S. H. Hong
Nagoya Math. J.
Vol. 96 (1984), 71-82

ON A GENERALIZATION OF THE ABSTRACT MORSE COMPLEX AND ITS APPLICATIONS

SUK HO HONG

Introduction

Klingenberg refers in [4] the fact that the homology group of the space Λ of closed H^1 curves on a manifold is isomorphic to that of the Morse complex. In this paper, we generalize the fact above and at the same time give a proof to it through cell decomposition method under a strong non degeneracy condition.

We first introduce so-called generalized Morse complex on a space X with an action of Lie group G and an invariant energy function E on X. The case of the space Λ of closed curves is obviously obtained through $G = S^1$.

Next we apply the Morse complex argument to the space Λ, where the isotropy group is closely related to the multiplicity. And we find the cycle $Z(c)$ constructed by Shikata-Klingenberg [1] is at most finite order in the homology of the Morse complex. Thus from a close investigation of the order of the cycle $Z(c)$ on $H_*(X)$, we deduce a relation between the torsion and the divisibility of multiplicities of a certain geodesic.

We would like to thank Professor Y. Shikata for many valuable discussions who gives me many helpful insight into the field of Morse complex.

§ 1. On G-action which generalizes S^1-action on Λ

1-1. Let X be a C^{r+1}-manifold ($r \geq 0$) with a G-action of a compact Lie group such that the isotropy group $I(p)$ at $p \in X$ is discrete for any $p \in X$. Suppose X admits an invariant Morse function E, i.e.,

$$E : X \longrightarrow \boldsymbol{R}$$

is c^r-function such that $E(gp) = E(p)$ for any $g \in G$ and let φ be the gradient flow of E, then φ is G-equivariant:

Received August 19, 1983.

$$\varphi(gp) = g\varphi(p)$$

for any $g \in G$ and $p \in X$.

For a critical point c of E, we set

$$S(c) = \{p \in X: \varphi_s(p) \longrightarrow c \quad \text{as} \quad s \longrightarrow \infty\}$$
$$U(c) = \{p \in X: \varphi_s(p) \longrightarrow c \quad \text{as} \quad s \longrightarrow -\infty\}$$

then $S(c)$ and $U(c)$ are called the stable and unstable manifolds respectively.

THEOREM 1. *If c is non-degenerated,* Codim $G \cdot S(c) = \text{index}\, c$.

Proof. Denote by $T_p(M)$ the tangent space of a submanifold M at p, then from the non degeneracy assumption above we have a natural spliting

$$T_p(U(c)) \oplus T_p(G \cdot S(c)) = T_p(X)$$

since

$$\dim U(c) = \text{index}\, c\,,$$

we have

$$\text{codim}\, G \cdot S(c) = \text{index}\, c\,.$$

We choose from each G-orbit $G \cdot c$ of a critical point c, a representative c and call them a pure critical point representing c and denote the set of pure critical point by Γ.

We introduce a polar coordinate system $(u, t)_c$ for $u \in S(T_c(U(c))$ and $t \in (0, \infty)$ where $S(T_c(U(c))), = \{u \in T_c(U(c)), \|u\| = 1\}$ in the unstable manifold $U(c)$ of a critical point c by mapping $(u, t)_c$ onto $\varphi_t(u)$. We deduce the following property for the polar coordinate easily:

LEMMA 2. $g(u, t)_c = (gu, t)_{gc}$ *where we used the notation gu also for the G-action on the tangent space.*

It is obvious that if any two flows $\varphi_t(u)$, $\varphi_{t'}(u')$ $(u \in T_c(U(c)), u' \in T_{c'}(U(c)))$ have an intersection for finite t, t', then they are agree entirely, therefore we may refer this fact as follows:

LEMMA 3. *If $(u, t)_c = (u', t')_{c'}$ for $c' \in G \cdot c$, then*

$$u = u', \; t = t' \quad \textit{and} \quad c = c'\,.$$

We refer the following property (P) at the strong non degeneracy of E:

(P): All the critical point c are non degenerate and for any critical points c, c', the stable and unstable manifolds have a generic intersection.

