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ON A CLASS OF PERFECT RINGS

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1. Introduction. In [**3**], the perfect rings of Bass [**1**] were characterized in terms of torsions in the following way:

A ring R is right perfect if and only if every (hereditary) torsion in the category **Mod R** *of all left R-modules is fundamental (i.e. generated by some minimal torsions) and closed under taking direct products; as a consequence, the number of all torsions in* **Mod R** *is finite and equal to 2^n for a natural n.*

Here, we present a simple description of those rings R which allow only two (trivial) torsions, viz. **0** and **Mod R** (and thus, are right perfect by [**3**]). Finite direct sums of these rings represent a natural generalization of completely reducible (i.e. artinian semisimple) rings (cf. Theorem 2) and we shall call them for that matter π-reducible rings. They can also be characterized in terms of their idempotent two-sided ideals. Moreover, we show in a simple manner a result of Courter [**2**] that π-reducible rings are precisely the rings R with the property that every R-module is rationally complete, i.e. has no proper rational extension (see [**4**]).

2. Preliminaries. Throughout the paper, all rings are associative and have unity, and all modules are left unital.

Given a ring R, by a *torsion* **T** in **Mod R** we shall always understand a hereditary torsion; thus, a torsion **T** in **Mod R** is a full subcategory of **Mod R** such that

(a) **T** is closed under taking submodules,

(b) for every $M \in$ **Mod R**, there is the greatest submodule (the **T**-torsion part) $\mathbf{T}(M)$ of M belonging to **T**, and

(c) $\mathbf{T}(M/\mathbf{T}(M)) = 0$ for every $M \in$ **Mod R**.

There is a one-to-one correspondence $\mathscr{K}$ between torsions in **Mod R** and certain sets of left ideals of R (cf. [**3**]): For every torsion $\mathbf{T}, \mathscr{K}(\mathbf{T})$ denotes the set of all left ideals L such that $R/L \in \mathbf{T}$. A torsion **T** is closed under taking direct products if and only if $\mathscr{K}(\mathbf{T})$ has the least element; the latter is then necessarily an idempotent two-sided ideal. Hence, there is a one-to-one correspondence between torsions closed under taking direct products and idempotent two-sided ideals.

All torsions in **Mod R** form an atomistic (i.e. each element contains an atom) complete lattice. The atoms $\mathbf{T}_\pi$, $\pi \in \Pi_R$, of this lattice will be called

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prime torsions; they are in one-to-one correspondence with the classes $\mathscr{W}_\pi$ of equivalent maximal left ideals (two maximal left ideals W_1 and W_2 are said to be equivalent if $R/W_1 \cong R/W_2$). Thus $\bigcup_{\pi \in \Pi_R} \mathscr{W}_\pi$ is the set of all maximal left ideals of R. The submodule $\mathbf{T}_\pi(M)$ of an R-module M will be called the *π-primary part* of M; if $\mathbf{T}_\pi(M) = M$, M will be called *π-primary.* A join of prime torsions in the lattice of all torsions will be called a *fundamental* torsion. All fundamental torsions form a (lattice) ideal of the lattice of all torsions which is isomorphic to the lattice of all subsets of Π_R (for details see [**3**]).

For a ring R, define the (left transfinite) *socle sequence*

$$0 = S^{(0)} \subseteq S^{(1)} \subseteq \ldots \subseteq S^{(\alpha)} \subseteq \ldots \subseteq R$$

(of the two-sided ideals $S^{(\alpha)}$) of R by

$$S^{(\alpha)}/S^{(\alpha-1)} = \text{Socle}(R/S^{(\alpha-1)}) \quad \text{for all non-limit and}$$

$$S^{(\alpha)} = \bigcup_{\beta<\alpha} S^{(\beta)} \quad \text{for all limit ordinals } 1 \leqq \alpha.$$

If $R = S^{(\delta)}$ for a certain δ, R is said to have a socle sequence (and the least δ with that property is called the *socle length* of R). Notice that R possesses a socle sequence if and only if it possesses a (left transfinite) *composition sequence*, i.e. a sequence of left ideals

$$0 = L^{(0)} \subset L^{(1)} \subset \ldots \subset L^{(\alpha)} \subset \ldots \subset L^{(\tau)} = R$$

such that

$$L^{(\alpha)}/L^{(\alpha-1)} \text{ is a simple factor for all non-limit}$$

and

$$L^{(\alpha)} = \bigcup_{\beta<\alpha} L^{(\beta)} \quad \text{for all limit ordinals } 1 \leqq \alpha \leqq \tau.$$

This, in turn, is equivalent to the fact that for every proper left ideal L of R, the socle of the monogenic R-module R/L is non-zero (or, that every non-zero R-module has a non-zero socle or, that every R-module possesses a socle or composition sequence). Moreover, recall that the latter holds if and only if all torsions in **Mod R** are fundamental (cf. [**3**]).

