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TAMING WILD SIMPLE CLOSED CURVES WITH MONOTONE MAPS

W. S. BOYD AND A. H. WRIGHT

1. Introduction. Hempel [**6**, Theorem 2] proved that if S is a tame 2-sphere in E^3 and f is a map of E^3 onto itself such that $f|S$ is a homeomorphism and $f(E^3 - S) = E^3 - f(S)$, then $f(S)$ is tame. Boyd [**4**] has shown that the converse is false; in fact, if S is any 2-sphere in E^3, then there is a monotone map f of E^3 onto itself such that $f|S$ is a homeomorphism, $f(E^3 - S) = E^3 - f(S)$, and $f(S)$ is tame.

It is the purpose of this paper to prove that the corresponding converse for simple closed curves in E^3 is also false. We show in Theorem 4 that if J is any simple closed curve in a closed orientable 3-manifold M^3, then there is a monotone map $f : M^3 \to S^3$ such that $f|J$ is a homeomorphism, $f(J)$ is tame and unknotted, and $f(M^3 - J) = S^3 - f(J)$.

In Theorem 1 of § 2, we construct a cube-with-handles neighbourhood of a simple closed curve in an orientable 3-manifold. This neighbourhood is a solid torus, sectioned into 3-cells, with a small cube-with-handles attached to each section to cover a small subarc of J associated with that section.

Theorem 1′ constructs an analogous neighbourhood for finite graphs.

In § 3 we extend the construction given in § 2 to give a cube-with-handles neighbourhood of a simple closed curve in which the simple closed curve is homotopic to a simple closed curve lying in the boundary of the solid torus portion of the neighbourhood. Similar extensions are given for neighbourhoods of finite graphs.

Sections 4 and 5 construct an infinite sequence of cube-with-handles neighbourhoods similar to those of Theorem 1, each lying "nicely" in the previous one. In the process of constructing these neighbourhoods, it is shown that if J is homologous to zero, then J bounds an open surface.

In § 6, the infinite sequence of neighbourhoods is used to construct the monotone map of the 3-manifold onto S^3 which carries a simple closed curve in the manifold onto a tame unknotted simple closed curve. In the case that the simple closed curve J has a solid torus neighbourhood in which it is homologous to a centreline, there is a monotone map of the manifold onto itself which tames J and which is the identity outside the solid torus neighbourhood.

In § 7, we show that any knot, link, or wedge of simple closed curves in an orientable 3-manifold which is homologous to zero (respectively, contractible to a point) in the 3-manifold, is homologous to zero (respectively, contractible to a point) in a cube-with-handles in the 3-manifold.

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By a *map* or *mapping* we will mean a continuous function. If each point inverse of a map is compact and connected, then the map will be called *monotone*.

A *surface* is a 2-manifold. An *open manifold* is a noncompact manifold without boundary, and a *closed manifold* is a compact manifold without boundary. We will denote the boundary of a manifold M by ∂M, and the interior of M by Int M. A surface S will be said to be *properly embedded* in a 3-manifold M if $\partial S \subset \partial M$ and $\text{Int}\, S \subset \text{Int}\, M$. We will assume that any manifold has a given metric, and we will denote this metric by the symbol ρ. The diameter of a set X will be denoted by diam(X).

A *punctured disk* is a disk D minus the interior of the union of a finite mutually disjoint collection of subdisks of the interior of D.

By a *graph*, we will mean a finite connected 1-complex. A vertex or 1-simplex v of a graph G has order n if v is a face of n 1-simplexes of G. The *star* of a vertex v of G is the closure of the union of the simplexes of G which have v as a face. An *n-frame* is the union of n arcs all intersecting at a common end point.

Let S be a 2-sided polyhedral surface in a 3-manifold M^3, and let A be an oriented polyhedral arc or simple closed curve which pierces S at each of its points of intersection with S. If A pierces S n more times in one direction than in the other, we call n the (unsigned) *algebraic intersection number* of A and S.

We use the fact that a polygonal simple closed curve in a 3-manifold M^3, which is homologous to zero in M^3, bounds a polyhedral orientable surface in M^3. Also, if two disjoint polygonal simple closed curves are homologous in M^3, then they bound a polyhedral orientable surface in M^3.

2. Neighbourhoods of finite graphs. In this section we construct neighbourhoods of finite graphs topologically embedded in a 3-manifold which are as close as we can make them to a regular neighbourhood. This neighbourhood is in fact the regular neighbourhood of a polygonal approximation to the graph with small cubes-with-handles attached along disks in the boundary of this regular neighbourhood to give a cube-with-handles neighbourhood of the topologically embedded graph. Near points where the topologically embedded graph is tame we do not need to attach the small cubes-with-handles. If the graph is polygonal our neighbourhood is, in fact, a regular neighbourhood of the graph. This neighbourhood will be used in §§ 4 and 5 to construct an infinite sequence of neighbourhoods of a simple closed curve which will in turn be applied in § 6 to define a monotone mapping carrying the simple closed curve to a tame unknotted simple closed curve in S^3.

THEOREM 1. *Let J be a simple closed curve topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, J has a cube-with-handles neighbourhood N with the following structure:*

(1) *There is a solid torus T with n meridional spanning disks $D_1, D_2, \ldots, D_n$ which divide T into n 3-cells $T_1, T_2, \ldots, T_n$ such that $D_i = T_i \cap T_{i+1}$ and $D_n = T_n \cap T_1$.*

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(2) *There are* n *points* $p_1, p_2, \ldots, p_n$ *on* J *which divide* J *into* n *closed subarcs* $J_1, J_2, \ldots, J_n$ *such that* $p_i = J_i \cap J_{i+1}$ *and* $p_n = J_n \cap J_1$.
(3) $p_i \in \operatorname{Int} D_i$, *for each* i.
(4) *Each* 3*-cell* T_i *has an associated cube-with-handles* H_i *such that* $T \cap H_i = T_i \cap H_i = (\partial T_i - D_i - D_{i-1}) \cap \partial H_i$ *is a disk* F_i.
(5) $J_i \subset T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}$.
(6) $\operatorname{diam}(T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}) < \epsilon$.
(7) $N = T \cup (\bigcup H_i)$.
(8) *If* J *is locally tame at each point* J_i, *then* $T_i \cap J = J_i$ *is an unknotted spanning arc of* T_i *(hence there is no need for* H_i*).*

Remark. If M^3 is non-orientable, the same theorem is true except that T may be a solid Klein bottle, so N is a cube with (possibly) non-orientable handles.

Proof of Theorem 1. Let $\delta < \epsilon/25$. Choose points $p_1, p_2, \ldots, p_n$ of J dividing J into subarcs $J_1, J_2, \ldots, J_n$ of diameter less than $\delta/3$ such that

$$p_i = J_i \cap J_{i+1}, \quad i = 1, \ldots, n$$

(subscripts are understood to be integers mod n), and $J_i \cap J_j = \emptyset$ if $j \neq i - 1$, i, or $i + 1$. The arcs $J_1, J_2, \ldots, J_n$ form the 2-skeleton of a curvilinear triangulation of J with vertices $p_1, p_2, \ldots, p_n$.

Let J' be a polygonal approximation to J, where $J' = J_1' \cup J_2' \cup \ldots \cup J_n'$ is a simple closed curve with J_i' $\delta/3$-homotopic to J_i by a homotopy keeping the endpoints of J_i fixed. By [**9**, Lemma 3], J_i' can be adjusted slightly near $J_i' \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1}))$ so that J_i' is disjoint from

$$\operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})).$$

Thus we will assume that J' has this property for each subarc J_i'.