1-2. We compute $H_*(X)$ through a cell decomposition of X. We first decompose $G \cdot U(c)$ into cells: Consider the covering space

$$\pi : G \longrightarrow G/I(c)$$

with the right hand $I(c)$-action and decompose the base manifold $G/I(c)$ into cells $\{\bar{\Delta}\}$ such that the covering π is trivial over each simplex $\bar{\Delta} \in \{\bar{\Delta}\}$. Then $A(\bar{\Delta}) = \pi^{-1}(\bar{\Delta})$ splits into a disjoint union $\{A_i(\bar{\Delta})\}$ of homeomorphic cells in G on which $I(c)$ acts effectively and transitively from the right. We choose and fix a representative $A_c(\bar{\Delta})$ from the inverse image $\{A_i(\bar{\Delta})\}$ of each cell $\bar{\Delta}$ in $\{\bar{\Delta}\}$.

LEMMA 4. *If there exist points p, p', q, q' such that*

$$p \in A_c(\bar{\Delta}), \quad p' \in A_c(\bar{\Delta}'), \quad q, q' \in U(c)$$

and

$$pq = p'q' \qquad \text{for cells } \bar{\Delta}, \bar{\Delta}' \in \{\bar{\Delta}\}$$

then we have

$$\bar{\Delta} = \bar{\Delta}', \; p = p' \quad \text{and} \quad q = q'\,.$$

In fact, in the polar coordinate on $U(c)$, we have

$$p(u, t)_c = p'(u', t')_c$$

therefore from Lemma 3, we see

$$pc = p'c$$

that is

$$p = p'x, \qquad x \in I(c)\,.$$

Since π is $I(c)$-covering, we have $x = \mathrm{id}$.

PROPOSITION 5. *The cell $A_c(\bar{\Delta})$ in G defines a cell $A_c(\bar{\Delta}) \cdot U(c)$ in $G \cdot U(c)$ which is homeomorphic to $A_c(\bar{\Delta}) \times U(c)$ in the interior.*

Proof. If (p, q), $(p', q') \in A_c(\bar{\Delta}) \times U(c)$ are mapped onto the same point through the multiplication, we have immediately from Lemma 4 that

$$p = p' \quad \text{and} \quad q = q'\,.$$

https://doi.org/10.1017/S0027763000021140 Published online by Cambridge University Press

Proposition 6. *The cells* $A_c(\bar{\Delta})\cdot U(c)$, $A_c(\bar{\Delta}')\cdot U(c)$ *have no interior intersection for* $\bar{\Delta} \neq \bar{\Delta}'$.

Proof. It is also obvious from Lemma 4 that the existence of the interior intersection

$$pq = p'q' \quad \text{for} \quad p \in A_c(\bar{\Delta}),\ p' \in A_c(\bar{\Delta}')\,,$$

$q, q' \in U(c)$ implies $p = p'$, $\bar{\Delta} = \bar{\Delta}'$.

Since $U(c) = gU(c)$ for any $g \in I(c)$ as sets we finally see that

$$\begin{aligned} G\cdot U(c) &= \bigcup_{\{\bar{\Delta}\}} \bigcup_{i} A_i(\bar{\Delta})\cdot U(c) \\ &= \bigcup_{\{\bar{\Delta}\}} \bigcup_{g\in I(c)} A_c(\bar{\Delta})\cdot gU(c) \\ &= \bigcup_{\{\bar{\Delta}\}} A_c(\bar{\Delta})\cdot U(c) \end{aligned}$$

that is, the cells $A_c(\bar{\Delta})\cdot U(c)$ for $\bar{\Delta} \in \{\bar{\Delta}\}$ cover $G\cdot U(c)$.

Theorem 7. *The cells* $A_c(\bar{\Delta})\cdot U(c)$ *give a cell decomposition of* $G\cdot U(c)$.

