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SOME NILPOTENT LIE ALGEBRAS OF EVEN DIMENSION

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For each even dimension greater than or equal to 8, an infinite family of 3-step nilpotent Lie algebras over $\mathbb{C}$ is constructed. In dimension m, the family contains isomorphism classes parameterised locally by approximately $m^3/48$ essential parameters.

One particular case is studied further to get some global information about the variety of all nilpotent Lie algebras of dimension 8. Using the results obtained here, and some known facts, it is shown that there is a component consisting of algebras not having minimal possible central dimensions.

I. Introduction

It is known that over the complex numbers there are only finitely many isomorphism classes of nilpotent Lie algebras of dimension less than or equal to 6 [**7**, **4**, **17**] whereas in higher dimensions there are infinite families of pairwise nonisomorphic nilpotent Lie algebras [**3**, **11**]. In dimension 7, each infinite family can be parameterised by a single complex modulus, upon which the structure constants depend analytically [**1**, **6**, **9**, **10**, **13**]. An open problem is to determine exactly how many analytic parameters F_n are needed to classify n-dimensional nilpotent Lie algebras. Thus $F_1 = \cdots = F_6 = 0$; $F_7 = 1$; $F_n \geqslant 1$ for $n \geqslant 8$.

The primary purpose of this paper is to demonstrate by examples that F_{2n+2} is at least $(n(n-1)(n+4))/6 - 3$ for $n \geqslant 3$. The examples are motivated by [**15**]. By examining the case $n = 3$ and using a few other known facts, we can derive some global information about the variety of nilpotent Lie algebras of dimension 8.

II. The Lie algebras $L_{\alpha,\sigma}$

Let $\{e_1, \cdots, e_n, f_1, \cdots, f_n, r, s\}$ be a basis of a complex vector space of dimension

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$2n+2$. Define an algebra structure by the following bracket relations:

$$\begin{aligned}
[e_i, e_j] &= \sum_{k=1}^{n} \alpha_{ij}^k f_k \quad i<j \\
[e_1, f_1] &= r \\
[e_2, f_2] &= s \\
[e_3, f_3] &= r+s \\
[e_i, f_i] &= r+\sigma_i s \quad i=4,\dots,n.
\end{aligned}$$

Use antisymmetry to define $[e_j, e_i]$, $[f_1, e_1]$, *et cetera*. Let all remaining brackets among basis vectors be zero, and extend linearly. This will define a Lie algebra $L_{\alpha,\sigma}$ if the Jacobi identities among all basis vectors are satisfied.

Those Jacobi identities involving central vectors r or s are trivially satisfied. $L_{\alpha,\sigma}$ has a grading, respected by brackets: $\deg(e_i)=1$, $\deg(f_j)=2$, $\deg(r)=\deg(s)=3$. By degree considerations, Jacobi identities involving one or more f_j's are also automatically satisfied. Thus $L_{\alpha,\sigma}$ is a Lie algebra if and only if all Jacobi identities among e_i's are satisfied. Each of these,

$$[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0,$$

yields two equations among the α_{ij}^k's and σ_i's: one for the coefficient of r and another for the coefficient of s. $\mathrm{Jac}(e_i, e_j, e_k)$ says that $p_{ijk}(\alpha)r + q_{ijk}(\alpha,\sigma)s = 0$ where p and q are polynomials. $\mathrm{Jac}(e_1, e_2, e_3)$ says $(-\alpha_{12}^3 - \alpha_{23}^1)r + (-\alpha_{12}^3 + \alpha_{13}^2)s = 0$. For $i \geqslant 4$, σ_i appears in q_{ijk} but not in any of the p's. Furthermore, α_{ij}^k appears only in $\mathrm{Jac}(e_i, e_j, e_k)$. An easy consequence of these facts is that the $2\begin{bmatrix} n \\ 3 \end{bmatrix}$ polynomial conditions on (α,σ) are algebraically independent.

We now consider only those $L_{\alpha,\sigma}$'s which are Lie algebras. For future reference, we define $g_i := [e_i, f_i]$ $i=1,\dots,n$ and assume these are pairwise linearly independent. This is the same as assuming the complex numbers $0, 1, \sigma_4, \dots, \sigma_n$ are distinct, and thus restricts σ to an open set. Another such assumption will be made later. Each $L_{\alpha,\sigma}$ is 3-step nilpotent.

