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QUASI-DETERMINANTS AND q-COMMUTING MINORS

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Abstract. We present two new proofs of the q-commuting property holding among certain pairs of quantum minors of a q-generic matrix. The first uses elementary quasi-determinantal arithmetic; the second involves paths in a directed graph. Together, they indicate a means to build the multi-homogeneous coordinate rings of flag varieties in other non-commutative settings.

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1. Introduction and main theorem. This paper arose from an attempt to understand the 'quantum shape algebra' of Taft and Towber [**15**], which we call the *quantum flag algebra* $\mathcal{F}\ell_q(n)$ here. One goal was to find quasi-determinantal justifications for the relations chosen for $\mathcal{F}\ell_q(n)$. A second goal was to find some hidden relations, within $\mathcal{F}\ell_q(n)$, known to hold in an isomorphic image. We save further remarks on the history of the problem and the present goals for after a statement of the theorem.

DEFINITION 1. Given two subsets $I, J \subseteq [n]$, we say J is *weakly separated* by I, written $J \curvearrowright I$, if (i) $|J| \le |I|$ and (ii) there exist disjoint subsets $\emptyset \subseteq J', J'' \subseteq J$ such that
- $J \setminus I = J' \,\dot\cup\, J''$,
- $j' < i$ for all $j' \in J'$ and $i \in I \setminus J$,
- $i < j''$ for all $i \in I \setminus J$ and $j'' \in J'$,

In this case, we put $\langle\langle J, I\rangle\rangle = |J''| - |J'|$.

Given an $n \times n$ *q-generic matrix* X and a subset $I \subseteq [n]$ with $|I| = d$, we write $[\![I]\!]$ for the quantum minor built from X by taking row-set I and column-set $[d]$.

THEOREM 2 (q-Commuting Minors). *Fix two subsets $I, J \subseteq [n]$. If $J \curvearrowright I$, then the quantum minors $[\![J]\!]$ and $[\![I]\!]$* q-commute. *Specifically,*

$$[\![J]\!][\![I]\!] = q^{\langle\langle J,I\rangle\rangle}[\![I]\!][\![J]\!]\,. \tag{1}$$

A proof in the case $I \cap J = \emptyset$ may be found in [**10**], while Leclerc and Zelevinsky [**13**] show that $[\![J]\!][\![I]\!] = q^a[\![I]\!][\![J]\!]$ for some $a \in \mathbb{Z}$ if and only if I and J are weakly separated. By now, many more commutation formulas are known for much larger collections of quantum minors (see [**3**, **6**]). The impetus for finding such results has been two-fold: (i) from the point of view of representation theory, such questions are intimately tied to the study of the canonical (or crystal) bases of Lustzig and Kashiwara [**1**, **8**, **14**]; (ii) from the point of view of non-commutative algebraic geometry, the study of quantum determinantal ideals provides non-commutative versions of the classical determinantal varieties [2, 7, 9]. Our goal is different.

Given a non-commutative algebra $\mathcal{A}$ with a 'quantum' determinant D, can we readily define an $\mathcal{A}$-analogue of $\mathcal{F}\ell_q(n)$ by specializing quasi-determinantal identities to the pair $(\mathcal{A}, D)$? Towards this goal, we analyse the gold standard $\mathcal{F}\ell_q(n)$ from a quasi-determinantal point of view. This idea leads to two new proofs of Theorem 2. The first proof ($\mathcal{Q}$) uses simple arithmetic involving quasi-determinants; the second ($\mathcal{G}$) involves counting weighted paths on a directed graph. Taken together, they imply that if $(\mathcal{A}, D)$ satisfies some version of Theorem 9, then quasi-Plücker relations indicate how to define the flag algebra for $\mathcal{A}$.

1.1. Useful notation. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $\binom{[n]}{d}$ denote the subsets of $[n]$ of size d. Given a set I and a subset $I' \subseteq I$, we sometimes write $I^{I'}$ for the set difference $I \setminus I'$. Given a set $I = \{i_1 < i_2 < \cdots < i_d\}$, we will view I as the d-tuple $(i_1, i_2, \ldots, i_d)$ when convenient. Fix $i \in [n]$ and suppose $I = \{i_1 < i_2 < \cdots < i_d\} \subseteq [n]$. If there is some $1 \leq k \leq d$ with $i_k = i$, we write $\mathrm{pos}_I(i) = k$ for the *position* of i in I.

Let $[n]^d$ denote the d-tuples (sequences) with entries chosen from $[n]$. Given a sequence $I \in [n]^d$ with distinct entries and a subsequence I', interpret $I^{I'}$ as the complementary subsequence. Given $I \in [n]^d$ with distinct entries, put $\mathrm{inv}(I) := \#\big\{(j,k) : j < k \text{ and } i_j > i_k\big\}$. Similarly, given sets or tuples I and J, put $\mathrm{inv}(I,J) := \#\big\{(i,j) : i \in I, j \in J, \text{ and } i > j\big\}$. Given $i \in [n]$, extend the definition of $\mathrm{pos}_I(i)$ to tuples $I \in [n]^d$ with distinct entries in the obvious manner. If I, J are two sets or tuples of sizes d, e, respectively, we define $I \lceil J$ to be the $(d+e)$-tuple $(i_1, \ldots, i_d, j_1, \ldots, j_e)$.

Let A be an $n \times n$ matrix whose rows and columns are indexed by the sets R and C, respectively. For any $R' \subseteq R$ and $C' \subseteq C$, we let $A^{R',C'}$ denote the submatrix built from A by deleting row-indices R' and column-indices C'. Let $A_{R',C'}$ be the complementary submatrix. In case $R' = \{r\}$ and $C' = \{c\}$, we may abuse notation and write, e.g., A^{rc}. Given d-tuples I and J chosen from R and C, respectively, we let $A_{I,J}$ denote the matrix built from A in the obvious manner: repeating or rearranging the rows and columns of A as necessary.

