SOME THEOREMS ON ABSOLUTE SUMMABILITY

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A summation method defined by the linear transformation

A: $$y_r = \sum_{k=0}^{\infty} a_{rk}x_k$$

will be called an *l-l* method if $\sum|y_r| < \infty$ whenever $\sum|x_k| < \infty$; if in addition we have $\sum y_r = \sum x_k$ whenever $\sum|x_k| < \infty$ we shall say the method is *absolutely regular*. (It should be observed that we are dealing with series-to-series methods, not sequence-to-sequence as usual.) It was shown by K. Knopp and G. G. Lorentz [3] that a necessary and sufficient condition for A to be an *l-l* method is that there exist a constant M such that

(1) $$\sum_r |a_{rk}| < M \qquad (k = 0, 1, \ldots),$$

and a necessary and sufficient condition for absolute regularity is that in addition to (1) the equations

$$\sum_r a_{rk} = 1 \qquad (k = 0, 1, \ldots)$$

hold.

The purpose of this note is to point out that the procedure developed by S. Mazur [5] and S. Banach [2, p. 90-95] for use with regular methods in the ordinary (Toeplitz) sense can readily be adapted to the *l-l* methods, and yields a result of considerable generality (Theorem 1). We also consider methods effective for the class of series $\sum u_k$ such that $\sum u_k z^k$ has its radius of convergence greater than a given value R, obtaining results related to those of R. P. Agnew [1], and conclude with the application to Euler-Knopp summability.

Suppose now that $y_r = \sum_k a_{rk}x_k$ is an *l-l* method. We denote the sequences $\{x_k\}$, $\{y_r\}$ by x, y, and denote by (A) the set of all sequences x such that $y \in l$, that is, $\sum|y_k| < \infty$. For each $x \in (A)$ we define $A(x) = \sum y_k$. We represent the column totals of A by $\alpha_k = \sum_r a_{rk}$ $(k = 0, 1, \ldots)$; then $|\alpha_k| < M$, and if $x \in l$ we have $A(x) = \sum \alpha_k x_k$.

Similarly, if $z_r = \sum_k b_{rk}x_k$ is another *l-l* method we write $B(x) = \sum z_r$ for $x \in (B)$.

If $(B) \supset (A)$ we say that B is absolutely not weaker than A, and write simply B $\succ$ A.

If $B(x) = A(x)$ for $x \in (A) \cdot (B)$, we say B is absolutely consistent with A.

If $\sum_r b_{rk} = \alpha_k$, so that $B(x) = A(x)$ for $x \in l$, we write B $\sim$ A.

The method A is said to be *reversible* if for each $y \in l$ the equations $y_r = \sum_k a_{rk}x_k$ have a unique solution $x \in (A)$.

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The method A is said to be of type M*, if for every bounded sequence $\{\theta_r\}$ the conditions

(2) $$\sum_r \theta_r a_{rk} = 0 \qquad (k = 0, 1, \ldots)$$

imply

(3) $$\theta_r = 0 \qquad (r = 0, 1, \ldots).$$

(The usual definition of "type M" requires that (2) imply (3) for every sequence $\{\theta_r\} \in l$.) We shall use the following equivalent formulation of type M*: for every bounded sequence $\{t_r\}$ the conditions

$$\sum_r t_r a_{rk} = \sum_r a_{rk} \qquad (k = 0, 1, \ldots)$$

imply

$$t_r = 1 \qquad (r = 0, 1, \ldots).$$

THEOREM 1. *In order that a reversible l-l method* A *be absolutely consistent with every l-l method* B *such that* B ≻ A, B ∼ A, *it is necessary and sufficient that* A *be of type M**.

Remark. It is not necessary that A be normal (that is, $a_{rk} = 0$ $(k > r)$, $a_{rr} \neq 0$) or regular.