III. Deformation Theoretic View (see, for example, [2] or [4])

A Lie algebra structure is a multiplication table relative to a particular basis. If $\{v_1, \dots, v_{\dim(L)}\}$ is a basis for L,

$$[v_i, v_j] = \sum_{k=1}^{\dim(L)} c_{ij}^k v_k.$$

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The matrix (c_{ij}^k) of structure constants is an element of $\mathbb{C}^{\dim(L)^3}$. The Jacobi identities and antisymmetry impose algebraic conditions on the structure constants. These are the defining equations of the variety of Lie algebra structures. The condition that L is k-step nilpotent (that is, k or less) is also algebraic, given by polynomials of degree k. In particular, for L of dimension n, L is nilpotent if and only if

$$\sum_{j_1,\dots,j_{n-2}} c_{i_1 i_2}^{j_1} c_{j_1 i_3}^{j_2} \cdots c_{j_{n-2} i_n}^{j_{n-1}} = 0 \text{ for all } i_1, \dots, i_{n+1}.$$

The nilpotent algebras form a subvariety $\mathcal{N}_n$ in the variety of all Lie algebras of dimension n.

$GL(n)$ acts on the variety of Lie algebra structures of dimension n, by changing bases. An orbit consists of algebra structures isomorphic to one another. A Lie algebra L is said to degenerate to L' (equally L' deforms to L) if L' lies in the closure of the orbit of L. For instance, given a structure c_{ij}^k for L relative to the basis $\{v_1, \dots, v_n\}$, the corresponding structure relative to $\{\lambda v_1, \dots, \lambda v_n\}$ is (λc_{ij}^k). Letting $\lambda \to 0$, one can see that L degenerates to the abelian Lie algebra. An algebra is said to be rigid if its orbit is open in the variety (of n-dimensional Lie algebras) and nilpotent-rigid if its orbit is open in $\mathcal{N}_n$. It is known [4] that there is one nilpotent-rigid orbit which is dense in the variety of 5-dimensional nilpotent Lie algebras. In particular, this variety has only one algebraic component. The same is true of 6-dimensional nilpotent Lie algebras [**17**, **18**, **14**].

IV. Isomorphisms among the $L_{\alpha,\sigma}$'s

We have not yet seen which $L_{\alpha,\sigma}$'s lie in distinct isomorphism classes. Bearing this in mind, let us call a basis $\{e_1, \dots, e_n, f_1, \dots, f_n, r, s\}$ for $L_{\alpha,\sigma}$ *allowable* if the resulting multiplication table is that of (possibly) another $L_{\alpha',\sigma'}$. First, we consider a basis $\{e_i' = \varepsilon_i e_i, f_i' = \zeta_i f_i, r' = \rho r, s' = \tau s\}$.

$$[e_1', f_1'] = \varepsilon_1\zeta_1[e_1, f_1] = \varepsilon_1\zeta_1 r = \varepsilon_1\zeta_1\rho^{-1}(r')$$
$$[e_2', f_2'] = \varepsilon_2\zeta_2[e_2, f_2] = \varepsilon_2\zeta_2 s = \varepsilon_2\zeta_2\tau^{-1}(s')$$
$$[e_3', f_3'] = \varepsilon_3\zeta_3[e_3, f_3] = \varepsilon_3\zeta_3(r+s) = \varepsilon_3\zeta_3\rho^{-1}(r') + \varepsilon_3\zeta_3\tau^{-1}(s')$$
$$[e_i', f_i'] = \varepsilon_i\zeta_i\rho_{-1}(r') + \varepsilon_i\zeta_i\sigma_i\tau^{-1}(s') \quad i = 4, \dots, n.$$

Allowability implies that $\varepsilon_1\zeta_1 = \varepsilon_2\zeta_2 = \cdots = \varepsilon_n\zeta_n = \rho = \tau$. This in turn implies that $\sigma' = \sigma$.

Secondly, let us determine which of the above bases actually yield the original $L_{\alpha,\sigma}$. This is the same as determining which maps $e_i \mapsto \varepsilon_i e_i$, $f_i \mapsto \zeta_i f_i$, $r \mapsto \rho r$, $s \mapsto \tau s$ are automorphisms of the Lie algebra $L_{\alpha,\sigma}$. Assume we have such an automorphism (that is, $(\alpha') = (\alpha)$).

$$[e_i', e_j'] = \varepsilon_i\varepsilon_j[e_i, e_j] = \sum_k \varepsilon_i\varepsilon_j\alpha_{ij}^k(f_k) = \sum_k \varepsilon_i\varepsilon_j\alpha_{ij}^k\zeta_k^{-1}(f_k') = \sum_k {\alpha'}_{ij}^k(f_k').$$

If $(\alpha) = (\alpha')$ and $\alpha_{ij}^k \neq 0$ then $\varepsilon_i\varepsilon_j = \zeta_k$. Let us assume that enough α_{ij}^k's are nonzero (for example, all of them) to ensure $\varepsilon_i\varepsilon_j = \zeta_k$ for all i, j, k. This places (α) (and hence $L_{\alpha,\sigma}$) in an open set. Then we must have $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$, $\zeta_1 = \cdots = \zeta_n = \varepsilon^2$, and $\rho = \tau = \varepsilon^3$.