2. Preliminaries for $\mathcal{Q}$-proof.

2.1. Quasi-determinants. The quasi-determinant [5] was introduced by Gelfand and Retakh as a replacement for the determinant over non-commutative rings $\mathcal{R}$. Given an $n \times n$ matrix $A = (a_{ij})$ over $\mathcal{R}$, the quasi-determinant $|A|_{ij}$ (there is one for each position (i,j) in the matrix) is not a polynomial in the entries a_{ij} but rather a rational expression. We collect here those definitions and results that are needed in the coming section. Further details may be found in [4, 10, 12]. Note that the phrase 'when defined' is implicit throughout.

Definition 3. Given A and $\mathcal{R}$ as above, if A^{ij} is invertible over $\mathcal{R}$, then the (i,j) *quasi-determinant* is defined and given by

$$|A|_{ij} = a_{ij} - \rho_i \cdot (A^{ij})^{-1} \cdot \chi_j\,,$$

where ρ_i is the ith row of A with column j deleted and χ_j is the jth column of A with row i deleted. In particular, $|A|_{ij}^{-1} = (A^{-1})_{ji}$ when both sides are defined.

THEOREM 4 (Homological Relations). *Let A be a square matrix and let $i \neq j$ $(k \neq l)$ be two row (column) indices. We have*

$$-|A^{jk}|_{il}^{-1} \cdot |A|_{ik} = |A^{ik}|_{jl}^{-1} \cdot |A|_{jk}.$$

THEOREM 5 (Muir's Law of Extensible Minors). *Let $A = A_{R,C}$ be a square matrix. Fix $R_0 \subsetneq R$ and $C_0 \subsetneq C$. Say a rational expression $\mathcal{I} = \mathcal{I}(A, R_0, C_0)$ in the quasi-minors $\{|A_{R',C'}|_{rc} : r \in R' \subseteq R_0,\ c \in C' \subseteq C_0\}$ is an* identity *if the equation $\mathcal{I} = 0$ is valid. Fix subsets $L \subseteq R \setminus R_0$ and $M \subseteq C \setminus C_0$. If $\mathcal{I}$ is an identity, then the expression $\mathcal{I}'$ built from $\mathcal{I}$ by extending all minors $|A_{R',C'}|_{rc}$ to $|A_{L\cup R',M\cup C'}|_{rc}$ is also an identity.*

DEFINITION 6. Let B be an $n \times m$ matrix. For any $i, j, k \in [n]$ and $M \subseteq [n] \setminus \{i\}$ with $|M| = d - 1$, define $r_{ji}^{M} = r_{ji}^{M}(B) := |B_{(j\lceil M),[d]}|_{jk} |B_{(i\lceil M),[d]}|_{ik}^{-1}$. This ratio is independent of k and is called a *right-quasi-Plücker* coordinate for B.

REMARK. Note that the r_{ij}^{M} aren't ratios of minors of B, as defined. It is easy to see that $|B_{(j\lceil M),[d]}|_{jk} |B_{(i\lceil M),[d]}|_{ik}^{-1} = |B_{j\cup M,[d]}|_{jk} |B_{i\cup M,[d]}|_{ik}^{-1}$ when $j \notin M$. We choose to work with generalised minors such as $|B_{(j\lceil M),[d]}|_{jk}$ for book-keeping purposes in the coming proofs.

THEOREM 7 (Quasi-Plücker Relations). *Fix an $n \times n$ matrix A, subsets $M, L \subseteq [n]$ with $|M| + 1 \leq |L|$, and $i \in [n] \setminus M$. We have the* quasi-Plücker relation $(\mathcal{P}_{L,M,i})$

$$1 = \sum_{j \in L} r_{ij}^{L\setminus j} r_{ji}^{M}.$$

2.2. Quantum determinants. We collect standard results about the quantum determinant that may be found in the literature [**16**]. An $n \times n$ matrix $X = (x_{ab})$ is said to be q-generic if its entries satisfy the relations

$$\begin{aligned}
(\forall i,\ \forall k < l) \quad & x_{il}x_{ik} = q x_{ik}x_{il} \\
(\forall i < j,\ \forall k) \quad & x_{jk}x_{ik} = q x_{ik}x_{jk} \\
(\forall i < j,\ \forall k < l) \quad & x_{jk}x_{il} = x_{il}x_{jk} \\
(\forall i < j,\ \forall k < l) \quad & x_{jl}x_{ik} = x_{ik}x_{jl} + (q - q^{-1})x_{il}x_{jk}.
\end{aligned}$$

Fix a field $\Bbbk$ of characteristic 0 and a distinguished invertible element $q \in \Bbbk$ not equal to a root of unity. Let $\mathrm{M}_q(n)$ be the $\Bbbk$-algebra with n^2 generators x_{ab} subject to the relations making X a q-generic matrix. It is known [**9**] that $\mathrm{M}_q(n)$ is a (left) Ore domain with (left) field of fractions $D_q(n)$.

DEFINITION 8. Given any $d \times d$ matrix A, define the row determinant $\det_q A$ by

$$\det_q A = \sum_{\sigma \in \mathfrak{S}_d} (-q)^{-\mathrm{inv}(\sigma)} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(d),d}.$$

When $A = X_{R,C}$ is a submatrix of X, it is known that: (i) $\det_q A$ agrees with the analogous expression modelled after the column-permutation definition of the determinant; (ii) swapping two adjacent rows of A introduces a q^{-1}; and (iii) allowing

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any row of A to appear twice yields zero. Properties (i)–(iii) have the following important consequence.