We see that a subdivision of the decomposition of $G\cdot U(c)$ induces a decomposition on $bd(G\cdot U(c))$ as follows: First, property (P) yields that $S(T_c(U(c)))$ is divided into cells by its intersection with the (weak) stable manifold $S(c_-)$ of critical points c_- of lower indexes than c, in fact the intersection

$$S(T_c(U(c))) \cap S(c_-)$$

is an open submanifold $S(T_c(U(c)))$ of dimension

$$\text{index}\, c - \text{index}\, c_- - 1$$

and the boundary of each one of the submanifold again splits into a union of submanifolds of this kind.

Thus taking product by small cell $\Delta \subset G$ to these cells, we can divide $\Delta\cdot S(T_c(U(c)))$ into cells. Therefore for a sufficiently fine decomposition $\{\bar{\Delta}\}$ of G we see that the decomposition of $A_c(\bar{\Delta})\cdot S(T_c(U(c)))$ defines a natural decomposition of $A_c(\bar{\Delta})\cdot U(c)$ through the polar coordinate. Take a decomposition $\{\bar{\Delta}\}$ of G so fine that the covering projection $\pi : G \to G/I(c_-)$ is trivial over $\bar{\Delta}$ for any pure critical point c_- such that $S(c_-) \cap S(T_c(U(c))) \neq \phi$, then we see that $\{\bar{\Delta}\cdot bdU(c)\}$ decomposes $G\cdot bdU(c)$ into cells, because $bdU(c)$ is ω-limit of $S(T_c(U(c)))$.

Let $X(n)$ denote the union of (weak) unstable manifolds over pure critical points of index lower than n or equal to n.

$$X(n) = \bigcup_{c \in \Gamma(n)} G \cdot U(c)$$
$$\Gamma(n) = \{c \in \Gamma\,, \qquad \text{index } c \leq n\}\,.$$

Then it is easy to see that $X(n)$ can be decomposed into cells in the method above and

$$X = \bigcup_n X(n)\,, \qquad X(n) \subset X(n+1)\,.$$

Since any k-submanifold in X is pushed down into $X(k)$ by the flow.

THEOREM 8. *The homology $H_k(X)$ may be computed as the homology $H_k(X(n))$ of $X(n)$ $(k < n)$ which is obtained as homology of a cell decomposition given by a subdivision of the cells $A_c(\bar{\Delta}) \cdot U(c)$.*

1-3. We construct an abstract chain complex $\mathscr{M}$ which is equivalent to the chain group over the cell complex above and we call it a generalized Morse complex. We fix an orientation on each cell of $\{\bar{\Delta} \cdot U(c)\}$ by choosing an orientation in $U(c)$ and also one in $\bar{\Delta} \in \{\bar{\Delta}\}_c$ for each pure critical point $c \in \Gamma$. We then have an graded chain group $C(X)$ of oriented cell $\{\bar{\Delta} \cdot U(c)\}$ by defining

$$\begin{aligned} \deg \bar{\Delta} \cdot U(c) &= \dim \bar{\Delta} \cdot U(c) \\ &= \dim \bar{\Delta} + \text{index}\, c\,. \end{aligned}$$

Let X^n be the union of cells in $X(m)$ of dimension lower than or equal to n $(n \geq m)$ and take the boundary operator ∂ in the exact sequence for the triple (X^n, X^{n-1}, X^{n-2}):

$$\partial\colon\ H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

then it is known that $C_n(X) = H_n(X^n, X^{n-1})$ and $C(X)$ turns out to be a chain complex together with the boundary ∂ (see [2], [6]), whose homology is equal to that of $X(m)$, thus we have

PROPOSITION 9. *Under the non degeneracy condition, we have a chain complex $C(X)$ over graded cells $\{\bar{\Delta} \cdot U(c)\}$ so that*

$$H_*(C(X)) = H_*(X)\,.$$

COROLLARY 10. *Under the same non degeneracy condition above, we* *see that the homology* $H_*(C(X))$ *of the cell complex is independent of the cell decomposition of* X, *especially that of* G.