Take a polygonal solid torus neighbourhood T of J' and a disjoint collection of meridional disks $D_1, D_2, \ldots, D_n$ such that $D_i \cap J' = \{p_i\}$, $i = 1, 2, \ldots, n$, and $p_i \in \operatorname{Int} D_i$. If T_i is the closure of the component of $T - \bigcup \{D_i : i = 1, 2, \ldots, n\}$ containing $D_{i-1} \cup D_i$, then T_i is a 3-cell for each $i = 1, 2, \ldots, n$. The D_i's and T may be chosen so that $\operatorname{diam}(T_i) < \delta/3$, and because

$$J_i' \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})) = \emptyset,$$

we may assume that T and the D_i's were chosen so that

$$T_i \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})) = \emptyset, \quad i = 1, 2, \ldots, n.$$

The latter condition insures that $J_i \cap \operatorname{Cl}(T - (T_{i-1} \cup T_i \cup T_{i+1})) = \emptyset$ for each i.

Consider the collection of sets $J_1 \cap \partial T, J_2 \cap \partial T, \ldots, J_n \cap \partial T$. This is a collection of mutually exclusive compact subsets of ∂T such that no component

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of any $J_i \cap \partial T$ separates a neighbourhood of itself in ∂T. Hence, there is a collection

$$\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \ldots \cup \mathscr{D}_n$$

of mutually exclusive disks in ∂T such that $\mathscr{D}_i$ is a mutually exclusive collection of disks containing $J_i \cap \partial T$ in the union of their interiors and $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ if $i \neq j$. Furthermore, $\mathscr{D}_i$ can be chosen so that the union $\mathscr{D}_i^*$ of the disks in $\mathscr{D}_i$ lies in $\partial(T_{i-1} \cup T_i \cup T_{i+1})$ missing the two end disks D_{i-2} and D_{i+1}, and since $J_i \cap \partial T$ is a $\delta/3$-set we may assume that each disk of $\mathscr{D}_i$ has diameter less than $\delta/3$.

By "sliding" each ∂D_i along ∂T we may adjust $\bigcup D_i$ so that $(\bigcup D_i) \cap \mathscr{D}^* = \emptyset$, no point of D_i is moved more than $\delta/3$, and $\mathscr{D}_i^*$ lies in $T_{i-1} \cup T_i \cup T_{i+1}$. We do this adjustment so close to each component of $\mathscr{D}$ that p_i is still in the adjusted D_i and the resulting T_i's retain the property that J_i does not meet any T_j unless $j = i - 1$, i, or $i + 1$, and $\operatorname{diam}(T_i) < \delta$.

As in [**7**, Lemma 2], let $\mathscr{D}^*_{i,j}$ denote the set $\mathscr{D}_i^* \cap \partial T_j$ for $j = i - 1$, i, $i + 1$. There are three mutually disjoint disks on $\partial T_j - D_{j-1} - D_j$, namely $B_{j-1,j}$, $B_{j,j}$, $B_{j+1,j}$ so that $\mathscr{D}^*_{j-1,j} \subset \operatorname{Int} B_{j-1,j}$, $\mathscr{D}^*_{j,j} \subset \operatorname{Int} B_{j,j}$ and $\mathscr{D}^*_{j+1,j} \subset \operatorname{Int} B_{j+1,j}$. Thus $\mathscr{D}_i^* \subset B_{i,i-1} \cup B_{i,i} \cup B_{i,i+1}$ and $B_{i,i-1} \subset \partial T_{i-1}$, $B_{i,i} \subset \partial T_i$, $B_{i,i+1} \subset \partial T_{i+1}$. There are two arcs, one joining $B_{i,i-1}$ to $B_{i,i}$ intersecting D_{i-1} precisely once, and one joining $B_{i,i}$ to $B_{i,i+1}$ intersecting D_i precisely once; both arcs are disjoint from any other $B_{j,k}$'s and lie in $\partial(T_{i-1} \cup T_i)$ and $\partial(T_i \cup T_{i+1})$, respectively.

It is easy to see that there is a disjoint collection of such arcs in ∂T such that each arc intersects $\bigcup \partial D_i$ precisely once and joins some $B_{i,i}$ to $B_{i,i-1}$ or some $B_{i,i}$ to $B_{i,i+1}$ and each $B_{i,i}$ is joined to $B_{i,i-1}$ by one such arc and to $B_{i,i+1}$ by one. Replacing these arcs by thin disks we obtain disks $F_1, F_2, \ldots, F_n$ on ∂T such that

$$\mathscr{D}_i^* \subset B_{i,i-1} \cup B_{i,i} \cup B_{i,i+1} \subset F_i,$$

$$F_i \subset \partial(T_{i-1} \cup T_i \cup T_{i+1}) - D_{i-2} - D_{i+1}, \quad \text{and} \quad F_i \cap F_j = \emptyset$$

if $i \neq j$. We now adjust the disks $D_1, D_2, \ldots, D_n$ near ∂T to slip them off $\bigcup F_i$ so that $F_i \subset \partial T_i - D_{i-1} - D_i$. This adjusts $T_1, T_2, \ldots, T_n$, also. We now have the structure of (1), (2), and (3) of the conclusions to the theorem. Since

$$\operatorname{diam}(F_i) \leqq \operatorname{diam}(T_{i-1} \cup T_i \cup T_{i+1}) < 3\delta$$

before this last adjustment, then

$$\operatorname{diam} T_i < \delta + 2(3\delta) = 7\delta$$

after pushing the D_i's off the F_i's.

Let M_i' be a compact 3-manifold with connected boundary intersecting T in a collection of punctured disks in the boundary of each of M_i' and T, with $M_i' \cap T \subset F_i$, $J_i - T \subset \operatorname{Int} M_i'$, $\operatorname{diam}(M_i') < \delta/3$, and $M_i' \cap M_j' = \emptyset$ if $i \neq j$. Fatten the disk F_i slightly into the complement of T and add the resulting cell to M_i' to obtain a compact 3-manifold with connected boundary M_i intersecting T in exactly the disk F_i.

There is a collection $\mathscr{A}_i$ of arcs in M_i such that each arc lies in Int M_i except that its endpoints lie in $\partial M_i - F_i$ and such that M_i minus a small tubular neighbourhood of every arc of $\mathscr{A}_i$ is a cube-with-handles. Such arcs exist by [**9**, Lemma 1]. By [**9**, Lemma 3], the collection of arcs $\mathscr{A}_i$ may be adjusted near $\mathscr{A}_i^* \cap J_i$ so that $J_i \cap \mathscr{A}_i^* = \emptyset$. Let H_i be the cube-with-handles obtained by removing small tubular neighbourhoods of these adjusted arcs of $\mathscr{A}_i$ from M_i. Then $H_i \cap T = F_i$ and

$$\begin{aligned}\operatorname{diam}(H_i) &\leqq \operatorname{diam}(M_i') + \operatorname{diam}(F_i)\\ &< \delta/3 + 3\delta = 3\tfrac{1}{3}\delta.\end{aligned}$$

The cube-with-handles H_i is the one promised in (4); and (5) follows. We let $N = T \cup (\bigcup H_i)$ and note that

$$\begin{aligned}\operatorname{diam}(T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}) &\leqq \operatorname{diam} T_{i-1} + \operatorname{diam} T_i + \operatorname{diam} H_i\\ &\quad + \operatorname{diam} T_{i+1}\\ &< 7\delta + 7\delta + 3\tfrac{1}{3}\delta + 7\delta\\ &= 24\tfrac{1}{3}\delta < \epsilon.\end{aligned}$$

To obtain (8) we assume without loss of generality by [**3**, Theorem 9] that J is locally polyhedral mod its set of wild points. If J is locally tame at each point of J_i, then J_i is polyhedral and we can choose $J_i' = J_i$. It then follows that T, T_i, D_i and D_{i-1} can be so chosen as in (8). This completes the proof of Theorem 1.