Thirdly let us consider all $L_{\alpha',\sigma'}$'s resulting from $L_{\alpha,\sigma}$ by allowable changes of basis. Let Z be the centre of $L_{\alpha,\sigma}$ and let Z^2 be the second centre. In each $L_{\alpha',\sigma'}$ the vectors f_j are special $\pmod Z$ in that $\dim(\mathrm{Image}(ad_{f_j})) = 1$ whereas $\dim(\mathrm{Image}(ad_f)) = 2$ for a linear combination f of 2 or more f_j's. This speciality must be preserved by an automorphism. The f_j's must be permuted, up to scalar multiplication, $\pmod Z$. Similarly, because $\dim([e_i, Z^2]) = 1$, the e_i's must also be permuted, up to scalar multiplication, $\pmod{Z^2}$. An allowable basis must have r', s' $\in Z$, and $f_j' \in Z^2$. Modulo terms of higher degree (which don't change α or σ) an allowable basis can only be obtained by permuting and scaling the original basis vectors (or choosing r and s from among multiples of other g_i's). This shows that those (α', σ')'s for which $L_{\alpha',\sigma'} \cong L_{\alpha,\sigma}$ lie in finitely many components, one of which consists of those (α', σ')'s resulting from bases $\{\varepsilon_i e_i, \zeta_i f_i, \rho r, \tau s\}$ first considered in this section. (There is one component for each permutation of the indices. The union of these components has the same dimension as the dimension of any one of them, so for our modest intentions we need not worry about the other components.)

V. Dimension Counting

There are $n\begin{bmatrix} n \\ 2 \end{bmatrix}$ structure constants α_{ij}^k, where $i < j$. There are $(n-3)$ σ_i's for $i = 4, \dots, n$. Thus (α, σ) lies in a space of dimension $n\begin{bmatrix} n \\ 2 \end{bmatrix} + (n-3)$. The Jacobi identities among the e_i's yield $2\begin{bmatrix} n \\ 3 \end{bmatrix}$ algebraic conditions on (α, σ), each further reducing by 1 the dimension of the variety of (α, σ)'s which correspond to Lie algebras. We have a family of Lie algebras indexed by $n\begin{bmatrix} n \\ 2 \end{bmatrix} + n - 3 - 2\begin{bmatrix} n \\ 3 \end{bmatrix}$ essential parameters. Fix (α, σ). Those (α, σ)'s for which $L_{\alpha',\sigma'} \cong L_{\alpha,\sigma}$ lie in a subspace of dimension n, since (# scaling factors) – (# allowability conditions) – dim{allowable scaling automorphisms of $L_{\alpha,\sigma}$} $= (2n+2) - (n+1) - 1 = n$. The isomorphism classes of $L_{\alpha,\sigma}$

algebras are parameterised by $n\begin{bmatrix} n \\ 2 \end{bmatrix} + n - 3 - 2\begin{bmatrix} n \\ 3 \end{bmatrix} - n = n(n-1)(n+4)/6 - 3$ essential parameters.

VI. Statement of Theorem and Further Discussion

Theorem. *$F_{2n+2} \geqslant n(n-1)(n+4)/6 - 3$ where F_n is the number of essential parameters needed to classify n-dimensional nilpotent Lie algebras over $\mathbb{C}$.*

In general, an n-dimensional algebra has n^3 structure constants c_{ij}^k. A nilpotent Lie algebra has a nicely ordered basis, so that $c_{ij}^k = 0$ if $k \leqslant i$ or $k \leqslant j$. Antisymmetry says $c_{ij}^k = -c_{ji}^k$. We see that a first estimate for F_n is that $F_n \leqslant \begin{bmatrix} n \\ 3 \end{bmatrix} < n^3/6$. The result of this paper is roughly that F_n grows at least as fast as $n^3/48$. Work of Higman and Sims, reinterpreted in this context, indicates that F_n grows roughly as fast as $2n^3/27$ [**5**, **16**]. It is as yet unknown exactly what terms of order lower than n^3 appear in the function F_n [**8**].