Finally, given a ring R, denote by R_m the ring of all $m \times m$ matrices over R. The ring R_m can be written as the direct sum $\bigoplus_{i=1}^m C_i$ of (left) column ideals C_i and, for every i, there is a one-to-one correspondence between the left ideals of R and the left ideals of R_m contained in C_i. As a consequence, R possesses a socle sequence if and only if R_m possesses a socle sequence.

3. π-reducible ring.

First, let us prove the following simple result.

THEOREM 1. *The following properties of a ring R are equivalent:*

(i) *There are only two torsions in* **Mod R**;

(i′) *R is π-primary;*

(ii) *R possesses a composition sequence with R-isomorphic factors;*

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(ii′) *R possesses a socle sequence with homogeneous factors of the same type;*
(iii) *R is isomorphic to the ring of all $m \times m$ matrices over a local (i.e. with a unique maximal left ideal) ring possessing a socle sequence (i.e. satisfying one of the preceding equivalent properties).*

Proof. The equivalence of (i) and (i′), as well as of (ii) and (ii′) is obvious. Also, assuming (i), we can observe that the torsions are necessarily fundamental and thus readily obtain (ii).

Now, assuming (ii), by an argument of Bass [**1**], it is easy to see that the Jacobson radical Rad R of a ring with a composition sequence is nil. Moreover, since all the factors of the sequence are R-isomorphic,

$$\text{Socle}(R/\text{Rad}\, R) = R/\text{Rad}\, R;$$

hence, $R/\text{Rad}\, R$ is completely reducible. This follows from the following simple result.

Lemma (*cf.* [**3**, *proof of Theorem* B]). *Let $\bar{R}$ be a ring such that* Rad $\bar{R} = 0$. *Then either*

$$\text{Socle}(\bar{R}) = \bar{R},$$

or there is a maximal left ideal $\bar{W}$ of $\bar{R}$ such that no factor of Socle$(\bar{R})$ *is $\bar{R}$-isomorphic to $\bar{R}/\bar{W}$.*

Proof of the lemma. Assume that Socle$(\bar{R}) \neq \bar{R}$ and take a (proper) maximal left ideal $\bar{W}$ of $\bar{R}$ containing Socle$(\bar{R})$; $\bar{W}$ is obviously essential in $\bar{R}$ in the sense that it intersects non-trivially every non-zero left ideal of $\bar{R}$. On the other hand, taking an arbitrary non-zero minimal left ideal L of $\bar{R}$, we can see easily that there is a maximal left ideal W of $\bar{R}$ such that

$$L \cap W = 0;$$

hence W is not essential in $\bar{R}$ and, moreover, L is $\bar{R}$-isomorphic to $\bar{R}/W$. As a consequence, L is not $\bar{R}$-isomorphic to $\bar{R}/\bar{W}$.

In order to complete the proof of the implication (ii) $\Rightarrow$ (iii), observe that the idempotents can be lifted modulo Rad R and

$$R = \bigoplus_{i=1}^{m} L_i,$$

where, for each $1 \leqq i \leqq m$, L_i is a left ideal of R containing a unique left ideal K_i of R maximal in L_i. Therefore, all L_i are R-isomorphic, the endomorphism ring $\text{End}_R(L_i)$ of L_i is a local ring with the unique maximal ideal $\{\varphi \mid \varphi \in \text{End}_R(L_i) \text{ and } L_i\varphi \subseteq K_i\}$, and thus, (ii) implies (iii).

Finally, if the ring R has a structure described in (iii), then there are obviously only fundamental torsions in **Mod R** and only two of these. Hence (i) follows and the proof of Theorem 1 is completed.