Remark. Let p be a point of a (possibly wild) simple closed curve J and let U be a neighbourhood of p. Then there is a disk D in U such that $\partial D \cap J = \emptyset$ and any polygonal approximation of J which is homotopic to J in the complement of ∂D intersects D algebraically once. Just choose D to be a D_i of a sufficiently close neighbourhood N of J as constructed in Theorem 1.

A *special decomposition* P of a graph G is a decomposition of G into vertices, 1-simplexes, and n-frames obtained as follows from a triangulation of the graph which is so fine that the star of two vertices of order greater than 2 do not intersect: At each vertex v of order $n > 2$, replace v and each 1-simplex containing v with the n-frame star of v. The 1-simplexes and n-frames of the decomposition will be called 1-*elements*. The special decomposition P' of the graph G is a *subdivision* of P if each vertex of P is also a vertex of P'.

THEOREM 1'. *Let G be a finite graph topologically embedded in an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood N with the following structure:*

(1) *There is a special decomposition P of G and a cube-with-handles $T = \bigcup \{T_\sigma : \sigma$ is a 1-element of $P\}$, where each T_σ is a 3-cell associated with σ.*

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(2) $T_\sigma \cap T_{\sigma'} = \emptyset$ *if* $\sigma \cap \sigma' = \emptyset$ *and* $T_\sigma \cap T_{\sigma'} = D_\tau$, *where* D_τ *is a disk in the boundary of each of* T_σ *and* $T_{\sigma'}$ *if* $\tau = \sigma \cap \sigma'$ *is a vertex* P. *In this case,* $\tau \in \operatorname{Int} D_\tau$.

(3) *Each* 3-*cell* T_σ *has an associated cube-with-handles* H_σ *such that* $T \cap H_\sigma = T_\sigma \cap H_\sigma = (\partial T_\sigma - \bigcup \{D_\tau : \tau \text{ is a vertex of } \sigma\}) \cap \partial H_\sigma$ *is a disk* F_σ.

(4) *If* σ *is a* 1-*element of* P, *then* $\sigma \subset T_\sigma \cup H_\sigma \cup (\bigcup \{T_{\sigma'} : \sigma' \text{ is a } 1\text{-element of } P \text{ and } \sigma' \cap \sigma \neq \emptyset\})$.

(5) *If* σ *is a* 1-*element of* P, *then* $\operatorname{diam}(\bigcup \{T_{\sigma'} \cup H_{\sigma'} : \sigma' \text{ is a } 1\text{-element of } P \text{ and } \sigma' \cap \sigma \neq \emptyset\}) < \epsilon$.

(6) $N = T \cup (\bigcup H_\sigma)$.

(7) *If* G *is locally tame at each point of the* 1-*element* σ, *then* $T_\sigma \cap G = \sigma$ *and* σ *lies in* T_σ *as the cone from an interior point of the* 3-*cell* T_σ *to a finite collection of points of* ∂T_σ. *In this case, there is no* H_σ.

Remark. If M^3 is non-orientable, T (and hence N) may be a cube with non-orientable handles; with this exception Theorem 1′ holds for a non-orientable M^3.

Proof of Theorem 1′. The proof is essentially the same as that of Theorem 1, except at the vertices of G of order $r > 2$. We indicate here how to modify the proof of Theorem 1. We take first of all a special decomposition P of the graph G instead of the triangulation of J. We choose a polygonal approximation G' of G $\delta/3$-homotopic to G keeping the vertices of P fixed; in particular, each vertex of G is also a vertex of G'. Instead of a solid torus neighbourhood of J', as in Theorem 1, we choose a regular neighbourhood T of G' and a collection of spanning disks D_τ of T, one for each vertex τ of P, which divides T into 3-cells satisfying (1) and (2). Note that there is a 3-cell T_σ for each 1-element σ of P and each T_σ is separated from "adjacent" $T_{\sigma'}$'s by a disk D_τ. If σ is an n-frame, note that T_σ is "adjacent" to more than two $T_{\sigma'}$'s, and is separated from them by a collection of disks $\{D_\tau : \tau \text{ is a vertex of } \sigma\}$, where there is one D_τ for each $T_{\sigma'}$.

The rest of the proof is the same as the proof for Theorem 1 with the appropriate change in notation.

3. In this section we take the neighbourhood N of Theorem 1 and modify it so that the simple closed curve J (respectively, graph G) is homotopic in N to a homeomorphic copy of itself in ∂N. As a consequence we can rename subsets of the new neighbourhood so that it satisfies the conclusion of Theorem 1 (respectively, Theorem 1′) except that possibly (5) holds in a slightly weaker form, but in which J (respectively, G) is homotopic in N to a spine of T. There is a corresponding version of each theorem for non-orientable 3-manifolds, which holds with little or no change in proof.

THEOREM 2. *Let* J *be a simple closed curve in the interior of an orientable* 3-*manifold* M^3. *For any* $\epsilon > 0$, J *has a cube-with-handles neighbourhood*

$N = T \cup H_1 \cup H_2 \cup \ldots \cup H_n$ as given in Theorem 1 with the additional property that J is ϵ-homotopic in N to a simple closed curve L on ∂N which crosses each D_i precisely once and has no other points of intersection with $\bigcup D_i$.

Proof. Let N be a cube-with-handles neighbourhood of J as given by Theorem 1 for $\epsilon/3$. Denote $T_i \cup H_i$ by N_i. Then $J_i \subset T_{i-1} \cup N_i \cup T_{i+1}$, $\text{diam}(T_{i-1} \cup N_i \cup T_{i+1}) < \epsilon/3$, $p_{i-1} \in \text{Int}\, D_{i-1}$, $p_i \in \text{Int}\, D_i$, where p_{i-1}, p_i are the endpoints of J_i, and D_i is the disk $N_i \cap N_{i+1} = T_i \cap T_{i+1}$.

Let us consider $N_i = T_i \cup H_i$. Let $E_1, E_2, \ldots, E_k$ be a collection of handles for H_i. That is, $E_1, E_2, \ldots, E_k$ is a mutually exclusive collection of disks properly embedded in H_i such that the closure of H_i minus a sufficiently close regular neighbourhood of $\bigcup E_j$ is a 3-cell. By choosing the E_j to miss the disk $F_i = T_i \cap H_i$, we ensure that $\bigcup E_j$ is a collection of handles for the cubes-with-handles $T_{i-1} \cup N_i \cup T_{i+1}$ and also for the cube-with-handles N_i.