THEOREM 9 (Quantum Determinantal Identities). *Let $A = X_{R,C}$ be a $d \times d$ submatrix of X. Then for all $i, j \in R$ and $k \in C$, we have:*

$$\sum_{c \in C} A_{jc} \cdot \left\{(-q)^{\mathrm{pos}_R(i) - \mathrm{pos}_C(c)} \det_q A^{ic}\right\} = \delta_{ij} \cdot \det_q A\,,$$

$$[\det_q A,\ A_{ik}] = 0.$$

In particular, every submatrix of X is invertible in $D_q(n)$. After Definition 3, we are free to use the relations in Section 2.1 on matrices built from X. Properties (ii) and (iii) allow us to uniquely define the quantum determinant of $A = X_{I,C}$ for any $I \in [n]^d$ and $C \in \binom{[n]}{d}$. In case $C = [d]$, we introduce the shorthand notation $\det_q A = [\![I]\!]$. The link between quasi- and quantum-determinants is as follows: for all $I \in [n]^d$ with distinct entries

$$|X_{I,[d]}|_{i,d} = (-q)^{d - \mathrm{pos}_I(i)} [\![I]\!] \cdot [\![I^i]\!]^{-1}. \tag{2}$$

Moreover, the factors on the right commute.

REMARK. Note, again, that our indexing sets I are d-tuples, not subsets of $[n]$. In the coming proofs, the reader should find it easier to keep track of powers of q with this convention.

Theorems 4 and 9 are sufficient to give the next result ([**12**], Proposition 10).

THEOREM 10. *Given any $i, j \in [n]$, $\{j\} \curvearrowright \{i\}$. For any $M \subseteq [n]$, the quantum minors $[\![j \lceil M]\!]$ and $[\![i \lceil M]\!]$ q-commute according to* (1).

3. $\mathcal{Q}$-proof of theorem. Our first proof of Theorem 2 proceeds by induction on $|J|$ and rests on two key lemmas.

LEMMA 11. *Given $I \subseteq [n]$ and $j \in [n] \setminus I$, suppose $\{j\} \curvearrowright I$. Then $[\![j]\!][\![I]\!] = q^{\langle\langle j,I \rangle\rangle}[\![I]\!][\![j]\!]$.*

Proof. From $(\mathcal{P}_{I,\emptyset,j})$ and (2) we have

$$1 = \sum_{i \in I} [\![j \lceil I^i]\!][\![i \lceil I^i]\!]^{-1}[\![i]\!][\![j]\!]^{-1}\,,$$

or

$$[\![j]\!] = \sum_{i \in I} [\![j \lceil I^i]\!][\![i \lceil I^i]\!]^{-1}[\![i]\!]. \tag{3}$$

Theorem 10 tells us that $[\![j \lceil I^i]\!]$ and $[\![i \lceil I^i]\!]$ q-commute, so we may clear the denominator in (3) on the left and get

$$[\![I]\!][\![j]\!] = \sum_{i \in I} (-q)^{\mathrm{inv}(i,I^i)} q^{-\langle\langle j,I \rangle\rangle} [\![j \lceil I^i]\!][\![i]\!]. \tag{4}$$

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In the other direction, Theorem 9 tells us that $[\![i\lceil I^i]\!]$ and $[\![i]\!]$ commute; clearing (3) on the right yields

$$[\![j]\!][\![I]\!] = \sum_{i\in I}(-q)^{\mathrm{inv}(i,I^i)}[\![j\lceil I^i]\!][\![i]\!]. \tag{5}$$

Compare (4) and (5) to conclude that $[\![j]\!]$ and $[\![I]\!]$ q-commute as desired. □

LEMMA 12. *Given $I, J, M \subseteq [n]$, if $[\![J]\!]$ and $[\![I]\!]$ q-commute, then $[\![J\cup M]\!]$ and $[\![I\cup M]\!]$ do as well. Moreover, they do so with the same q exponent.*

Proof. An easy consequence of (2) and Muir's Law (Theorem 5). □

We are now ready for the first advertised proof of Theorem 2.

Proof of Theorem 2. Fix $J, I \subseteq [n]$ and suppose $J \frown I$. Note that, by definition of 'weakly separated', $J\cup M \frown I\cup M$ for all $M \subseteq [n]\setminus(I\cup J)$. After Lemma 12, we may thus assume $I\cap J = \emptyset$. We proceed by induction on $|J|$, the base case being handled in Lemma 11.

Let j be the least element of J, i.e. $\mathrm{inv}(j, J^j) = 0$, and consider $(\mathcal{P}_{I,J^j,j})$:

$$1 = \sum_{i\in I} r_{ji}^{I\setminus i} r_{ij}^{J^j}.$$

In terms of quantum determinants, we have

$$[\![j\lceil J^j]\!] = \sum_{i\in I}[\![j\lceil I^i]\!][\![i\lceil I^i]\!]^{-1}[\![i\lceil J^j]\!].$$

By induction, we may clear the denominator to the right and get

$$[\![j\lceil J^j]\!][\![I]\!] = q^{\langle\!\langle J^j,I^i\rangle\!\rangle}\sum_{i\in I}(-q)^{\mathrm{inv}(i,I^i)}[\![j\lceil I^i]\!][\![i\lceil J^j]\!]. \tag{6}$$

On the otherhand, we may clear the denominator on the left at the expense of $q^{-\langle\!\langle j,i\rangle\!\rangle}$:

$$[\![I]\!][\![j\lceil J^j]\!] = q^{-\langle\!\langle j,i\rangle\!\rangle}\sum_{i\in I}(-q)^{\mathrm{inv}(i,I^i)}[\![j\lceil I^i]\!][\![i\lceil J^j]\!]. \tag{7}$$

We are nearly done. First note that

$$q^{\langle\!\langle J^j,I^i\rangle\!\rangle} = q^{\langle\!\langle J^j,I\rangle\!\rangle}, \quad q^{-\langle\!\langle j,i\rangle\!\rangle} = q^{-\langle\!\langle j,I\rangle\!\rangle}, \quad \text{and} \quad q^{\langle\!\langle J,I\rangle\!\rangle} = q^{\langle\!\langle j,I\rangle\!\rangle}q^{\langle\!\langle J^j,I\rangle\!\rangle}.$$

Using these observations to compare (6) and (7) finishes the proof. □

4. Preliminaries for $\mathcal{G}$-proof.

4.1. Quantum flag algebra. The algebra $\mathcal{F}\ell_q(n)$, as it is presented below, first appeared in [**15**]. An equivalent presentation due to Lakshmibai and Reshetikhin appeared concurrently [**11**].