VII. Deformation Theoretic Results

The general linear group acts on the variety of Lie algebra structures as described above. The stabiliser of a point is the automorphism group of the corresponding Lie algebra. The dimension of the automorphism group is the same as the dimension of the algebra of derivations of the Lie algebra. Thus the dimension of an orbit can be determined by computing $\dim(\mathrm{Der}(L))$. In our setting, $L_{\alpha,\sigma}$ has dimension $2n+2$. Suppose $\delta \in \mathrm{Der}(L_{\alpha,\sigma})$. Then $\delta(Z) \subset Z$ and $\delta(Z^2) \subset Z^2$. An easy computation, starting from

$$\delta(e_i) = \sum \varepsilon_{ij} e_j + \sum \zeta_{ij} f_j + \chi_i r + \psi_i s$$
$$\delta(f_j) = \sum \eta_{jk} f_k + \chi'_j r + \psi'_j s.$$

For $i \neq j$, $[e_i, f_j] = 0$. Hence $0 = \delta([e_i, f_j]) = [\delta(e_i), f_j] + [e_i, \delta(f_j)] = \varepsilon_{ij} g_j + \eta_{ji} g_i$ and therefore $\varepsilon_{ij} = \eta_{ji} = 0$ for $i \neq j$. Thus we can simplify:

$$\delta(e_i) = \varepsilon_i e_i + \sum \zeta_{ij} f_j + \chi_i r + \psi_i s$$
$$\delta(f_j) = \eta_j f_j + \chi'_j r + \psi'_j s$$
$$\delta(r) = [\delta(e_1), f_1] + [e_1, \delta(f_1)] = (\varepsilon_1 + \eta_1) r$$
$$\delta(s) = [\delta(e_2), f_2] + [e_2, \delta(f_2)] = (\varepsilon_2 + \eta_2) s.$$

Easy computations, starting from $\delta([e_i, e_j]) = [\delta(e_i), e_j] + [e_i, \delta(e_j)]$, show that

1. $\forall i, j, k \;\; \eta_k = \varepsilon_i + \varepsilon_j \Rightarrow \varepsilon_1 = \cdots = \varepsilon_n \; (= \varepsilon)$ and each $\eta_j = 2\varepsilon$.

2. The ζ_{ij}'s are determined by the χ_j's and ψ_j's.
3. $\chi_1, \dots \chi_n, \psi_1, \dots \psi_n, \chi'_1, \dots \chi'_n, \psi'_1, \dots \psi'_n$ and ε can be chosen independently.

Thus δ is determined by these $4n+1$ coefficients and $\dim(\mathrm{Der}(L_{\alpha,\sigma})) = 4n+1$. Finally, we see that each $L_{\alpha,\sigma}$ sits in an orbit of dimension $(2n+2)^2 - (4n+1) = 4n^2 + 4n + 3$.

The degenerate case $n = 3$ makes sense, and although σ is indexed by the empty set, all of the above results hold for $n = 3$. (Note, however, that 4 is somewhat less than $8^3/48$.) There is a 4-parameter family of nonisomorphic 8-dimensional algebras, each having an orbit of dimension 51.

The most general 8-dimensional nilpotent Lie algebras (in some sense) are Umlauf's N_γ algebras, $\gamma \neq 1$. The nonzero brackets are given by

$$\begin{aligned}
[v_1, v_i] &= v_{i+1} \quad i = 2, \dots, 7 \\
[v_2, v_3] &= v_5 \\
[v_2, v_4] &= v_6 \\
[v_2, v_5] &= \gamma v_7 + v_8 \\
[v_2, v_6] &= (2\gamma - 1) v_8 \\
[v_3, v_4] &= (1 - \gamma) v_7 - v_8 \\
[v_3, v_5] &= (1 - \gamma) v_8 .
\end{aligned}$$

These algebras have centres Z, Z^2, *et cetera* of minimal possible dimension (an open condition), and among all such algebras, each of the others has some other closed, or degenerate, property (such as the existence of a 6-dimensional abelian ideal). It is impossible for any of the N_γ algebras to be a degeneration of any other 8-dimensional nilpotent Lie algebra. The union $\mathcal{U}$ of the orbits of the N_γ's constitutes a Zariski-open set in the variety of all nilpotent Lie algebra structures of dimension 8. A few pages of straightforward computation show that each N_γ, $\gamma \neq 1$, has a derivation algebra of dimension 10 and therefore an orbit of dimension 54. The closure of $\mathcal{U}$ is an algebraic component of dimension 55. There is in fact another algebraic component of dimension 55 containing a dense open subset of filiform Lie algebras [**1**, **12**]. (A nilpotent Lie algebra is *filiform* when there is an element whose centraliser is 2-dimensional.) All filiform algebras lie in these two components.

The union $\mathcal{L}$ of the $L_{\alpha,\sigma}$ orbits also has a closure of dimension 55. Because of dimension considerations, $\mathcal{L}$ cannot lie in either of the two filiform components. The number of algebraic components is at least 3, and in particular there is a component consisting of algebras which are not filiform.

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