Let $\delta > 0$ be less than half the distance between any two of the disks $E_1, E_2, \ldots, E_k$. Let J_i'' be any polygonal arc which is δ-homotopic to J_i in $T_{i-1} \cup N_i \cup T_{i+1}$ by a homotopy keeping p_{i-1} and p_i fixed. Suppose that J_i and J_i'' are oriented from p_{i-1} to p_i and that each E_j is oriented. Suppose further that J_i'' pierces each disk E_j at each point of intersection.

We assign a letter to each crossing of J_i'' with one of the disks E_j as follows:

e_j is a positive crossing through the disk E_j,
e_j^{-1} is a negative crossing through the disk E_j.

Using this convention we can write out a word in the letters $e_1, e_2, \ldots, e_k$ representing J_i''. For example, if J_i'' were represented by the word $e_1 e_3^{-1} e_2$ it would mean that proceeding along J_i'' from p_{i-1} to p_i, J_i'' crosses E_1 in the positive direction, then E_3 in the negative direction, then E_2 in the positive direction and that there are no other intersections of J_i'' with any E_j.

If J_i'' is represented by the word

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_l^{\epsilon_l}$$

where $\epsilon_j = \pm 1$ and $x_j = e_r$ for some $r \in \{1, 2, \ldots, k\}$, we can obtain (uniquely) a reduced word

$$\alpha_1^{\eta_1} \alpha_2^{\eta_2} \ldots \alpha_m^{\eta_m}$$

where $\eta_j = \pm 1$, α_j is some e_r, and no symbol $\alpha_j^{\eta_j} \alpha_{j+1}^{\eta_{j+1}}$ is of the form $\alpha^{-1}\alpha$ or $\alpha\alpha^{-1}$. We do this by successively deleting such combinations from the word $x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_l^{\epsilon_l}$ until none occur. (If J_i'' were a closed curve at p_{i-1} this procedure would yield the reduced word of J_i'' in the fundamental group based at p_{i-1} of the cube-with-handles $T_{i-1} \cup N_i \cup T_{i+1}$ in a presentation of this group.) Note that for any two approximations to J_i such as J_i'' we obtain the same reduced word which we will refer to as the word of J_i. This follows from our choice of δ.

Now suppose that $\alpha_1^{\eta_1} \alpha_2^{\eta_2} \ldots \alpha_m^{\eta_m}$ is the (reduced) word of J_i. We emphasize that this word is unique. We would like to construct an arc on

$\partial N_i - (D_{i-1} \cup D_i)$ whose word is also $\alpha_1^{\eta_1}\alpha_2^{\eta_2} \ldots \alpha_m^{\eta_m}$. In general, this is not possible. However, there is an oriented polygonal curve L' in ∂N_i running from a point p'_{i-1} in ∂D_{i-1} to a point p_i' in ∂D_i with $\text{Int}\, L' \subset \partial N_i - (D_{i-1} \cup D_i)$ such that the word of L' is the same as that of J_i and L' has a finite number of self-intersections, all crossing points. We may assume that each such crossing is a double point and that none of the double points lies in the boundary of one of the disks E_j, $j = 1, 2, \ldots, k$. We will drill out holes in N_i to make a new cube-with-handles in N_i for which it is possible to find an L' with no self-intersections.

Let q denote one of the double points of L' and let A', A'' be two subarcs of L' which cross at q and contain no other singularities of L. Let A' and A'' have the orientations inherited from L'. If E is a small disk in N_i with $E \cap \partial N_i = \partial E \cap \partial N_i = A'$ and $E \cap (\cup E_j) = \emptyset$, then we could drill out a tube in N_i along the arc $A = \partial E - \text{Int}\, A'$ and obtain a new cube-with-handles $N_i' \subset N_i$ in place of N_i. By replacing the subarc A' of L' with the arc A on the tube we can reduce the number of singularities of L' by one. However, the new curve L'' so obtained would have a different word in N_i' because of the crossing of A'' through the disk E, which must now be taken as a handle of $T_{i-1} \cup N_i' \cup T_i$ along with $E_1, E_2, \ldots, E_k$.

To compensate for this we choose E so that J_i also has this letter in its (reduced) word. Suppose that the new letter e corresponding to passage of L'' through E is between $\alpha_s^{\eta_s}$ and $\alpha_{s+1}^{\eta_{s+1}}$ in the word of L'. The word of L'', then, is the word of L' with e inserted between $\alpha_s^{\eta_s}$ and $\alpha_{s+1}^{\eta_{s+1}}$:

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2} \ldots \alpha_s^{\eta_s} e \alpha_{s+1}^{\eta_{s+1}} \ldots \alpha_m^{\eta_m}.$$

J_i can be divided into 3 subarcs B_1, B_2, B_3 such that the word of B_1 is $\alpha_1^{\eta_1}\alpha_2^{\eta_2} \ldots \alpha_s^{\eta_s}$, the word of B_2 is the identity (i.e., B_2 does not intersect any of the disks $E_1, E_2, \ldots, E_k$), and the word of B_3 is $\alpha_{s+1}^{\eta_{s+1}} \ldots \alpha_m^{\eta_m}$.

To obtain the arcs B_1, B_2 and B_3 we first take a regular neighbourhood U of $\cup E_j$ in N_i missing the endpoints of J_i. There is a finite collection of mutually disjoint closed subintervals of J_i, the union of the interiors of which cover $J_i \cap (\cup E_j)$, such that each interval lies in U. We can read a word for J_i from these intervals as follows: If the endpoints of an interval are separated in U by some E_k, then that interval represents a (net) crossing of J_i through E_k in either a positive direction or a negative direction. In the first case we associate the letter e_k with the interval; in the second case, the letter e_k^{-1}. If the endpoints of the interval are not separated in U by any E_k, then that interval represents a (net) crossing of J_i through $\cup E_k$ of zero. If the interval intersects some E_k (it can intersect at most one), associate the letter t with that interval; otherwise, just eliminate it from the collection. The word for J_i is the word obtained by traversing J_i from p_{i-1} to p_i writing down the letter associated with each interval as we come to it. It follows that, if we treat t as equal to the trivial word, then this word reduces to the (reduced) word of J_i obtained before. Furthermore, reduction can be accomplished geometrically by replacing the two intervals corresponding to an $e_k e_k^{-1}$ (or an $e_k^{-1} e_k$) by a longer interval equal to the union of these two intervals with the subinterval of J_i lying between them, associating the letter t with it (and ignoring it, henceforth, in the reduction as it represents the trivial word). It follows that the (reduced) word of J_i is represented by some subcollection of the original collection of intervals. We now have a collection $\mathscr{A}$ of closed intervals of J_i remaining (including the intervals associated with letter t). Now eliminate from $\mathscr{A}$ any interval which is contained in some other interval in $\mathscr{A}$. Note that each of these eliminated intervals has the letter t associated with it. Then $\mathscr{A}$ is a collection of mutually disjoint closed intervals and the union of the interiors of these intervals covers $J_i \cap (\bigcup E_j)$. Furthermore, the (reduced) word of J_i can be read directly from the intervals of $\mathscr{A}$ if letter t is ignored in each occurrence. Therefore there is a point x of J_i such that the

