2(\mathbb{T})\otimes H^2(\mathbb{T})$ and $\tilde{\mathcal{H}} = L^2(\mathbb{T})\otimes\mathcal{K} = L^2(\mathbb{T},\mathcal{K})$; we have that $\tilde{\mathcal{H}} = F^{-1}(\ell^2(\mathbb{H}^+))$. We will use the same symbol for the restriction of F to $\tilde{\mathcal{H}}$.

Let $\tilde{W} = F^{-1}L_wF, \tilde{U} = F^{-1}L_uF, \tilde{V} = F^{-1}L_vF$ (acting on $\tilde{\mathcal{H}}$) and $\mathcal{L} = F^{-1}\mathcal{T}_L(\mathbb{H}^+)F$. For a fixed $\xi\in\mathbb{T}$, let $V_\xi = A_\xi\otimes S\in\mathcal{B}(H^2\otimes H^2)$, where $S = T_{\zeta_1}$ is the shift on H^2 and A_ξ is given by $(A_\xi f)(z) = f(z/\xi)$, $f\in H^2$.

Write μ for the normalized Lebesgue measure on $\mathbb{T}$. We consider the Hilbert space $\tilde{\mathcal{H}}$ as a direct integral over the measure space $(\mathbb{T},\mu)$ of the constant field $\xi\to\mathcal{K}(\xi)=\mathcal{K}$ of Hilbert spaces. Thus, an operator T is decomposable [**3**] with respect to this field if and only if it belongs to $\mathcal{M}\otimes\mathcal{B}(\mathcal{K})$, where $\mathcal{M}$ denotes the multiplication MASA of $L^\infty(\mathbb{T})$; we write

$$T = \int_{\mathbb{T}} T(\xi)\,\mathrm{d}\mu(\xi).$$

We note that $\tilde{W}$, $\tilde{U}$ and $\tilde{V}$ are decomposable. In the next proposition we identify their direct integrals.

Proposition 5.1. *When* $\tilde{\mathcal{H}}$ *is identified with the direct integral over* $(\mathbb{T},\mu)$ *of the constant field* $\xi\to\mathcal{K}$ *of Hilbert spaces, we have*

$$\tilde{W} = \int_{\mathbb{T}}\xi(I\otimes I)\,\mathrm{d}\mu(\xi),\qquad \tilde{U} = \int_{\mathbb{T}}(S\otimes I)\,\mathrm{d}\mu(\xi),\qquad \tilde{V} = \int_{\mathbb{T}}V_\xi\,\mathrm{d}\mu(\xi).$$

Proof. We identify the elements of $\tilde{\mathcal{H}} = L^2(\mathbb{T},\mathcal{K})$ with functions on three variables, $f = f(\xi,z_1,z_2)$, such that for almost every $\xi\in\mathbb{T}$, the function on two variables $f(\xi,\cdot,\cdot)$ is analytic. To show that

$$\tilde{W} = \int_{\mathbb{T}}\xi(I\otimes I)\,\mathrm{d}\mu(\xi),$$

note that if $f\in\tilde{\mathcal{H}}$, then $\tilde{W}f(\xi,z_1,z_2) = \xi f(\xi,z_1,z_2)$, $\xi,z_1,z_2\in\mathbb{T}$.

The claim concerning $\tilde{U}$ is immediate from its definition. For $\tilde{V}$ we argue as follows: let $f(\xi,z_1,z_2) = \xi^n z_1^k z_2^m$ (that is, $f = F^{-1}(w^n\otimes u^k\otimes v^m)$); then

$$\begin{aligned}\tilde{V}f &= \tilde{V}F^{-1}(w^n\otimes u^k\otimes v^m)\\ &= F^{-1}L_v(w^n\otimes u^k\otimes v^m)\\ &= F^{-1}(w^{n-k}\otimes u^k\otimes v^{m+1})\end{aligned}$$

and thus $\tilde{V}f(\xi, z_1, z_2) = \xi^{n-k} z_1^k z_2^{m+1}$. On the other hand, the direct integral

$$\int_{\mathbb{T}} (A_\xi \otimes I)\, \mathrm{d}\mu(\xi)$$

transforms the function f into the function $g(\xi, z_1, z_2) = \xi^{n-k} z_1^k z_2^m$. We thus have that

$$\tilde{V} = (I \otimes S) \int_{\mathbb{T}} (A_\xi \otimes I)\, \mathrm{d}\mu(\xi) = \int_{\mathbb{T}} (A_\xi \otimes S)\, \mathrm{d}\mu(\xi).$$

□

For $\xi \in \mathbb{T}$, let $\mathcal{L}_\xi \subseteq \mathcal{B}(\mathcal{K})$ be the weak-$*$ closed subalgebra generated by $S \otimes I$ and V_ξ. The operators $A_\xi, S \in \mathcal{B}(H^2)$ are easily seen to satisfy the assumptions of Corollary 3.6 with $\lambda = \bar{\xi}$. It follows that $\mathcal{L}_\xi$ is reflexive; in particular, it is weakly closed. We note that the algebra $\mathcal{L}_\xi$ was studied by Hasegawa in [**11**], where a class of invariant subspaces of $\mathcal{L}_\xi$ was exhibited.