Proof. (i) Necessity of the condition. Suppose A is not of type M*, and let $\{t_r\}$ be a bounded sequence, with some $t_r \neq 1$, and such that $\sum_r t_r a_{rk} = \sum_r a_{rk}$, for each k. Now choose $\bar{y} \in l$ such that $\sum t_r \bar{y}_r \neq \sum \bar{y}_r$; then since A is reversible there is a unique sequence $\bar{x} \in (A)$ with $\bar{y}_r = \sum_k a_{rk}\bar{x}_k$. The method T $= (t_r a_{rk})$ is an l-l method with T ≻ A, T ∼ A, but $T(\bar{x}) \neq A(\bar{x})$.

(ii) Sufficiency of the condition. We have to show that if A is of type M* and B ≻ A, B ∼ A, then $B(x) = A(x)$ for each $x \in (A)$. We note first that if B ≻ A, then $B(x)$ is a linear functional of y. For since A is by hypothesis reversible, each term x_k of x is a linear functional of y [2, p. 49]. It follows that $z_r = \sum_k b_{rk}x_k$ and $B(x) = \sum z_r$ are also linear functionals of y [2, p. 23, Theorem 4]. Thus corresponding to each l-l method B ≻ A, there is a bounded sequence $\{t_r\}$ such that [2, p. 67]

(4) $$B(x) = \sum t_r y_r$$

for each $x \in (A)$.

Now, if B ∼ A it follows from (4), by considering the sequences

$$(1, 0, 0, \ldots),\ (0, 1, 0, \ldots),\ \ldots,$$

that

$$\sum_r a_{rk} = \sum_r t_r a_{rk} \qquad (k = 0, 1, \ldots);$$

then since A is of type M*, we have $t_r \equiv 1$. Hence $B(x) = \sum y_r = A(x)$ for each $x \in (A)$. This completes the proof.

For simple examples of methods which do or do not belong to type M*, we

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may observe that the matrix giving the series-to-series form of the $(C,1)$ method, namely

$$\begin{pmatrix} 1 & & & \\ 0 & 1/(1.2) & & \\ 0 & 1/(2.3) & 2/(2.3) & \\ 0 & 1/(3.4) & 2/(3.4) & 3/(3.4) \\ \cdot & \cdot & \cdot & \cdot \cdots \end{pmatrix}$$

is of type M*, while the matrix

$$\begin{pmatrix} \frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 0 & \frac{1}{2} & \frac{1}{2} & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot \cdots \end{pmatrix}$$

is not of type M*, though it is of type M.

It is also interesting to consider the situation where $\alpha_0 = \alpha_1 = \ldots = 0$ (so that A is a "multiplicative zero" method). In this case A is not of type M*, since we may take $t_r \equiv 2$; and since A is reversible it is easily seen that (A) properly includes l, and that A is not absolutely consistent with the method $B = 2A$ which has $B \succ A$, $B \sim A$.

We now introduce the class $C(R)$ (where $R \geqslant 0$) of sequences $\{u_k\}$ such that $\sum u_k z^k$ has its radius of convergence greater than R. We shall use the transformation

G: $$y_r = \sum_k g_{rk} u_k.$$

R. P. Agnew [1] found necessary and sufficient conditions on the matrix $G = (g_{rk})$ in order that $y = \{y_r\}$ should converge whenever $u = \{u_k\} \in C(R)$. By an easy application of the preceding work we shall find necessary and sufficient conditions on G in order that $y \in l$ whenever $u \in C(R)$, and shall show that "type M*" enters in the same way as before.

If $y \in l$ whenever $u \in C(R)$ we shall speak of G as a $C(R) - l$ method. If in addition $\sum y_r = \sum u_k$ we shall say that G is regular $[C(R) - l]$.

THEOREM 2. *A necessary and sufficient condition for* G *to be a* $C(R) - l$ *method is that the inequalities*

(5) $$\sum_r |g_{rk}| < M(\rho)\rho^k \qquad (k = 0, 1, \ldots)$$

hold for each $\rho > R$, $M(\rho)$ *being independent of* r, k. *A necessary and sufficient condition for* G *to be regular* $[C(R) - l]$ *is that in addition to* (5) *the equations*

$$\sum_r g_{rk} = 1 \qquad (k = 0, 1, \ldots)$$

hold.