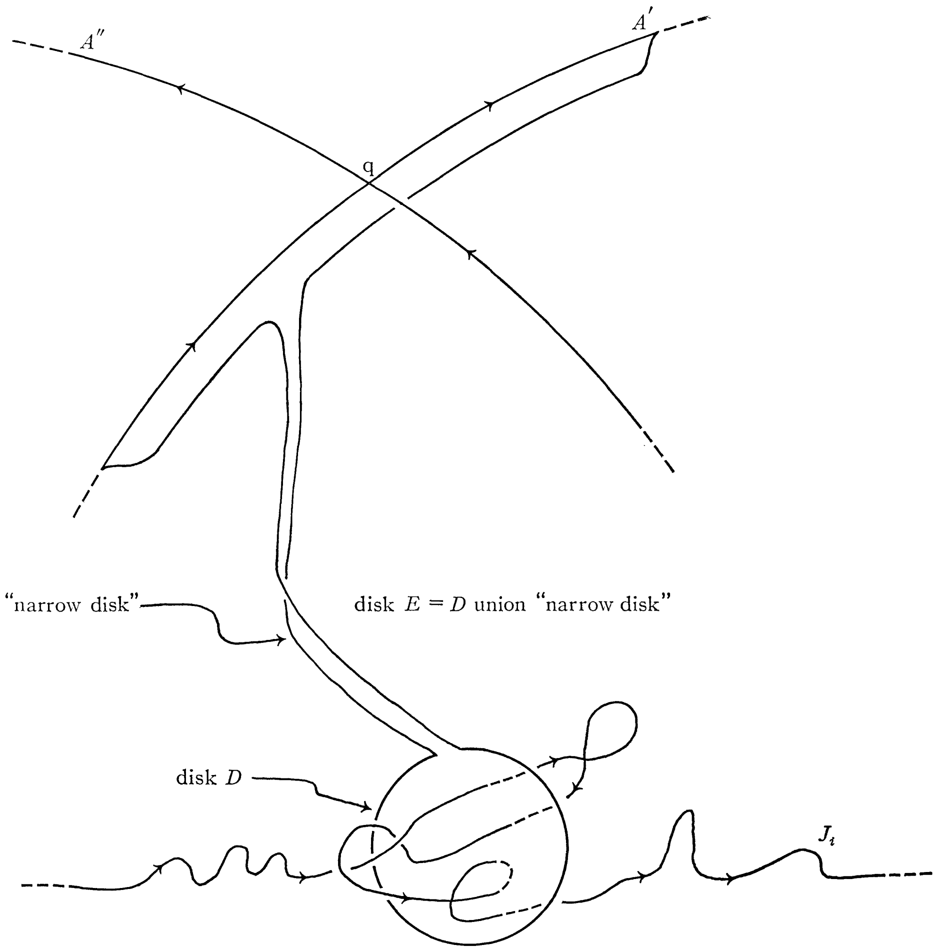

FIGURE 1

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word of J_i from p_{i-1} up to x is the word $\alpha_1^{\eta_1} \ldots \alpha_s^{\eta_s}$ and x does not lie in any of the intervals of $\mathscr{A}$. Let B_2 be a subarc of J_i containing x which is disjoint from each interval of $\mathscr{A}$. Let B_1 and B_3 be the closures of the appropriate components of $J_i - B_2$.

Let D be a small disk in $\operatorname{Int} N_i$ missing $\bigcup E_j \cup (J - \operatorname{Int} B_2)$ such that any sufficiently close polygonal approximation to J intersects D algebraically once. Let β be a polygonal arc joining ∂D to $q(=A' \cap A'')$ with $\operatorname{Int} \beta \subset \operatorname{Int} N_i - (J \cup (\bigcup E_j))$. That β can be chosen to miss $\bigcup E_j$ follows because $N_i - \bigcup E_j$ is homeomorphic to a 3-cell less a finite disjoint collection of disks in its boundary. Replace β with a narrow disk (see Figure 1) intersecting L' in the subarc A' and D in a subarc of ∂D. Let the disk E be the union of D and the narrow disk. Drill a small tubular hole out of N_i along the arc $\partial E - A'$ to obtain the cube-with-handles N_i'. A set of handles for N_i' is $E, E_1, \ldots, E_k$. Let L'' be the arc obtained from L' by replacing the subarc A' with an arc running along the tube and not intersecting E. If the narrow disk is given the appropriate "twist" before attaching to D to form E, then J_i has the same word

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2} \ldots \alpha_s^{\eta_s} e \alpha_{s+1}^{\eta_{s+1}} \ldots \alpha_m^{\eta_m}$$

in N_i' as does L''.

It is clear that we can apply the above technique at each point of singularity of L' and obtain a cube-with-handles $N_i' \subset N_i$ by drilling out small tubular holes in N_i. We can replace the singular arc L' by a non-singular arc L_i lying in $\partial N_i' - (D_{i-1} \cup D_i)$ except for its endpoints $p'_{i-1} \in \partial D_{i-1}$ and $p_i' \in \partial D_i$. Furthermore, the construction gives a set of handles $E_1^i, E_2^i, \ldots, E_{k_i}^i$ for N_i' consisting of the handles for N_i together with the handles such as E introduced at the points of singularity of L'. The word of L_i in N_i' is the same as the word of J_i. Notice also that $J - \operatorname{Int} J_i$ is disjoint from each handle E_j^i.

Let $L = \bigcup \{L_i : i = 1, 2, \ldots, n\}$. Then L is a simple closed curve on $\partial N' = \partial(\bigcup N_i')$. All that remains is to show that L and J are ϵ-homotopic in $N' = \bigcup N_i'$.

Join p_i to p_i' by an arc l_i in D_i and orient l_i from p_i to p_i'. Then $J_i l_i L_i^{-1} l_{i-1}^{-1}$ is an oriented loop in $N'_{i-1} \cup N_i' \cup N'_{i+1}$ based at p_i. Because this loop does not intersect the handles of N'_{i-1} or of N'_{i+1} and its word in N_i' is zero, this loop represents the trivial word in $N'_{i-1} \cup N_i' \cup N'_{i+1}$ and thus bounds a singular disk in this cube-with-handles. By piecing together these singular disks, we obtain an ϵ-homotopy in N' from J to L.

Dropping the primes, we have the structure required in Theorem 2.

COROLLARY 1. *Let J be a simple closed curve in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, J has a cube-with-handles neighbourhood $N = T \cup H_1 \cup H_2 \cup \ldots \cup H_n$ as given in Theorem* 1, *except that conclusion* (5) *becomes:*

(5) *there are 3-cells $C_i \subset N_i$ with $C_i \cap \partial N_i \supset D_{i-1} \cup D_i$ and $J_i \subset C_{i-1} \cup N_i \cup C_{i+1}$ (where $N_i = T_i \cup H_i$),*

and with the additional property that J is ϵ-homotopic in N to a geometric centre-line of T.

Proof. Let $N = N_1 \cup N_2 \cup \ldots \cup N_n$ be the neighbourhood constructed in Theorem 2 for J. Let T_i be a regular neighbourhood in N_i of $L_i \cup D_{i-1} \cup D_i$ and let H_i be the closure of $N_i - T_i$. Let $T = \bigcup T_i$.

Let the 3-cells C_i be N_i minus a regular neighbourhood of $\bigcup E_j^i$ which is so close to $\bigcup E_j^i$ that it is disjoint from $J - \text{Int}\, J_i$.