In the next theorem, we use the notion of a direct integral of non-self-adjoint operator algebras developed in [**3**].

Theorem 5.2. *The algebra $\mathcal{T}_L(\mathbb{H}^+)$ is reflexive.*

Proof. By definition, $\mathcal{L} = F^{-1}\mathcal{T}_L(\mathbb{H}^+)F$ is generated, as a weak-$*$ closed algebra, by the operators $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ and $\tilde{W}^{-1}$.

Note that $\mathcal{L} \subseteq \mathcal{M} \otimes \mathcal{B}(\mathcal{K})$; moreover, $\mathcal{L}$ is weakly closed, since $\mathcal{T}_L(\mathbb{H}^+)$ is a commutant (Theorem 4.7). Hence, by [**3**], $\mathcal{L}$ gives rise to a direct integral

$$\int_{\mathbb{T}} \mathcal{A}(\xi)\, \mathrm{d}\mu(\xi),$$

where $\mathcal{A}(\xi)$ is the weakly closed algebra generated by $\tilde{U}(\xi)$, $\tilde{V}(\xi)$, $\tilde{W}(\xi)$ and $\tilde{W}^{-1}(\xi)$. Since the operators $\tilde{W}(\xi)$ and $\tilde{W}^{-1}(\xi)$ are scalar multiples of the identity, we have that $\mathcal{A}(\xi) = \mathcal{L}_\xi$. On the other hand, since $\mathcal{M} \otimes I_{\mathcal{K}} \subseteq \mathcal{L}$, all diagonal operators of the integral decomposition are contained in $\mathcal{L}$. Proposition 3.3 of [**3**] shows that an operator

$$T = \int_{\mathbb{T}} T(\xi)\, \mathrm{d}\mu(\xi)$$

belongs to $\mathcal{L}$ if and only if almost all $T(\xi)$ belong to $\mathcal{L}_\xi$. As observed above, $\mathcal{L}_\xi$ is reflexive for each $\xi \in \mathbb{T}$. Proposition 3.2 of [**3**] now implies that $\mathcal{L}$ is reflexive. Therefore, so is $\mathcal{T}_L(\mathbb{H}^+)$. □

6. Other representations

Until now we were concerned with the left regular representation of the Heisenberg semigroup. In this section, we consider another class of representations defined as follows. Let $\lambda = \mathrm{e}^{2\pi \mathrm{i}\theta}$ with θ irrational and let $\alpha\colon \mathbb{T} \to \mathbb{T}$ be the rotation corresponding to θ, that is, the map given by $\alpha(z) = \lambda z$. We let ν be a Borel probability measure on $\mathbb{T}$

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which is quasi-invariant (that is, $\nu(E) = 0$ implies $\nu(\alpha(E)) = 0$, for every measurable set $E \subseteq \mathbb{T}$) and ergodic (that is, $f \circ \alpha^k = f$ for all $k \in \mathbb{Z}$ implies that f is constant, for every $f \in L^\infty(\mathbb{T}, \nu)$). Let $\mathcal{W}_\pi(\mathbb{H}^+)$ be the weak-$*$ closed subalgebra of $\mathcal{B}(L^2(\mathbb{T}, \nu))$ generated by the operators

$$\pi(u) = M_{\zeta_1}, \quad \pi(v)f = \sqrt{\frac{\mathrm{d}\nu_\lambda}{\mathrm{d}\nu}}(f \circ \alpha) \quad \text{and} \quad \pi(w) = \lambda I,$$

where M_{ζ_1} is the operator of multiplication by the function ζ_1 on $L^2(\mathbb{T}, \nu)$ (recall that $\zeta_n(z) = z^n$) and $\nu_\lambda(A)$ is the Borel measure on $\mathbb{T}$ given by $\nu_\lambda(A) = \nu(\alpha(A))$.