DEFINITION 13 (Quantum Flag Algebra). The quantum flag algebra $\mathcal{F}\ell_q(n)$ is the $\Bbbk$-algebra generated by symbols $\{f_I : I \in [n]^d,\ 1 \le d \le n\}$ subject to the relations

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indicated below. (Recall that to a subset $\{i_1 < i_2 < \cdots < i_d\} \in \binom{[n]}{d}$, we associate the d-tuple $(i_1, i_2, \ldots, i_d)$.)

- *Alternating* relations $(\mathcal{A}_I)$: For any $I \in [n]^d$ and $\sigma \in \mathfrak{S}_d$,

$$f_{\sigma I} = \begin{cases} 0, & \text{if } I \text{ contains repeated indices,} \\ (-q)^{-\mathrm{inv}(\sigma)} f_I, & \text{if } I = (i_1 < i_2 < \cdots < i_d). \end{cases} \tag{8}$$

- *Young symmetry* relations $(\mathcal{Y}_{I,J})_{(a)}$: Fix $1 \le a \le d \le e \le n-a$. For any $I \in \binom{[n]}{e+a}$ and $J \in \binom{[n]}{d-a}$,

$$0 = \sum_{\Lambda \subseteq I, |\Lambda| = a} (-q)^{-\mathrm{inv}(I^\Lambda, \Lambda)} f_{I^\Lambda} f_{\Lambda \int J}. \tag{9}$$

- *Monomial straightening* relations $(\mathcal{M}_{J,I})$: For any $I, J \subseteq [n]$, $|J| \le |I|$,

$$f_J f_I = \sum_{\Lambda \subseteq I, |\Lambda| = |J|} (-q)^{\mathrm{inv}(\Lambda, I^\Lambda)} f_{J \int I^\Lambda} f_\Lambda. \tag{10}$$

In their article, Taft and Towber construct an algebra map $\phi : \mathcal{F}\ell_q(n) \to \mathrm{M}_q(n)$ taking f_I to $[\![I]\!]$ and show that ϕ is monic, with image the subalgebra of $\mathrm{M}_q(n)$ generated by the quantum minors $\{[\![I]\!] : I \in [n]^d,\ 1 \le d \le n\}$. We have already seen that the minors $[\![I]\!]$ often q-commute. This relation does not appear above, so it must be a consequence of (8)–(10). The coming proof explicitly demonstrates this connection.

Abbreviate the right-hand side of (9) by $Y_{I,J;(a)}$. Also, we abbreviate the difference $(lhs - rhs)$ in (10) by $M_{J,I}$, and the difference $(lhs - rhs)$ in (1) by $C_{J,I}$ (replacing $[\![\text{-}]\!]$ by $f_{\text{-}}$). As (1), (9) and (10) are all homogeneous, $C_{J,I}$ must be some $\Bbbk$-linear combination of the expressions $M_{K,L}$ and $Y_{M,N;(a)}$, modulo (8).

Example ($\{1\} \curvearrowright \{2,3,4\}$). We calculate the expressions $C_{1,234}$, $M_{1,234}$ and $Y_{1234,\emptyset;(1)}$ and arrange them as rows in Table 1. Viewing the table column by column, we readily see that $C_{1,234} = M_{1,234} + q^2 Y_{1234,\emptyset;(1)}$.

While the idea for our second proof of Theorem 2 is simple ('perform Gaussian elimination'), the proof itself is not. We separate out the combinatorial component below.

4.2. Weighted paths in a directed graph. Given $I, J \subseteq [n]$ such that $J \curvearrowright I$, we build the edge-weighted directed graph $\Gamma(J) = \Gamma(J; I)$ as follows. Its vertex set $\mathcal{V}$ is the power set $\mathcal{P}(J)$ and its edge set is $\{(A, B) \mid A, B \in \mathcal{V},\ A \subsetneq B\}$. The *weight* of an edge $(A, B) \in \Gamma$ depends on $|I|$, carrying the value

$$\alpha_A^B = (-q)^{-\mathrm{inv}(J^B, B^A) - \mathrm{inv}(B^A, A) + \left(2|J^B| - |I|\right)|(B^A) \cap J'|}, \tag{11}$$

with J' as in Definition 1.

Table 1. Finding the relation $f_1 f_{234} - q^{-1} f_{234} f_1 = 0$.

$C_{1,234}$	$f_1 f_{234}$				$-q^{-1} f_{234} f_1$
$M_{1,234}$	$f_1 f_{234}$	$-q^2 f_{123} f_4$	$+q^1 f_{124} f_3$	$-q^0 f_{134} f_2$	
$Y_{1234,\emptyset;(1)}$		$f_{123} f_4$	$-q^{-1} f_{124} f_3$	$+q^{-2} f_{134} f_2$	$-q^{-3} f_{234} f_1$

EXAMPLE. If $|J| = m$, then $\Gamma(J)$ has 2^m vertices and $\sum_{k=1}^{m} \binom{m}{k}(2^k - 1) = 3^m - 2^m$ edges. In Figure 1, we illustrate $\Gamma(\{1, 6\})$ and $\Gamma(\{1, 5, 6\})$, omitting three edges and many edge weights in the latter for legibility.