Theorem 2. *The following properties of a ring R are equivalent and characterize a π-reducible ring:*

(i) *R is a finite direct sum $\bigoplus_{i=1}^{n} R_i$ of rings R_i described in Theorem* 1;
(ii) *Every R-module is a direct sum of its π-primary parts;*
(ii′) *R is a direct sum of its π-primary parts;*
(iii) *R is a right perfect ring whose idempotent two-sided ideals form a sublattice of the lattice of all left ideals of R;*
(iv) *Every R-module is rationally complete, i.e. no R-module is a rational extension of its proper submodule;*
(iv′) *No monogenic* (*i.e. one-generator*) *R-module is a rational extension of its proper submodule.*

Proof. Clearly, (i) implies (ii); for, given an R-module M,

$$M = \bigoplus_{i=1}^{n} R_i M$$

is the decomposition into its π-primary parts. The implication (ii) $\Rightarrow$ (ii′) is trivial. And, since a π-primary part of a ring R is a ring described in Theorem 1 and since, in view of the existence of unity in R, the direct sum of (ii′) is finite, (ii′) implies readily (i).

Now, (iii) follows easily from (i); for, all possible direct sums of some of the rings R_i are just all idempotent two-sided ideals. On the other hand, assuming (iii) and taking all atoms I_i in the lattice of all idempotent ideals, we can easily see that their number is finite and that

$$R = \bigoplus_{i=1}^{n} I_i.$$

Furthermore, I_i are obviously rings of the type described in Theorem 1.

In order to show that (i) implies (iv), it is evidently sufficient to show that no R_i-module M is a rational extension of its proper R_i-submodule N. But this follows immediately from the fact that all simple R_i-submodules, and for that matter those of M/N and of N, are R_i-isomorphic. The implication (iv) $\Rightarrow$ (iv′) is trivial.

Finally, let us complete the proof of Theorem 2 by showing that (iv′) implies (i). First, take a proper left ideal L of R and a maximal left ideal W of R such that

$$L \subseteq W \subset R.$$

Considering the monogenic R-module R/L, the submodule W/L is either a direct summand of it or R/L is an essential extension of W/L; but then, in view of our hypothesis, there must be a simple submodule V/L of W/L such that $V/L \cong R/W$. Thus, in either case, R/L has a non-zero socle, and consequently R possesses a socle sequence. Applying Bass' argument of [**1**], we see that the Jacobson radical $\operatorname{Rad} R$ of R is nil, and hence that the idempotents can be lifted modulo $\operatorname{Rad} R$. But $R/\operatorname{Rad} R$ is completely reducible. This follows easily from our lemma. For, if

$$\operatorname{Socle}(R/\operatorname{Rad} R) \neq R/\operatorname{Rad} R,$$

then $\bar{R} = R/\text{Rad}\, R$ would be a rational extension of $\overline{W}$. Therefore, the (left) principal indecomposable ideals L_i of R in

$$R = \bigoplus_{i=1}^{k} L_i$$

contain a unique maximal left ideal K_i of R $(1 \leqq i \leqq k)$. Now, take i_0 $(1 \leqq i_0 \leqq k)$ and denote by $\mathbf{T}_0$ the prime torsion in **Mod R** corresponding to the class $\mathscr{W}_0$ of equivalent maximal left ideals of R containing the ideal

$$W_0 = \bigoplus_{\substack{1 \leqq i \leqq k;\\ i \neq i_0}} L_i \oplus K_{i_0}.$$

Since the monogenic R-module L_{i_0} is not a rational extension of K_{i_0}, the $\mathbf{T}_0$-torsion part $\mathbf{T}_0(L_{i_0})$ of L_{i_0} is not zero. Assume that $\mathbf{T}_0(L_{i_0}) \neq L_{i_0}$. Then observe that $\mathbf{T}_0(L_{i_0}) \subset K_{i_0}$, $\mathbf{T}_0(L_{i_0}) \neq K_{i_0}$; for,

$$\mathbf{T}_0(L_{i_0}/\mathbf{T}_0(L_{i_0})) = 0.$$

And, since $L_{i_0}/\mathbf{T}_0(L_{i_0})$ is not a rational extension of $K_{i_0}/\mathbf{T}_0(L_{i_0})$, we obtain, by the same argument, a contradiction. Hence, $\mathbf{T}_0(L_{i_0}) = L_{i_0}$. Therefore, for $1 \leqq i < j \leqq k$,

$$\text{either } L_i \cong L_j \text{ or } \text{Hom}_R(L_i, L_j) = 0,$$

and the proof is completed.

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