THEOREM 2′. *Let G be a finite graph topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood $N = T \cup (\bigcup H_\sigma)$ as given by Theorem 1′ with the additional property that J is ϵ-homotopic in N to a polygonal finite graph L in ∂N which is homeomorphic to G.*

Proof. The proof is essentially the same as that of Theorem 2. If P is the special decomposition of G used in constructing a neighbourhood N as in Theorem 1′, each 1-element σ of P has an associated cube-with-handles $N_\sigma = T_\sigma \cup H_\sigma$. We drill tubes out of each N_σ to form an N_σ' which is a cube-with-handles and construct a homotopy of σ in $\bigcup \{N_{\sigma'} : \sigma' \text{ is a 1-element of } P \text{ and } \sigma' \cap \sigma \neq \emptyset\}$ onto a copy of σ in $\partial N_\sigma'$ as in the proof of Theorem 2. The main difference occurs when σ is an n-frame.

Suppose that σ is an n-frame of P with $n > 2$. That is, σ is a 1-element of P containing a point v of G of order greater than 2. Let $\tau_1, \tau_2, \ldots, \tau_m$ be the vertices of σ. Let $E_1, E_2, \ldots, E_k$ be a set of handles for N_σ. Choose an arc γ in $N_\sigma - (G - \{v\}) - \bigcup E_j$ joining v to a point v' of

$$\partial N - \bigcup \{D_{\tau_i} : i = 1, 2, \ldots, m\}.$$

The n-frame σ is the union of 1-simplexes σ_i, $i = 1, 2, \ldots, m$, such that one vertex of σ_i is v and the other is τ_i. Construct, as in Theorem 2, singular arcs L'_{σ_i} on $\partial N_\sigma - \bigcup \text{Int}\, D_{\tau_i}$ joining v' to a point $\tau_i' \in \partial D_{\tau_i}$, such that the word of L'_{σ_i} is the (reduced) word of σ_i in N_σ. Each L'_{σ_i} may cross itself and other L'_{σ_j}'s as well. We can drill tubes in N_σ as before to obtain a new N_σ' on which each L'_{σ_i} can be replaced by a non-singular L_{σ_i} such that $L_{\sigma_i} \cap L_{\sigma_j}$ is the point v' and the word of L_{σ_i} in N_σ' is the same as the word of σ_i in N_σ'. To do this involves only a simple generalization of the technique of Theorem 2 to the case of a finite collection of arcs.

If l_i is an arc in D_{τ_i} from τ_i to τ_i' which misses $G - \{\tau_i\}$, then $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$ is a simple loop which lies in N_σ' except for part of σ_i. The part of σ_i outside of N_σ' does not intersect any handles of any $N'_{\sigma'}$, $\sigma' \neq \sigma$, so the word of $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$ in

$$\bigcup \{N_{\sigma'}' : \sigma' \cap \sigma \neq \emptyset\}$$

is the same as its word in N_σ', namely zero. Thus it bounds a singular disk in $\bigcup \{N'_{\sigma'} : \sigma' \cap \sigma \neq \emptyset\}$. Piecing together along γ all the singular disks obtained

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in this way (one for each $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$), we obtain an ϵ-homotopy in N_σ' of $\sigma = \bigcup \sigma_i$ onto $\bigcup L_{\sigma_i}$ in $\partial N_\sigma'$. Let $L_\sigma = \bigcup L_{\sigma_i}$.

By piecing together all the L_σ's, we obtain a homeomorphic copy L of G on the boundary of $N' = \bigcup \{N_\sigma' : \sigma \text{ is a 1-element of } P\}$. By piecing together the homotopies, we obtain an ϵ-homotopy of G onto L in N'.

COROLLARY 2. *Let G be a finite graph topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood $N = T \cup (\bigcup H_\sigma)$ as given in Theorem 1′ except that conclusion (5) becomes:*

(5) *there exist 3-cells $C_\sigma \subset N_\sigma$ with $C_\sigma \cap \partial N_\sigma \supset \bigcup \{D_\tau : \tau \text{ is a vertex of } \sigma\}$ and $\sigma \subset N_\sigma \cup (\bigcup \{C_{\sigma'} : \sigma' \text{ is a 1-element of } P \text{ and } \sigma' \cap \sigma = \emptyset\})$ (where $N_\sigma = T_\sigma \cup H_\sigma$),*

and with the additional property that G is ϵ-homotopic in N to a 1-spine of T.

Proof. Let $N = \bigcup N_\sigma$ be the neighbourhood constructed in Theorem 2′ for G. Let T_σ be a regular neighbourhood of $L_\sigma \cup (\bigcup \{D_\tau : \tau \text{ is a vertex of } \sigma\})$ in N_i and let H_σ be the closure of $N_\sigma - T_\sigma$. Let $T = \bigcup \{T_\sigma : \sigma \text{ is a 1-element of } P\}$. Then the T_σ and H_σ give N the required structure.

The 3-cell C_σ is N_σ minus a regular neighbourhood of the handles of N_σ which is sufficiently close to these handles to not intersect $G - \operatorname{Int} \sigma$.

4. A second smaller neighbourhood. Let M^3 be an orientable 3-manifold, and let J be a simple closed curve in Int M^3 which is homologous to zero in M^3. In this section, we will take the neighbourhood N of J given in Theorem 1, and construct a smaller neighbourhood N^1 which lies "nicely" in N. We will also construct a spanning surface S in $N - \operatorname{Int} N^1$. This will be the inductive step in constructing an infinite sequence of neighbourhoods in the next section.

Let S be a polyhedral surface in a 3-manifold, and let δ be a polydehral arc which intersects S only in its endpoints. There is a 3-cell B such that $B \cap S$ consists of two disks D_1 and D_2 on ∂B, Int $\delta \subset$ Int B, and the endpoints of δ are in Int D_1 and Int D_2, respectively. We can now add a handle to S by replacing $(\operatorname{Int} D_1) \cup (\operatorname{Int} D_2)$ with $\partial B - (D_1 \cup D_2)$. We call this operation *adding a handle to S along δ*. Note that if S is orientable and two-sided, and if δ approaches S on the same side at both endpoints of δ, then the handle added to S is orientable.

Step 1. *Let N be a cube-with-handles neighbourhood of J as given in Theorem* 1. *Then there is an orientable surface $S^0 \subset M^3 - \operatorname{Int} N$ where $S^0 \cap N = \partial S^0 = L$ is a simple closed curve which is homologous to J in N. Furthermore, S^0 can be chosen so that $\partial S^0 = L$ intersects every disk D_i exactly once.*

Remark. If J is not homologous to zero in M^3, there is still a simple closed curve L on ∂N so that J is homologous to L in N and such that L intersects each D_i exactly once.

Proof. As in the proof of Theorem 1, we can choose a polygonal approximation J' to J which is homotopic to J in Int N. Furthermore, we can assume that the points $p_1, p_2, \ldots, p_n$ are on J' and that they divide J' into subarcs $J_1', J_2', \ldots, J_n'$ where $J_i' \subset T_{i-1} \cup T_i \cup H_i \cup T_{i+1}$ (mod n), and where J' pierces the disk D_i at the point p_i. It is now easy to see that J' intersects each D_i algebraically once.

Since J' is homologous to zero in M^3, J' bounds an orientable (and hence two-sided) surface S' in Int M^3. Suppose that S' is in general position with respect to ∂N and each ∂D_i. Then $S' \cap \partial N$ is a 1-cycle in ∂N which intersects each ∂D_i algebraically once on ∂N. If $S' \cap \partial D_i$ contains more than one point, there is a subarc δ_i of ∂D_i which intersects S' only in its endpoints. Furthermore, δ_i can be chosen so that it approaches S' on the same side at both endpoints. Thus, we can add an orientable handle to S' along δ_i. By adding handles of this type, we can insure that $S' \cap \partial N$ intersects each ∂D_i exactly once.