For the remainder of the section, we assume that $J \curvearrowright I$ with $J \cap I = \emptyset$. We write $J = J' \dot{\cup} J'' = \{j_1 < \cdots < j_{r'}\} \cup \{j_{r'+1} < \cdots < j_{r'+r''}\}$ (with $j_{r'} < j_{r'+1}$), and we let $r = r' + r'' = |J|$, $s = |I|$ and $s - r = t$. In the graph $\Gamma(J; I)$, we consider *paths* π on p steps $(0 < p < r)$ defined as follows:

$$\mathfrak{P}_0 = \big\{\pi = (A_1, A_2, \ldots, A_p) \mid \emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p \subsetneq J\big\}.$$

We form $\mathfrak{P} = \mathfrak{P}_0$ by adjoining the unique path $\hat{0} = ()$ on zero steps and the special path $\hat{1}$ on r steps given by

$$\hat{1} = \big(\{j_{r'+1}\}, \{j_{r'+1}, j_{r'+2}\}, \ldots, J'', \{j_{r'}, \ldots, j_r\}, \ldots, \{j_2, \ldots, j_r\}, J\big).$$

The *weight* $\alpha(\pi)$ of a path $\pi \in \mathfrak{P}$ is given by $\alpha(\hat{0}) = \alpha_{\emptyset}^{J}$, for $\pi = \hat{0}$, and otherwise

$$\alpha(\pi) = \alpha_{\emptyset}^{A_1} \cdot \alpha_{A_1}^{A_2} \cdots \alpha_{A_{p-1}}^{A_p} \cdot \alpha_{A_p}^{J}.$$

The aim of the present discussion is to divide the paths $\mathfrak{P}$ into two equinumerous camps via a bijection $\wp$ satisfying $\alpha(\wp(\pi)) = \alpha(\pi)$. This will be useful in Section 5, where it will make a rather unwieldy sum (14) collapse to a single term. We divide $\mathfrak{P}$ into two parts using the function $\mathsf{mM}(\text{-})$ defined as follows. Fix $K \subseteq J$. If $K \cap J' \neq \emptyset$, put $\mathsf{mM}(K) = \mathsf{min}(K \cap J')$. Otherwise, put $\mathsf{mM}(K) = \mathsf{max}(K \cap J'')$.

DEFINITION 14. A path $(A_1, \ldots, A_p) \in \mathfrak{P}$ shall be called *regular* (or *regular at position* i_0), if $p > 0$ and there exists $1 \leq i_0 \leq p$ satisfying:

(a) $|A_i| = i \ \ (\forall\, 1 \leq i \leq i_0)$;

(b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1})$.

Here and below, we take $A_0 = \emptyset$ and $A_{p+1} = J$, as needed. A path is *irregular* if it is nowhere regular. (Note $\hat{0}$ is irregular and $\hat{1}$ is regular.)

PROPOSITION 15. *The regular and irregular paths in* $\mathfrak{P}$ *are equinumerous.*

Given an irregular path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}$, we construct a regular path $\wp(\pi)$ by inserting a new step B. If $\pi = \hat{0}$, put $\wp(\hat{0}) = (\{j_1\})$. Otherwise:

(1) Find the unique i_0 satisfying: $(|A_i| = i \quad \forall i \leq i_0)$ and $(|A_{i_0+1}| > i_0 + 1)$.

(2) Compute $b = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0})$

(3) Put $B = A_{i_0} \cup \{b\}$.

(4) Define $\wp(\pi) := (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A_p)$.

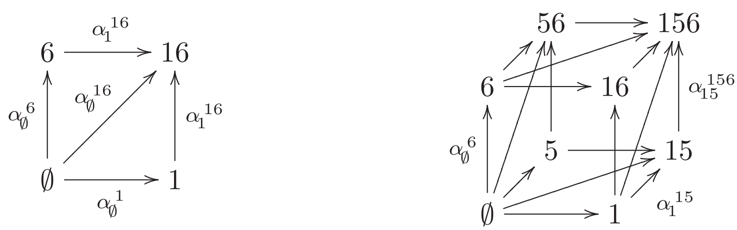

Figure 1. The graphs $\Gamma(\{1, 6\})$ and $\Gamma(\{1, 5, 6\})$ (partially rendered).

EXAMPLE. Table 2 illustrates the action of $\wp$ on $\mathfrak{P}$ when $J = J' \cup J'' = \{1\} \cup \{5, 6\}$.

Proof of Proposition 15. Let $\mathfrak{P}'$ and $\mathfrak{P}''$ denote the irregular and regular paths, respectively. We reach a proof in three steps.

Claim 1: $\wp(\mathfrak{P}') \subseteq \mathfrak{P}''$.

Given $\pi \in \mathfrak{P}'$, the effect of $\wp$ (namely, adding a step B to the path π) is to insert a regular point, so the claim is proven if we can show that $\wp(\pi) \in \mathfrak{P}$.

Since $\wp(\hat{0})$ belongs to $\mathfrak{P}$, we turn to the irregular paths $\pi = (A_1, A_2, \ldots, A_p)$ in $\mathfrak{P}_0$. The only concern is that the inserted step may be $B = J$, which would put $\wp(\pi)$ in $\mathfrak{P}$ only if $\wp(\pi) = \hat{1}$.

Case $p < r - 1$: At some point $1 \leq i_0 < p$, there is a jump in set-size greater than one when moving from A_{i_0} to A_{i_0+1}. Hence, the B to be inserted will not come at the end, but rather immediately after A_{i_0}.

Case $p = r - 1$: Let $\hat{1} = (A_1, A_2, \ldots, A_r)$. One checks that $(A_1, A_2, \ldots, A_{r-1})$ is nowhere regular, and that this is the only path on $r - 1$ steps with this feature. Since $\wp((A_1, A_2, \ldots, A_{r-1})) = \hat{1}$, we are done.

Claim 2: $\wp$ is 1-1.

Suppose $\wp(A_1, \ldots, A_p) = \wp(A'_1, \ldots, A'_{p'})$, and suppose we insert B and B' respectively. By the nature of $\wp$, we have $p = p'$ and $i_0 \neq i'_0$. Take $i_0 < i'_0$ and notice that $(A'_1, \ldots, A'_{p'}) = (A_1, \ldots, A_{i_0}, B, A_{i_0+1}, \ldots, A'_{i'_0}, \ldots A'_{p'})$. In particular, B is a regular point of $(A'_1, \ldots, A'_{p'})$, and consequently, $(A'_1, \ldots, A'_{p'}) \notin \mathfrak{P}'$.

Claim 3: $\wp$ is onto.