We will need the following two lemmas; the results are probably known in some form, but we have been unable to locate a precise reference and so we include their proofs. Below, the terms *singular* and *absolutely continuous* are understood with respect to Lebesgue measure μ.

Lemma 6.1.

(i) *The measure ν is either absolutely continuous or singular.*

(ii) *If ν is absolutely continuous, it is equivalent to Lebesgue measure.*

(iii) *If ν is singular and not continuous, it is supported on an orbit of α.*

Proof. (i) Denote by ν_{a} (respectively, ν_{s}) the absolutely continuous (respectively, singular) part of ν. Suppose that $\nu_{\mathrm{s}} \neq 0$ and $\nu_{\mathrm{a}} \neq 0$ and let A be a Borel set of Lebesgue measure zero such that $\nu_{\mathrm{s}}(\mathbb{T} \setminus A) = 0$. Then $\bigcup_{n\in\mathbb{Z}} \alpha^n(A)$ is an invariant set of positive ν-measure. On the other hand, the Lebesgue measure of $\bigcup_{n\in\mathbb{Z}} \alpha^n(A)$ is zero and hence $\bigcup_{n\in\mathbb{Z}} \alpha^n(A)$ is not of full ν-measure. This contradicts the ergodicity of ν.

(ii) Let $E \subseteq \mathbb{T}$ be the set on which the Radon–Nikodỳm derivative $\mathrm{d}\nu/\mathrm{d}\mu$ vanishes; clearly, $\nu(E) = 0$. Setting $F = \bigcup_{n\in\mathbb{Z}} \alpha^n(E)$, we have that F is invariant and $\nu(F) = 0$. By the ergodicity of μ, either $\mu(F) = 0$ or $\mu(\mathbb{T} \setminus F) = 0$. However, if $\mu(\mathbb{T} \setminus F) = 0$, then $\nu(\mathbb{T} \setminus F) = 0$ and hence $\nu = 0$. Thus, $\mu(F) = 0$ and hence $\mu(E) = 0$. It follows that ν is equivalent to μ.

(iii) Let $z_0 \in \mathbb{T}$ be such that $\nu(\{z_0\}) \neq 0$. Then the orbit $X = \{\alpha^n(z_0) : n \in \mathbb{Z}\}$ of z_0 is an invariant set of positive ν-measure and it follows from ergodicity that its complement is ν-null. □

Note that the following lemma could also be deduced from the results of Wermer [**24**]. (We thank the referee for bringing this reference to our attention.) We include a direct proof using the F. and M. Riesz Theorem.

Lemma 6.2. *Let ν be a singular continuous measure. Then the weak-$*$ closed hull of the linear span of the set $\{M_{\zeta_n} : n = 1, 2 \dots\}$ is equal to $\{M_f : f \in L^\infty(\mathbb{T}, \nu)\}$.*

Proof. Let $f \in L^1(\mathbb{T}, \nu)$ be such that

$$\int f \zeta_n d\nu = 0 \quad \text{for all } n = 1, 2, \dots .$$

It follows from the F. and M. Riesz Theorem that the measure $f\,\mathrm{d}\nu$ is absolutely continuous. Since ν is singular, we obtain that $f = 0$ ν-almost everywhere, and hence it is equal to 0 as an element of $L^1(\mathbb{T}, \nu)$. □

The next theorem describes completely the operator algebras arising from the class of representations that we consider.

Theorem 6.3. *Let* $\mathcal{N} = \{\zeta_k H^2 : k \in \mathbb{Z}\}$.

(i) *If* ν *is equivalent to Lebesgue measure, then the algebra* $\mathcal{W}_\pi(\mathbb{H}^+)$ *is unitarily equivalent to the nest algebra* $\mathrm{Alg}\,\mathcal{N}$.

(ii) *If* ν *is singular and not continuous, then* $\mathcal{W}_\pi(\mathbb{H}^+)$ *is again unitarily equivalent to* $\mathrm{Alg}\,\mathcal{N}$.

(iii) *If* ν *is singular and continuous, then* $\mathcal{W}_\pi(\mathbb{H}^+) = \mathcal{B}(L^2(\mathbb{T}, \nu))$.

Proof. (i) Since ν is equivalent to Lebesgue measure, we may assume that $\mathcal{W}_\pi(\mathbb{H}^+)$ acts on $L^2(\mathbb{T})$, $\pi(u) = M_{\zeta_1}$ and $\pi(v)f = f \circ \alpha$.