In order to describe the boundary operator ∂, we start with a small cell $e = \tilde{\Delta} \cdot U(c)$ in $H_n(X^n, X^{n-1})$, which is represented as the image of a (relative) product homeomorphism $\varphi = \varphi_1 \times \varphi_2$ of

$$\varphi_1 \colon I^k \longrightarrow X$$

and

$$\varphi_2 \colon I^m \longrightarrow X$$

such that $\pi \colon G \to G/I(c)$ is trivial over $\varphi_1(I) = \tilde{\Delta}$:

$$\begin{array}{ccc} I^k \times I^m & \xrightarrow{\text{characteristic map}} & G \times U(c) \\ & \underset{\varphi=\varphi_1\times\varphi_2}{\searrow} & \big\downarrow \text{multiplication} \\ & & X \end{array}$$

Since φ_* commutes with the boundary homeomorphism, we see that $\partial e = j_* \varphi_* \partial_* f$ for the fundamental class f in $H_n(I^k \times I^m, \operatorname{bd}(I^k \times I^m))$, as is seen from the following diagram:

$$\begin{array}{ccc} H_n(I^k \times I^m, \operatorname{bd}(I^k \times I^m)) & \xrightarrow[\partial_*]{} & H_{n-1}(\operatorname{bd}(I^k \times I^m)) \\ \big\downarrow \varphi_* & & \big\downarrow \varphi_* \\ H_n(X^n, X^{n-1}) & \xrightarrow[\partial_*]{} & H_{n-1}(X^{n-1}) \\ & \underset{\partial}{\searrow} & \big\downarrow j_* \\ & & H_{n-1}(X^{n-1}, X^{n-2}) \end{array}$$

The fundamental class f splits into a cross product $f_1 \times f_2$ of

$$\begin{aligned} & f_1 \in H_k(I^k, \operatorname{bd} I^k)\,, \\ & f_2 \in H_m(I^m, \operatorname{bd} I^m) \\ & k = n - m = \dim \tilde{\Delta} \\ & m = \operatorname{index} c \end{aligned}$$

corresponding to $\tilde{\Delta}$ and to $U(c)$, respectively, therefore from the naturality as the boundary formula of the cross product, we have that

$$\begin{aligned} \partial e &= j_* \varphi_* \partial f \\ &= j_* \varphi_* (\partial_* f_1 \times f_2 + (-1)^k f_1 \times \partial_* f_2) \end{aligned}$$

$$= (j_*\varphi_{1*}\partial_* f_1 \times j_*\varphi_{2*}f_2) + (-1)^k(j_*\varphi_{1*}f_1) \times (j_*\varphi_{2*}\partial_* f_2)$$
$$= \partial e_1 \times e_2 + (-1)^k e_1 \times \partial e_2$$

Here the classes

$$e_1 = j_*\varphi_{1*}f_1 \in H_k(X^k, X^{k-1})$$
$$e_2 = j_*\varphi_{2*}f_2 \in H_m(X^m, X^{m-1})$$

may be regarded as the classes representing $\tilde{\Delta}$ and $U(c)$ respectively. Moreover we may replace the cross product above by the multiplication of G on X because every cell under consideration acts effectively on $U(c)$, thus we see that

PROPOSITION 11. *The boundary operator ∂ in the cell decomposition of Theorem 8 in Section 1 satisfies that*

$$\partial(\tilde{\Delta}\cdot U(c)) = (\partial\tilde{\Delta})\cdot U(c) + (-1)^k\tilde{\Delta}\cdot\partial U(c)\,, \qquad \textit{where } k = \dim \tilde{\Delta}\,.$$

Finally we investigate $\tilde{\Delta}$, $U(c)$ geometrically.