Consider a path $\pi = (A_1, \ldots, A_p) \in \mathfrak{P}''$. If $p = 1$, then it is plain to see that the only irregular path is $\pi = (\{j_1\})$, which is the image of $\hat{0}$ under $\wp$. So assume $p > 1$. Note that $|A_1| = 1$, for otherwise π cannot have any regular points. Now, locate the first $1 \leq i_0 \leq p$ with (a) $|A_{i_0}| = i_0$; and (b) $A_{i_0} \setminus A_{i_0-1} = \mathsf{mM}(A_{i_0+1} \setminus A_{i_0-1})$. The path $\pi' = (A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_k)$ belongs to $\mathfrak{P}'$ and, moreover, $\wp(\pi') = \pi$. □

The map $\wp$ we have used has an additional nice property.

PROPOSITION 16. *The bijection $\wp$ is path-weight preserving.*

The result rests on the following result.

LEMMA 17. *Let $\emptyset \subseteq A \subseteq B \subseteq C \subseteq J$. Writing $\hat{B} = B \setminus A$ and $\hat{C} = C \setminus B$, we have*

$$\alpha_A^B \alpha_B^C = \left[(-q)^{2\,\mathrm{inv}(\hat{B}\cap J', \hat{C}) - 2\,\mathrm{inv}(\hat{C}, \hat{B}\cap J'')}\right] \alpha_A^C. \tag{12}$$

Proof. From the definition of α_*^*, we have

$$\begin{aligned}
\alpha_A^B &= (-q)^{-\mathrm{inv}(J^B, \hat{B}) - \mathrm{inv}(\hat{B}, A) + \left(2|J^A| - 2|\hat{B}| - |I|\right)|\hat{B}\cap J'|},\\
\alpha_B^C &= (-q)^{-\mathrm{inv}(J^C, \hat{C}) - \mathrm{inv}(\hat{C}, \hat{B}\cup A) + \left(2|J^A| - 2|\hat{B}\cup\hat{C}| - |I|\right)|\hat{C}\cap J'|},\\
\alpha_A^C &= (-q)^{-\mathrm{inv}(J^C, \hat{B}\cup\hat{C}) - \mathrm{inv}(\hat{B}\cup\hat{C}, A) + \left(2|J^C| - 2|\hat{B}\cup\hat{C}| - |I|\right)|(\hat{B}\cup\hat{C})\cap J'|}.
\end{aligned}$$

Table 2. The pairing of irregular and regular paths via $\wp$.

π	$\hat{0}$	(5)	(6)	(15)	(16)	(56)	(5,56)
$\wp(\pi)$	(1)	(5,15)	(6,16)	(1,15)	(1,16)	(6,56)	$\hat{1}$

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Now compare exponents on either side of (12), using identities such as

$$|\hat{C}||\hat{B}\cap J'| = \mathrm{inv}(\hat{C}, \hat{B}\cap J') + \mathrm{inv}(\hat{B}\cap J', \hat{C}),$$
$$\mathrm{inv}(\hat{C}, \hat{B}) = \mathrm{inv}(\hat{C}, \hat{B}\cap J')\,\mathrm{inv}(\hat{C}, \hat{B}\cap J'').$$

□

Proof of Proposition 16. Suppose that $\pi = (\ldots, A, C, \ldots)$ and $\wp(\pi)$ inserts B immediately after A. Putting $B = A \cup \mathsf{mM}(C \setminus A) = A \cup \{b\}$, (12) implies that

$$\alpha(\wp(\pi)) = \left[(-q)^{2\,\mathrm{inv}(b\cap J', \hat{C}) - 2\,\mathrm{inv}(\hat{C}, b\cap J'')}\right]\cdot \alpha(\pi)\,,$$

where $\hat{B}$ and $\hat{C}$ are as in the lemma. Now, if $b\cap J' \neq \emptyset$, then b is the smallest element in $C\setminus A$, and, in particular, $\mathrm{inv}(b, \hat{C}) = 0$. In this same case, $b \cap J'' = \emptyset$, so $\mathrm{inv}(\hat{C}, b\cap J'') = 0$ too. An analogous argument works for the case $b\cap J' = \emptyset$. □

Before leaving path weights behind, we compute the weight of $\wp^{-1}(\hat{1})$ explicitly.

PROPOSITION 18. *Given, I, J, J', J'' and $\hat{1}$ as above, we have*

$$\alpha(\wp^{-1}(\hat{1})) = (-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J}. \tag{13}$$

Proof. Recall that $\pi = \wp^{-1}(\hat{1})$ is the path

$$\pi = \big(\{j_{r'+1}\}, \{j_{r'+1}, j_{r'+2}\}, \cdots, J'', \{j_{r'}, \ldots, j_r\}, \ldots, \{j_2, \ldots, j_r\}\big).$$

Applying (12) repeatedly to $\alpha(\pi)$ we see that

$$\begin{aligned}
\alpha(\pi) &= \alpha_{\emptyset}^{j_{r'+1}} \alpha_{j_{r'+1}}^{j_{r'+1}j_{r'+2}} \left(\alpha_{j_{r'+1}j_{r'+2}}^{j_{r'+1}\cdots j_{r'+3}} \cdots \alpha_{j_2\cdots r}^{J}\right)\\
&= \left[(-q)^{-2(1)}\right] \alpha_{\emptyset}^{j_{r'+1}j_{r'+2}} \left(\alpha_{j_{r'+1}j_{r'+2}}^{j_{r'+1}\cdots j_{r'+3}} \cdots \alpha_{j_2\cdots r}^{J}\right)\\
&\;\;\vdots\\
&= \left[(-q)^{-2(1+2+\cdots|J''|-1)}\right] \alpha_{\emptyset}^{J''} \left(\alpha_{J''}^{j_{r'}\cdots j_r} \cdots \alpha_{j_2\cdots r}^{J}\right),
\end{aligned}$$

and continuing,

$$\begin{aligned}
&= (-q)^{-|J''|(|J''|-1)} \left[(-q)^{2(1)}\right] \alpha_{\emptyset}^{j_{r'}\cdots j_r} \left(\alpha_{j_{r'}\cdots j_r}^{j_{r'-1}\cdots j_r} \cdots \alpha_{j_2\cdots r}^{J}\right)\\
&\;\;\vdots\\
&= (-q)^{-|J''|(|J''|-1)} \left[(-q)^{2(1+2+\cdots+|J'|-1)}\right] \left(\alpha_{\emptyset}^{J}\right).
\end{aligned}$$