If $a = (a_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, let D_a be given by $\widehat{D_a f}(n) = a_n \hat{f}(n)$; thus, D_a is the image, under conjugation by the Fourier transform, of the diagonal operator on $l^2(\mathbb{Z})$ given by $(x_j) \to (a_j x_j)$. Let $\mathcal{D} = \{D_a : a \in \ell^\infty(\mathbb{Z})\}$; clearly, $\mathcal{D}$ is a MASA on $L^2(\mathbb{T})$. Since the map $\sigma \to D_{(\sigma^n)_n}$ is weak-$*$ continuous from $\mathbb{T}$ into $\mathcal{B}(L^2(\mathbb{T}))$ and $\{\lambda^k : k \in \mathbb{Z}_+\}$ is dense in $\mathbb{T}$, the weak-$*$ closed linear span of $\{D_{(\lambda^{kn})_n} : k \in \mathbb{Z}_+\} = \{\pi(v)^k : k \in \mathbb{Z}_+\}$ contains $\{D_{(\sigma^n)_n} : \sigma \in \mathbb{T}\}$; it is hence a self-adjoint algebra and so must equal $\mathcal{D}$ by the Bicommutant Theorem. On the other hand, if $a \in \ell^\infty(\mathbb{Z})$ and $p \geqslant 0$, the matrix of $\pi(u)^p D_a$ with respect to the basis $\{\zeta_k\}_{k\in\mathbb{Z}}$ has the sequence a at the pth diagonal and zeros elsewhere. It follows that all lower triangular matrix units belong to the algebra $\mathcal{W}_\pi(\mathbb{H}^+)$, and hence this equals $\mathrm{Alg}\,\mathcal{N}$.

(ii) By Lemma 6.1 (iii), ν is supported on the orbit of a point $z_0 \in \mathbb{T}$. For $k \in \mathbb{Z}$, write $z_k = \alpha^{-k}(z_0)$ and $\beta_k^2 = \nu(\{z_k\})$. Since $\nu_\lambda(\{z_k\}) = \nu(\{\alpha(z_k)\}) = \nu(\{z_{k-1}\})$ we have $\beta_{k-1} = \beta(z_k)\beta_k$, where β is the function determined by the identity $\beta^2 = \mathrm{d}\nu_\lambda/\mathrm{d}\nu$. If $f_k = \chi_{\{z_k\}}/\beta_k$, then $\{f_k : k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{T}, \nu)$ and we have $\pi(v)\chi_{\{z_k\}} = \beta \cdot (\chi_{\{z_k\}} \circ \alpha) = \beta\chi_{\{z_{k+1}\}}$. Thus,

$$\pi(v)f_k = \beta\frac{\chi_{\{z_{k+1}\}}}{\beta_k} = \frac{\beta_k}{\beta_{k+1}}\frac{\chi_{\{z_{k+1}\}}}{\beta_k} = f_{k+1},$$

and so $\pi(v)$ is the bilateral shift with respect to $\{f_k\}$. Also $\pi(u)f_k = z_k f_k = \bar{\lambda}^k z_0 f_k$ for each k and hence, as in the proof of (i), the weak-$*$ closed linear span of the positive powers of $\pi(u)$ contains all operators diagonalized by $\{f_k\}$. It follows as in (i) that $\mathcal{W}_\pi(\mathbb{H}^+)$ consists of all operators which are lower triangular with respect to $\{f_k\}$; hence, it is unitarily equivalent to $\mathrm{Alg}\,\mathcal{N}$.

(iii) By Lemma 6.2, the algebra $\mathcal{W}_\pi(\mathbb{H}^+)$ contains a MASA, namely, the multiplication MASA of $L^\infty(\mathbb{T}, \nu)$. Since α acts ergodically, it is standard that $\mathcal{W}_\pi(\mathbb{H}^+)$ has no non-trivial invariant subspaces. It follows from [**2**] that it is weak-$*$ dense in, and hence equal to, $\mathcal{B}(L^2(\mathbb{T}, \nu))$. □

Remark 6.4. Note the different roles of $\pi(u)$ and $\pi(v)$ in (i) and (ii) of Theorem 6.3: in (i), the diagonal MASA is generated by (the non-negative powers of) $\pi(v)$; in (ii) the MASA is generated by $\pi(u)$. These two representations generate inequivalent representations of the irrational rotation algebra, as the corresponding measures are not equivalent [**4**].