They may be considered as the homology classes represented as the classes of the boundaries

$$\partial\tilde{\Delta} = \partial e_1 = j_*(\varphi_1)_*\partial_* e_1 \in H_{k-1}(X^{k-1}, X^{k-2})$$
$$\partial U(c) = \partial e_2 = j_*(\varphi_2)_*\partial_* U(c) \in H_{m-1}(X^{m-1}, X^{m-2})\,.$$

Therefore $\partial U(c)$ can be regarded as the sum of $(m-1)$ cells appearing on the boundary of $U(c)$ with the suitable coefficient, which we can count as the intersection number of $S(T_c(U(c)))$ with the (weak) stable manifold $S(\tilde{\Delta}c_-)$ of codimension $m-1$ for a cell $\tilde{\Delta} \in G$ of dimension index $c_- - (m-1)$.

LEMMA 12. *Let $[\tilde{\Delta}c_-, c]$ be the intersection number of $S(T_c(U(c)))$ and the stable manifold $S(\tilde{\Delta}c_-)$ of codimension $m-1$, then we have*

$$\partial U(c) = \sum [\tilde{\Delta}c_-, c]\tilde{\Delta}U(c_-)\,.$$

We introduce an abstract chain complex $\mathscr{M}$ over the set Γ of the pure critical points as the chain group generated over formal elements

$$\{\tilde{\Delta}c/\tilde{\Delta}\colon \text{cell in } G, c \in \Gamma\}$$

with the degree given by

$$\text{degree } \tilde{\Delta}c = \dim \tilde{\Delta} + \text{index } c$$

and define the boundary operator ∂ as follows:

$$\partial c = \sum [\tilde{\Delta} c_-, c]\tilde{\Delta} c_-$$
$$\partial \tilde{\Delta} c = \partial \tilde{\Delta} c + (-1)^k \tilde{\Delta} \partial c \qquad \text{where } k = \dim \tilde{\Delta} .$$

Since we see easily that the chain complex $\mathscr{M}$ is chain homotopic to $C(X)$, we deduce the following from Lemma 12, Propositions 9, 11.

THEOREM 13. $H_*(\mathscr{M}) = H_*(X)$.

§ 2. Relations to torsion and divisibility

2-1. In case of the space Λ of closed curve, we have a natural S^1-action on Λ through the action on the parameter;

$$\theta \cdot \alpha(t) = \alpha(\theta + t) \quad t, \ \theta \in S^1, \quad \alpha \in \Lambda .$$

If we remove the point curves Λ_0 from Λ, we have the S^1-action on $\Lambda - \Lambda_0$ such that the isotropy $I(x)$ is discrete for any $x \in \Lambda - \Lambda_0$ thus we may apply our method to the case $X = \Lambda - \Lambda_0$, $G = S^1$.

In this case, we have a well known relation between the order of Iso(x) and the multipilicity $m(x)$ of x defined as the maximal number m so that

$$x = \alpha \cdots \alpha = \alpha^m \qquad \text{for some } \alpha \in \Lambda .$$

LEMMA 14. $\operatorname{ord} I(x) = m(x)$.

We notice that when we consider the S^1-action on the Morse complex $\mathscr{M}$, then also have a notion of isotropy Iso(x) for a chain $x \in C$. In particular for a chain represented by a critical ponit c, we have Iso(c) other than $I(c)$.

LEMMA 15. $\operatorname{ord} \operatorname{Iso}(c) = \operatorname{ord} I(c)$ *or* $2 \operatorname{ord} I(c)$

$$= m(c) \quad \text{or} \quad 2m(c) .$$

In fact, if the multiplication by $g \in I(c)$ on $U(c)$ preserves the orientation in $U(c)$, we have the first case, otherwise we take double in order to preserve the orientation and we have the second case.

On the other hand, Klingenberg constructed a energy function E on the space Λ which satisfies the condition (C). (cf. Klingenberg [4]). Therefore if we assume further the strong degeneracy on E, we may apply Theorem 13 to the space $\Lambda - S(\Lambda_0)$, where $S(\Lambda_0)$ denotes the stable manifold over Λ_0 and we reproduce the Klingenberg's anouncement [4] on the homology of $\Lambda - S(\Lambda_0)$ with S^1-action.