Let $N_i = T_i \cup H_i$. Then each N_i is a cube-with-handles, $N = \bigcup N_i$, and $N_i \cap N_{i+1} = D_i$. Thus $S' \cap \partial N_i \cap \partial N$ is now an arc ξ_i from ∂D_{i-1} to ∂D_i, plus a finite collection of simple closed curves missing D_{i-1} and D_i. If there are any such simple curves in $S' \cap \partial N_i \cap \partial N$, there is an arc δ_i' from one of them to ξ_i on $\partial N_i \cap \partial N$. The arc δ_i' can be chosen so that it approaches S' on the same side at both endpoints. Then we can add an orientable handle to S' along δ_i', and this will reduce the number of simple closed curves of $S' \cap \partial N_i \cap \partial N$ by one. In this way, we can insure $S' \cap \partial N$ is one simple closed curve which intersects each D_i exactly once. Let

$$S^0 = S' \cap (M^3 - \text{Int } N).$$

Step 2. *Let N be a neighbourhood of J as given in Theorem* 1 *and let $\epsilon' > 0$. Then there is a neighbourhood N^1 of J in* Int N *with*

$$N^1 = (\bigcup T_j^1) \cup (\bigcup H_j^1)$$

where *T_j^1, H_j^1, J_j^1, p_j^1, and D_j^1 are as described in Theorem* 1. *Furthermore, if $p_i = J_i \cap J_{i+1}$* (mod n), *then for some $j = 1, 2, \ldots, n_1$, $p_i = p_j^1 = J_j^1 \cap J^1{}_{j+1}$* (mod n_1). *Also, each D_i can be adjusted in a neighbourhood of ∂T^1 so that*

$$p_i = p_j^1 \subset \text{Int } D_j^1 \subset D_j^1 \subset \text{Int } D_i.$$

Proof. We repeat the construction of Theorem 1 to construct N^1. The points $p_1^1, \ldots, p_{n_1}^1$ can be chosen so that each p_i is a p_j^1. Thus, if $p_j^1 = p_i$, the disk D_j^1 can be chosen initially so that it is a subdisk of D_i. For each adjustment of D_j^1 near ∂T^1 in the construction of Theorem 1, D_i can also be adjusted in the same way near ∂T^1 so that D_j^1 remains a subdisk of D_i.

Step 3. *Given neighbourhoods N and N^1 as in Step* 2, *there is a disjoint collection of orientable surfaces $E_1, \ldots, E_n$ such that $E_i \cap \partial N = \partial D_i$, and $E_i \cap N^1 = \partial D_j^1$* (*where D_j^1 is the special subdisk of D_i defined in Step* 2). *Each E_i can be obtained by adding handles to the annulus D_i −* Int *D_j^1.*

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Proof. Let J' be a polyhedral centreline for $T^1 \subset N^1$ so that each $p_i \in J'$ and is in general position with respect to each D_i. As in § 2, we can associate a word with the intersections of J' and $D_1, D_2, \ldots, D_n$. Thus for each disk D_i we have a letter e_i. Each time J' crosses D_i in a positive direction, the letter e_i appears in the words, and for each negative crossing of D_i, the letter e_i^{-1} appears. We consider this word a cyclic word; in other words, it is equivalent to any of its cyclic permutations. Since J' is homotopic in N to a simple closed curve which pierces each D_i exactly once, this word freely reduces to the word $e_1e_2 \ldots e_n$. Corresponding to each free reduction $e_ie_i^{-1}$ (or $e_i^{-1}e_i$) we can add an orientable handle to D_i. In this way, we obtain new surfaces, also called $D_1, D_2, \ldots, D_n$ so that $J' \cap D_i = p_i$. Since J' is a spine for T^1, there is an isotopy of N onto itself, fixed on ∂N, which pushes each D_i off T^1, except for the disks $D_j^1 \subset D_i$ (where D_j^1 is the meridional disk of T^1 containing $p_j^1 = p_i$).

For each $j = 1, 2, \ldots, n$ there is a wedge of simple closed curves in H_j^1 so that this wedge is a spine of H_j^1. We can assume that the wedge lies in the interior of H_j^1, except for the wedge point which lies in the interior of the disk $F_j^1 = H_j^1 \cap T^1$. Again, we can add orientable handles to the D_i's so that they do not intersect the wedge. Then there is an isotopy of N onto itself which pushes the D_i's off H_j^1. Therefore, we can assume that $D_i \cap N^1 = D_j^1$. Let

$$E_i = D_i - \operatorname{Int} D_j^1.$$

If N^1 is chosen sufficiently close to J, we can insure that each annulus with handles E_i constructed in this step lies in the union of the sections N_{i-1}, N_i, N_{i+1}, and N_{i+2} of the original neighbourhood N.

Step. 4. *Let N and N^1 be neighbourhoods of J as in Steps 2 and 3. Let L be a simple closed curve in ∂N which is homologous to J in N. Then there is an orientable surface $S \subset N - \operatorname{Int} N^1$ such that $S \cap \partial N = L$, $S \cap \partial N^1$ is a simple closed curve L^1 which is homologous to J in N^1, and $\partial S = L \cup L^1$. Furthermore, S can be chosen so that $S \cap E_i$ is an arc joining L to L^1.*

Proof. Let J'' be a polyhedral simple closed curve in N^1 which is homologous to J in N^1. Then L is homologous to J'' in N, so there is a surface S' such that $\partial S' = L \cup J''$. By the proof of Step 1 we can assume that $S' \cap \partial N^1$ is a simple closed curve L^1 which intersects each D_j^1 exactly once. Let $S = S' \cap (N - \operatorname{Int} N^1)$.

For each i, $S \cap E_i$ is an arc ξ_i joining the two boundary components of E_i, plus a finite number of simple closed curves. If this number of simple closed curves in $S \cap E_i$ is non-zero, there is an arc δ_i joining one of them to the arc ξ_i. The arc δ_i can be chosen so that it approaches S on the same side at both endpoints. We can add an orientable handle to S along δ_i, and this will reduce the number of simple closed curves in $S \cap E_i$ by one. Thus, we can assume that for each i, $S \cap E_i$ is an arc joining L to L^1.

Step 5. *Let K_i be the closure of the component of $N - (N^1 \cup (\bigcup_{i=1}^n E_i))$ such that $E_{i-1} \cup E_i \subset \operatorname{Cl}(K_i)$. Then K_i is a 3-manifold with connected boundary,* *and $S_i = S \cap K_i$ is an orientable surface with connected boundary which is properly embedded in K_i, and ∂S_i does not separate ∂K_i. Furthermore,* diam $K_i < \epsilon$.

Proof. This step just restates the results of the previous steps.

5. An infinite sequence of neighbourhoods. In Theorem 3 we construct an infinite sequence of cubes-with-handles neighbourhoods of the simple closed curve J, and an open surface S whose closure is $S \cup J$. In Theorem 3′, we construct a similar sequence of neighbourhoods for a finite graph.

The proof of Theorem 3 is contained in Steps 1–5 of the previous section.