Remark. It is easily seen by a change of variable that the case $R = 1$ gives conditions under which a power series is absolutely summable within its radius of convergence.

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Proof. Let $l(\rho)$ $(\rho > 0)$ be the set of all sequences $\{u_k\}$ such that $\sum u_k \rho^k$ converges absolutely. Then

$$C(R) = \bigcup_{\rho > R} l(\rho).$$

Now $l(\rho)$ may be put in one-to-one correspondence with l by letting $\{u_k\} \in l(\rho)$ correspond to $\{u_k \rho^k\} \in l$. In order that (g_{rk}) be an $l(\rho)$-l method it is necessary and sufficient that (g_{rk}/ρ^k) be an l-l method, or that $\sum_r |g_{rk}/\rho^k| < M(\rho)$ [see equation (1)]. For (g_{rk}) to be a $C(R) - l$ method, this must hold for all $\rho > R$. This gives (5), and the second part of the theorem is easily obtained.

In order to extend Theorem 1, we define absolute consistency, and the notation H $\succ$ G, H $\sim$ G, as before. It is easily verified that if G is a $C(R) - l$ method and $\gamma_k = \sum_r g_{rk}$ for each k, we have $G(u) = \sum \gamma_k u_k$ for each $u \in C(R)$. Then if H is another $C(R) - l$ method with H $\sim$ G, that is, $\sum_r h_{rk} = \gamma_k$ for each k, it follows that $H(u) = G(u)$ for each $u \in C(R)$.

THEOREM 3. *In order that a reversible,* $C(R) - l$ *method* G *be absolutely consistent with every* $C(R) - l$ *method* H *such that* H $\succ$ G, H $\sim$ G, *it is necessary and sufficient that* G *be of type* M*.

The proof, which follows exactly the proof of Theorem 1, is omitted.

We conclude by considering the Euler-Knopp series-to-series method $\mathfrak{E}(p)$ given by

$$y_r = \sum_{k=0}^{r} \binom{r}{k} p^{k+1}(1 - p)^{r-k} u_k.$$

We shall show that if $R \geqslant 1$, a necessary and sufficient condition for $\mathfrak{E}(p)$ to have the property that $\sum u_k$ is absolutely summable $\mathfrak{E}(p)$ whenever $\{u_k\} \in C(R)$, is that

$$(6) \qquad |p/R| + |1 - p| \leqslant 1.$$

(The same formula holds for ordinary summability; see [4]). We have

$$g_{rk} = \begin{cases} \binom{r}{k} p^{k+1}(1 - p)^{r-k} & (k \leqslant r) \\ 0 & (k > r). \end{cases}$$

Then

$$\sum_{r=0}^{\infty} |g_{rk}| = |p|^{k+1} \sum_{r=k}^{\infty} \binom{r}{k} |1 - p|^{r-k}$$

$$= \frac{|p|^{k+1}}{(1 - |1 - p|)^{k+1}} < M(\rho)\rho^k,$$

for each $\rho > R$, if and only if

$$\frac{|p|}{1 - |1 - p|} \leqslant R,$$

which gives (6). The result now follows by Theorem 2. Finally we shall

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show that $\mathfrak{E}(p)$ is of type M* for all values of p such that $|1-p| < 1$. Following Mazur [5, p. 49-50] we assume that $\{\theta_r\}$ is bounded, and that

$$(7) \qquad \sum_{r=0}^{\infty} \theta_r g_{rk} = p^{k+1} \sum_{r=k}^{\infty} \theta_r \binom{r}{k} (1-p)^{r-k} = 0 \qquad (k = 0, 1, \ldots).$$

Consider the function

$$f(z) = \sum_{r=0}^{\infty} \theta_r z^r, \qquad (|z| < 1).$$

We have

$$f^{(k)}(z) = k! \sum_{r=k}^{\infty} \theta_r \binom{r}{k} z^{r-k} = 0$$

when $z = 1 - p$, by (7). Hence $\theta_r = 0$ $(r = 0, 1, \ldots)$, and so $\mathfrak{E}(p)$ is of type M*.

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