THEOREM 16. *The homology* $H_*(\Lambda - S(\Lambda_0))$ *of* $\Lambda - S(\Lambda_0)$ *is obtained as the homology of the Morse complex associated with* $\Lambda - S(\Lambda_0)$ *and* E, *provided that* E *satisfies the strong non degeneracy condition.*

It may be possible to weaken the strong non degeneracy condition to a weak non degeneracy, that is, only assuming the non degeneracy of each critical point, for this we return in near future.

Our purpose in the remaining is to investigate a relation between a torsion property of homology $H_*(X)$, (reduced to the Morse complex) and a behavior of the multiplicities which is related to the order of isotropy as an application of what we have discussed.

Our point is that we can deduce a type of divisibility even for the Finsler case provided the strong non degeneracy because our method is entirely topological and does not use the ϑ-action which comes from Riemannian structure.

2-2. We investigate a torsion property of a cycle $Z(c)$ in $\mathscr{M}$ constructed by Shikata-Klingenberg [1]. We quickly review here how $Z(c)$ is constructed over a pure critical point $c \in \Gamma$. Let $\overline{m}$ be the order of isotropy of c, then we have

$$1/\overline{m} \cdot c = c\ ,$$

hence

$$1/\overline{m} \cdot \partial c = \partial c\ .$$

Thus we have an invariant chain ∂c in $\mathscr{M}$ under the action of a subgroup $G(\overline{m})$ of S^1 generated by $1/\overline{m}$ and therefore we can split ∂c into a sum of invariant chains x_i which is invariant under the action of a subgroup $H_i \supset G(\overline{m})$:

$$\partial c = \sum_1^n x_i$$

Then the fact that

$$h_i x_i = x_i \qquad \text{for } h_i \in H_i$$

implies that

$$\partial((1 - h_1) \cdots (1 - h_n)c) = 0\ ,$$

yielding a cycle

$$Z(c) = (1 - h_1) \cdots (1 - h_n)c\ .$$

https://doi.org/10.1017/S0027763000021140 Published online by Cambridge University Press

In order to investigate a further property of the cycle $Z(c)$, we consider the case $n = 1$, $H_1 \supsetneqq G(\overline{m})$.

Lemma 17. *Let* $h = 1/\text{ord}\, H_1$ *then*

$$Z(c) = (1 - h)c$$

is at most a torsion element of $\text{ord}\,(H_1)$.

In fact, take

$$\Delta = [0, h]$$

and let

$$y = \Delta c$$

then we have

$$\begin{aligned} \partial y &= \partial\Delta\cdot c - \Delta\partial c \\ &= (1 - h)c - \Delta x_1 \\ &= Z(c) - \Delta\cdot x_1 \,. \end{aligned}$$

Since x_1 is H_1-invariant, it is expressed as a sum over H_1:

$$x_1 = \sum_{k\in H_1} ku \,, \qquad u \in \mathscr{M} \,.$$

Therefore

$$\Delta x_1 = \sum_{k\in H_1} k\cdot\Delta u = (\sum_{k\in H_1} k\Delta)\cdot u$$

may be expressed as $\Delta x_1 = S^1\cdot u$.

Thus we have

$$\partial y = Z(c) - S^1\cdot u \,.$$

On the other hand, consider $v = S^1\cdot c$ then we see that

$$\begin{aligned} \partial v &= \partial S^1\cdot c - S^1\cdot\partial c \\ &= -\, S^1(\sum_{k\in H_1} k\cdot u) \\ &= -\sum_{k\in H_1} k(S^1\cdot u) \\ &= -\,(\sum_{k\in H_1} k)\cdot S^1\cdot u \\ &= -\,(\text{ord}\, H_1)S^1\cdot u \,. \end{aligned}$$

Hence we have that

$$(\text{ord}\, H_1)\partial y = (\text{ord}\, H_1)Z(c) + \partial v \,.$$

indicating that $Z(c)$ is at most of $\operatorname{ord}(H_1)$ torsion in $H_*(\mathscr{M})$.