This is the desired result. □

5. $\mathcal{G}$-proof of theorem. We keep the notations J', J'', r', r'', r, s, t from Section 4.2. We also assume that $J\cap I = \emptyset$. (Only minor changes to the coming proof are needed to give the more general result.) To express the q-commuting relations as a consequence of the flag relations, it is sufficient to show that

$$C_{J,I} - M_{J,I} = \sum_{\emptyset\subseteq K\subsetneq J} \beta_K \cdot Y_{I\cup J^K, K;(r-|K|)}$$

for some choice of coefficients β_K. We begin by writing the left-hand side as

$$C_{J,I} - M_{J,I} = -q^{\langle\langle J,I\rangle\rangle} f_I f_J + \left(\sum_{\Lambda\subseteq I,\, |\Lambda|=r} (-q)^{\mathrm{inv}(\Lambda, I^\Lambda)} f_{J\lceil I^\Lambda} f_\Lambda \right)$$

or, replacing $\mathrm{inv}(\Lambda, I^\Lambda)$ with $|I^\Lambda||\Lambda| - \mathrm{inv}(I^\Lambda, \Lambda)$ and $\mathrm{inv}(J, I^\Lambda)$ with $|J''||I| - \mathrm{inv}(J, \Lambda)$ and using $|\Lambda| = |J|$, as

$$C_{J,I} - M_{J,I} = (-q)^{|J'|t+|J''||J|} \left(\sum_{\Lambda\subseteq I} (-q)^{-\mathrm{inv}((I\cup J)^\Lambda, \Lambda)} f_{(I\cup J)^\Lambda} f_\Lambda \right) - q^{\langle\langle J,I\rangle\rangle} f_I f_J.$$

This is to be compared with the expressions

$$Y_{I\cup J^K, K;(r-|K|)} = \sum_{\substack{\Lambda\subseteq I\cup J^K \\ |\Lambda|=r-|K|}} (-q)^{-\mathrm{inv}((I\cup J)^{K\cup\Lambda}, \Lambda)} (-q)^{-\mathrm{inv}(\Lambda, K)} f_{(I\cup J)^{K\cup\Lambda}} f_{K\cup\Lambda}.$$

The alternating property of the symbols f_K and the product in $\mathcal{F}\ell_q(n)$ play no role in our proof, so we eliminate these distractions. Let V be the vector space over $\Bbbk$ with basis $\{e_{A,B} : A \cup B = I \cup J, A \cap B = \emptyset \text{ and } |B| = r\}$. We prove the theorem in two steps.

Proposition 19. *Given $I, J \subseteq [n]$, suppose $J \curvearrowright I$. Then there is a scalar θ so that the vector*

$$cm(\theta) := \left(\sum_{\Lambda\subseteq I} (-q)^{-\mathrm{inv}((I\cup J)^\Lambda, \Lambda)} e_{(I\cup J)^\Lambda, \Lambda} \right) - \theta e_{I,J}$$

is a linear combination of the vectors $\{y^K : \emptyset \subseteq K \subsetneq J\}$, with

$$y^K := \sum_{\substack{\Lambda\subseteq I\cup J^K \\ |\Lambda|=r-|K|}} (-q)^{-\mathrm{inv}((I\cup J)^{K\cup\Lambda}, \Lambda)} (-q)^{-\mathrm{inv}(\Lambda, K)} e_{(I\cup J)^{K\cup\Lambda}, K\cup\Lambda}.$$

Proposition 20. *In the notation above, $\theta = (-q)^{-|J'|t-|J''||J|} q^{\langle\langle J,I\rangle\rangle}$.*

The first step (Proposition 19) is not obvious: note that the dimension of V is $\binom{r+s}{r}$, while the span of the y^K has dimension (at most) $2^r - 1$. Nevertheless, this step follows fairly quickly from a triangularity argument and the fact that $J \curvearrowright I$. The second step (Proposition 20) will follow from the results of Section 4.2, together with a careful book-keeping in the proof of the first step.

The following total order on the basis of V will be used in the coming proofs: say $(A, B) \prec (C, D)$ if $B \cap J$ precedes $D \cap J$ in the dictionary (viewing the ordered sets as words on the letters $\{1, 2, 3, \ldots\}$), or if $B \cap J = D \cap J$ and $B \cap I$ precedes $D \cap I$. For example, if $I = \{2, 3, 4, 5\}$ and $J = \{1, 6, 7\}$, then

$$(1567, 234) \prec (1367, 245) \prec (1347, 256) \prec (2347, 156).$$

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Proof of Proposition 19. We begin with the observation that many of the basis vectors $e_{A,B}$ in the definition of y^K carry the same coefficient: for fixed $\Lambda' \subseteq J \setminus K$, $(-q)^{-\mathrm{inv}((I\cup J)^{K\cup(\Lambda'\cup\Lambda)},\Lambda'\cup\Lambda)}(-q)^{-\mathrm{inv}(\Lambda'\cup\Lambda,K)}$ is invariant as Λ varies in I. This is true because $J \curvearrowright I$. We collect terms with equal coefficients and define the auxiliary vectors

$$e^{K'} := \sum_{\Lambda\subseteq I, |\Lambda|=r-|K'|} (-q)^{-\mathrm{inv}((I\cup J)^{K'\cup\Lambda},\Lambda)}(-q)^{-\mathrm{inv}(\Lambda,K')} e_{(I\cup J)^{K'\cup\Lambda},K'\cup\Lambda}$$

for each $\emptyset \subseteq K' \subseteq J$. Given $K \subsetneq J$, by construction we have

$$y^K = \sum_{K\subseteq K'\subseteq J} \alpha_{K,K'}\, e^{K'}$$

for some scalars $\alpha_{K,K'} \in \Bbbk$. Note, also, that $cm(\theta) = e^{\emptyset} - \theta e^J$.