6.1. A non-reflexive representation

We now construct an example of a representation of $\mathbb{H}^+$ which generates a non-reflexive weakly closed operator algebra. This representation, ρ, acts on H^2 and is defined as follows: if $S = T_{\zeta_1}$ is the shift and $V \in \mathcal{B}(H^2)$ is the operator given by $(Vf)(z) = f(\lambda z) = (f \circ \alpha)(z)$, we define

$$\rho(u) = S, \quad \rho(v) = SV \quad \text{and} \quad \rho(w) = \lambda I$$

with $\lambda = \mathrm{e}^{2\pi \mathrm{i}\theta}$ and θ irrational. Let $\mathcal{A}$ be the weakly closed unital algebra generated by $\rho(u)$ and $\rho(v)$. Using Fourier transform, we identify H^2 with $\ell^2(\mathbb{N})$ and let $E \colon \mathcal{B}(H^2) \to \mathcal{D} \simeq \ell^\infty(\mathbb{N})$ be the usual normal conditional expectation onto the diagonal given by $E((a_{ij})) = (b_{ij})$, where $b_{ij} = a_{ij}\delta_{ij}$. Define E_k for $k \geqslant 0$ by $E_k(A) = E((S^*)^k A)$.

We recall that $[\mathcal{S}]$ denotes the linear span of a subset $\mathcal{S}$ of a vector space.

Proposition 6.5. *If $A \in \mathcal{A}$, then $E_m(A) \in [I, V, \dots, V^m]$.*

Proof. The operator A is the weak limit of polynomials of the form

$$\sum_{k,n \geqslant 0} c_{k,n} S^{k+n} V^n.$$

Thus, $E_m(A)$ is a weak limit of polynomials of the form

$$\sum c_{k,n} V^n,$$

where the summation is over all $k, n \in \mathbb{Z}_+$ with $k + n = m$ and hence $E_m(A) \in [I, V, \dots, V^m]$. $\square$

Proposition 6.6. *If $\mathcal{K} \in \operatorname{Lat}\{S, SV\}$, then in fact $\mathcal{K} \in \operatorname{Lat}\{S, V\}$ and hence $\mathcal{K} = \zeta_k H^2$ for some $k \in \mathbb{Z}_+$.*

Proof. Since $S(\mathcal{K}) \subseteq \mathcal{K}$ and $\mathcal{K} \subseteq H^2$, by Beurling's Theorem there is an inner function ϕ such that $\mathcal{K} = \phi H^2$. Since $SV(\mathcal{K}) \subseteq \mathcal{K}$, we have $SV(\phi) \in \mathcal{K} = \phi H^2$, so $z\phi(\lambda z)/\phi(z) \in H^\infty$. Thus, there exists $h \in H^\infty$ such that

$$z\phi(\lambda z) = h(z)\phi(z) \quad \text{for all } z \in \mathbb{D}. \tag{6.1}$$

Let ϕ_1 be an analytic function and l be a non-negative integer such that $\phi_1(0) \neq 0$ and $\phi(z) = z^l \phi_1(z)$ for all $z \in \mathbb{D}$. We obtain

$$z^{l+1}\lambda^l \phi_1(\lambda z) = h(z) z^l \phi_1(z) \quad \text{for all } z \in \mathbb{D} \tag{6.2}$$

and hence

$$z\lambda^l\phi_1(\lambda z) = h(z)\phi_1(z) \qquad \text{for all } z \in \mathbb{D}. \tag{6.3}$$

Setting $z = 0$ in (6.3), we obtain that $h(0) = 0$. Thus, there exists $h_1 \in H^\infty$ such that $h(z) = zh_1(z)$. The relation $z\phi(\lambda z) = h(z)\phi(z) = zh_1(z)\phi(z)$ implies $\phi \circ \alpha = h_1\phi$ and hence $(\phi \circ \alpha)H^2 \subseteq \phi H^2$. Therefore,

$$V(\mathcal{K}) = V(\phi H^2) = (\phi \circ \alpha)H^2 \subseteq \phi H^2 = \mathcal{K}.$$

Considering $\mathcal{K}$ as a subspace of $L^2(\mathbb{T})$, Theorem 6.3 (i) gives that $\mathcal{K} = \zeta_k H^2$ for some k (note that here ν equals Lebesgue measure); since $\mathcal{K} \subseteq H^2$, k must be non-negative. □

Theorem 6.7. *The algebra $\mathcal{A}$ is not reflexive; in fact* $\operatorname{Ref}\mathcal{A} = \operatorname{Alg}\mathcal{N}$, *where* $\mathcal{N} = \{\zeta_k H^2 \colon k \in \mathbb{Z}_+\}$.

Proof. By Proposition 6.6, $\operatorname{Ref}\mathcal{A} = \operatorname{Alg}\mathcal{N}$. It follows from Proposition 6.5 that $\mathcal{A}$ is strictly contained in $\operatorname{Ref}\mathcal{A}$. □

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