Next we take the case $n = 2$,

$$H_1 \supsetneqq G(\overline{m}), \qquad H_2 \supsetneqq G(\overline{m})$$

and

$$H_1 \cap H_2 \neq H_1,\ H_2 .$$

Lemma 18. *Let*

$$h_1 = 1/\operatorname{ord} H_1, \qquad h_2 = 1/\operatorname{ord} H_2$$

and $Z(c)$ is of the form

$$Z(c) = (1 - h_1)(1 - h_2)c$$

then it is zero in $H_*(\mathscr{M})$.

In fact, take $\Delta = [0, h]$ and let

$$y = \Delta(1 - h_2)c$$

then we see that

$$\begin{aligned} \partial y &= Z(c) - \Delta(1 - h_2)\partial c \\ &= Z(c) - \Delta(1 - h_2)(x_1 + x_2) \\ &= Z(c) - \Delta(1 - h_2)x_1 \\ &= Z(c) - (1 - h_2)S^1 \cdot u \end{aligned}$$

by the same u and by the same reasoning as in the case 1. Thus we see that

$$\partial y = Z(c) .$$

In general, from a similar computation, we see easily that for $n \geqq 2$, the homology class $Z(c)$ is zero, also we may remark that for the case $n = 1$ the homology classes $(1 - h)Z(c)$ is zero.

In [1] Shikata-Klingenberg deduced a modified divisibility lemma using a chain bounding the cycle $Z(c) + \vartheta Z(c)$, for the involution ϑ in Λ keeping E invariant. Thus their theory is related to the Riemannian structure of the underlying manifold at this point. But we can cut this point off from the Riemannian structure by taking $Z(c)$ or $(1 - h)Z(c)$.

Proposition 19. *We may apply Shikata-Klingenberg theory to the cycle $Z(c)$ or $(1 - h)Z(c)$ to have the divisibility lemma in the modified form* *even in case we do not have the involution ϑ, like in non symmetric Finsler space.*

Remark 1. Shikata-Klingenberg theory uses $\pi_1(\Lambda)=0$ on the way, therefore Katok's Finsler example on S^2 has nothing to do with the proposition above.

Remark 2. Shikata-Klingenberg's modified divisibility lemma is roughly as follows: Under a certain non degeneracy assumption as $\pi_1(\Lambda) = 0$, there exists a series $\{c_i\}$ of critical points in Λ, so that

$$m(c_i)\,|\,2m(c_{i+1}) \quad \text{or} \quad m(c_{i+1})\,|\,2m(c_i)$$

where the $m(c)$ is the multiplicity of the curve c in Λ and is related to the order of isotropy $I(c)$.

References

[1] W. Klingenberg and Y. Shikata, On a proof of the divisibility lemma, Closed geodesics in Tokyo University Symposium 1982.
[2] ——, On the existence theorems of infinitely many closed geodesics, Proc. Int. Conference on Topology, Moscow 1979.
[3] W. Klingenberg, Riemannian geometry, Walter de Gruyter 1982.
[4] ——, Lectures on closed geodesics, Springer Verlag 1978.
[5] Glen E. Bredon, Equivariant Cohomology Theories, Springer Verlag 1967.
[6] E. Spanier, Algebraic topology, McGraw Hill 1966.
[7] J. Milnor, Lectures on the h-cobordism theorem, Princeton University Press 1965.
[8] ——, Morse theory, Princeton University Press 1963.
[9] R. Thom, Quelques propriétés globales des variétés differentiables Comment. Math. Helv., **28** 1954.

Department of Mathematics
Nagoya University
Nagoya 464 Japan

Current address:
Department of Mathematics
Kon-kuk University
Seoul, Korea

https://doi.org/10.1017/S0027763000021140 Published online by Cambridge University Press