Since the least values of $e_{A,B}$ appearing in the supports of the vectors e^K are distinct, the latter are linearly independent (and span a subspace of V of dimension 2^r). Moreover, since the $\alpha_{K,K}$ above are identically equal to 1, we have

$$\mathrm{span}\{y^K : K \subsetneq J\} = \mathrm{span}\{e^K : K \subsetneq J\}$$

by triangularity. Finally, since we have no vector y^J to work with, we see that the vector $cm(\theta) = e^{\emptyset} - \theta e^J$ belongs to the span of the y^K for a unique coefficient θ. □

Proof of Proposition 20. In order to properly identify θ, we must first identify the coefficients $\alpha_{K,K'}$ in the previous proof.

Claim: The scalars $\alpha_{K,K'}$ appearing in the description of the vectors y^K are precisely the edge weights $\alpha_K^{K'}$ from Section 4.2.

We leave the proof of this claim to the reader. The next step is to perform Gaussian elimination on a certain matrix. Table 3 displays this matrix for $J = J' \cup J'' = \{1\} \cup \{5, 6\}$ and should make our intentions clear.

We know from Proposition 19 that we can clear most entries in the first row of this matrix, resulting in a new row $(y^{\emptyset})' = 1e^{\emptyset} + \theta e^J = cm(\theta)$ for some θ. Careful

Table 3. Writing the vectors y^K in terms of the $e^{K'}$.

	$e^{\emptyset}$	e^1	e^5	e^6	e^{15}	e^{16}	e^{56}	e^{156}
$y^{\emptyset}$	1	$\alpha_{\emptyset}^1$	$\alpha_{\emptyset}^5$	$\alpha_{\emptyset}^6$	$\alpha_{\emptyset}^{15}$	$\alpha_{\emptyset}^{16}$	$\alpha_{\emptyset}^{56}$	$\alpha_{\emptyset}^{156}$
y^1		1			α_1^{15}	α_1^{16}		α_1^{156}
y^5			1		α_5^{15}		α_5^{56}	α_5^{156}
y^6				1		α_6^{16}	α_6^{56}	α_6^{156}
y^{15}					1			α_{15}^{156}
y^{16}						1		α_{16}^{156}
y^{56}							1	α_{56}^{156}

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book-keeping shows that

$$\theta = \alpha_{\emptyset}^{J} - \left(\sum_{\emptyset \subsetneq K \subsetneq J} \alpha_{\emptyset}^{K} \alpha_{K}^{J} \right) + \left(\sum_{\emptyset \subsetneq K_1 \subsetneq K_2 \subsetneq J} \alpha_{\emptyset}^{K_1} \alpha_{K_1}^{K_2} \alpha_{K_2}^{J} \right) - \cdots$$
$$\cdots + (-1)^{r-1} \left(\sum_{\emptyset \subsetneq K_1 \subsetneq \cdots \subsetneq K_{r-1} \subsetneq J} \alpha_{\emptyset}^{K_1} \alpha_{K_1}^{K_2} \cdots \alpha_{K_{r-1}}^{J} \right). \tag{14}$$

In other words, θ is a signed sum of path weights $\alpha(\pi)$, with π running over all paths in $\mathfrak{P}$ save for $\hat{1}$. Note that the sign attached to π in (14) changes according to the number of steps in π. Since the bijection $\wp$ from Section 4.2 increases the number of steps by one and preserves path weight, we conclude that θ depends only on $\pi = \wp^{-1}(\hat{1})$. More precisely,

$$\begin{aligned} \theta &= (-1)^{|J|-1} \times \alpha\big(\wp^{-1}(\hat{1})\big) \\ &= (-1)^{|J|-1}(-q)^{|J'|(|J'|-1)-|J''|(|J''|-1)} \times \alpha_{\emptyset}^{J} \\ &= (-1)^{|J|-1}(-q)^{|J''|-|J'|}(-q)^{|J'||J'|-|J''||J''|-|I||J'|} \\ &= q^{\langle\langle J,I \rangle\rangle}(-q)^{-|J'|t-|J''||J|}, \end{aligned}$$

as desired. □

With Proposition 20 proven, Theorem 2 is finally demonstrated (modulo the Taft–Towber isomorphism ϕ). Moreover, we achieve the second goal stated in the introduction. A brief discussion of the first goal follows.

6. From quasi- to quantum determinantal varieties. The algebra $\mathcal{F}\ell_q(n)$ is a quantum deformation of the classic multi-homogeneous coordinate ring of the full flag variety over GL_n. In [**15**], it is admitted that finding the proper form of the relations was somewhat difficult. In [**3**] we see a completely different (equivalent) set of relations. One hopes to proceed in a less ad-hoc manner. Perhaps a theory of *non-commutative flag varieties* using quasi-Plücker coordinates could help explain the choices for the relations in $\mathcal{F}\ell_q(n)$. In [**12**], it is shown that any relation $(\mathcal{Y}_{I,J})_{(a)}$ has a quasi-Plücker coordinate origin. Section 3 shows that (1) does too. The second proof of Theorem 2 shows that a great many instances of $(\mathcal{M}_{J,I})$ do as well: to see this, note that the roles of $M_{J,I}$ and $C_{J,I}$ were interchangeable there. The question of whether and to what extent the gap (the case $J \not\curvearrowright I$) may be filled by finding new quasi-Plücker coordinate identities is an interesting one. For example, it could be used to provide flag algebras in a variety of familiar settings, such as Yangian or super algebras. Towards a partial answer, we leave the reader to verify that

$$(\mathcal{P}_{I,J^j,j}) \Rightarrow (\mathcal{M}_{J,I}),$$

whenever $I, J \subseteq [n]$ are such that $|J| \leq |I|$ and $J^j \subseteq I$.

Looking past flag algebras to more general determinantal varieties, the same question is valid. In this direction, one might look at Goodearl's article [**6**], departing from, say, the quasi-minor identities in [**10**]. Some of Goodearl's relations evidently have quasi-determinantal origins. A careful study of which relations have this property would be the subject of another paper.

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