THEOREM 3. *Let M^3 be an orientable 3-manifold, and let J be a simple closed curve in* Int M^3 *which is homologous to zero in M^3. Then there exist cubes-with-handles $N^1, N^2, N^3, \ldots$ and an open surface S such that:*

(1) $\text{Int}\, M^3 \supset N^1 \supset \text{Int}\, N^1 \supset N^2 \supset \text{Int}\, N^2 \supset \ldots \supset J$ *and* $J = \bigcap_{k=1}^{\infty} N^k$.

(2) $N^k - \text{Int}\, N^{k+1} = K_1^k \cup \ldots \cup K^k_{nk}$ *where each K_i^k is a cube-with-holes.*

(3) $K^k_{i+1} \cap K_i^k = E_i^k$ *where E_i^k is an annulus with orientable handles with one boundary component contained in ∂N^k and the other boundary component contained in ∂N^{k+1}.*

(4) $K_i^k \cap \partial N^k = \alpha_i^k$ *where α_i^k is an annulus with orientable handles.*

(5) $K_i^k \cap \partial N^{k+1} = \beta_i^k$ *where β_i^k is an annulus with orientable handles.*

(6) $\partial K_i^k = E^k_{i-1} \cup E_i^k \cup \alpha_i^k \cup \beta_i^k$.

(7) $S = S^0 \cup S^1 \cup S^2 \cup S^3 \cup \ldots$, *where, for each* $k \neq 0$, $S^k \subset N^k - \text{Int}\, N^{k+1}$ *is an annulus with orientable handles. One boundary component of S^k is contained in ∂N^k and one boundary component of S^k is contained in ∂N^{k+1}. The surface $S^0 \subset M^3 - \text{Int}\, N^1$ is a disk with handles, and* $\partial S^0 \subset \partial N^1$.

(8) $S^k \cap K_i^k = S_i^k$ *is a disk with orientable handles properly embedded in K_i^k. Furthermore, ∂S_i^k is made up of a spanning arc of E^k_{i-1}, a spanning arc of α_i^k, a spanning arc of E_i^k, and a spanning arc of β_i^k. (Thus, ∂S_i^k does not separate ∂K_i^k.)*

(9) *There exist points $p_i^k, \ldots, p^k_{nk}$ on J dividing J into segments $J_1^k, \ldots, J^k_{nk}$ with* $p_i^k = J_i^k \cap J^k_{i+1}$ (mod n_k).

(10) *Each J_j^{k+1} is contained in some J_i^k, and each α_j^{k+1} is contained in some β_i^k.*

(11) *If $p_i^k = p_j^{k+1}$, then $E_i^k \cap E_j^{k+1}$ is a simple closed curve in ∂N^{k+1}.*

(12) $\text{diam}(K_i^k \cup J_i^k) < 1/k$.

Definition. Let $J_1, J_2, \ldots, J_n$ be a collection of mutually exclusive simple closed curves in a space X. Let S be an open orientable surface in X with $S \cap (\bigcup J_j) = \emptyset$ and $\text{Cl}\, S = S \cup (\bigcup J_i)$. We say that $\bigcup J_j$ *bounds the open surface S* if there is a sequence $h_1, h_2, \ldots$ of disjoint disks with handles in S with the following properties:

(1) diam $h_i \to 0$ as $i \to \infty$

(2) $S - \bigcup h_i$ contains no non-separating simple closed curves.

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Note that if $h_1, h_2, \ldots$ is a finite sequence, then $\mathrm{Cl}(S)$ is a surface whose boundary is $\bigcup J_j$.

COROLLARY 3. *Let J be a simple closed curve topologically embedded in the interior of a 3-manifold M^3, and suppose that J is homologous to zero in M^3. Then J bounds an open surface S in M^3.*

Proof. If M^3 is orientable this is part of Theorem 3. The required open surface is $S = S^0 \cup S^1 \cup S^2 \cup \ldots$ and the null sequence of disks with handles are the S_i^k with a small annulus about ∂S_i^k removed to make them disjoint. If M^3 is non-orientable, Theorem 1 still is valid if T is allowed to be a solid Klein bottle. The construction of the sequence of neighbourhoods and the surface proceeds analogously as in Steps 1–5 of § 3 and Theorem 3.

Question. What are necessary and sufficient conditions for a simple closed curve to be the boundary of a compact surface?

COROLLARY 4. *Let J_1 and J_2 be disjoint simple closed curves topologically embedded in the interior of a 3-manifold M^3 with J_1 homologous to J_2 in M^3. Then $J_1 \cup J_2$ bounds an open surface S in M^3.*

Remark. By virtue of Conclusion (8) of Theorem 1, S may be chosen so that if $p \in J$ (respectively, $p \in J_1$ or $p \in J_2$) is a point at which the simple closed curve is locally tame, then the null sequence $h_1, h_2, \ldots$ of disks with handles of S does not cluster at p. In fact, $\lim_{i\to\infty} h_i$ lies in the set of wild points of J (respectively, $J_1 \cup J_2$). Thus if J (respectively, $J_1 \cup J_2$) is tame, then $h_1, h_2, \ldots$ is a finite sequence and $\mathrm{Cl}\, S$ is a surface whose boundary is J (respectively, $J_1 \cup J_2$).

COROLLARY 5. *Let J be a simple closed curve in the interior of a 3-manifold M^3 and let $p \in J$. Then there is a connected non-compact surface E with one simple closed curve boundary component such that $\mathrm{Cl}(E) = E \cup p$, $E \cap J = \emptyset$, and $\mathrm{Cl}(E)$ locally separates J at p.*

Proof. In the construction of the neighbourhood sequence in § 3, choose p to be a p_i^k. By Conclusion 11 of Theorem 3, $E = \bigcup \{E_j^l : p_j^l = p_i^k, l \geqq k\}$ is the required non-compact surface.

THEOREM 3′. *Let G be a finite graph topologically embedded in an orientable 3-manifold M^3. Then there exist cubes-with-handles $N^1, N^2, N^3, \ldots$ such that:*

(1) $\mathrm{Int}\, M^3 \supset N^1 \supset \mathrm{Int}\, N^1 \supset N^2 \supset \mathrm{Int}\, N^2 \supset \ldots \supset G$ *and* $G = \bigcap_{k=1}^{\infty} N^k$,
(2) *There is a sequence $P^1, P^2, P^3, \ldots$ of special decompositions of G so that each P^k is a subdivision of P^{k-1}.*
(3) *For each 1-element σ of P^k, there is an associated cube-with-holes K_σ^k.*
(4) $N^k - \mathrm{Int}\, N^{k+1} = \bigcup \{K_\sigma^k : \sigma$ *is a 1-element of* $P^k\}$.
(5) *If σ and σ' are two one elements of P^k which intersect in a vertex τ, there is an annulus with orientable handles E_τ^k so that $K_\sigma^k \cap K_{\sigma'}^k = E_\tau^k$. If $\sigma \cap \sigma' = \emptyset$, then $K_\sigma^k \cap K_{\sigma'}^k = \emptyset$.*

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(6) *E_τ^k is properly embedded in $N^k - \text{Int } N^{k+1}$. One component of ∂E_τ^k is contained in ∂N^k, and one component is contained in ∂N^{k+1}.*
(7) *If τ is a vertex of both P^k and P^{k+1}, then $E_\tau^k \cap E_\tau^{k+1} \subset \partial N^{k+1}$ is a simple closed curve.*
(8) *If σ is a 1-element of P^k, then $\text{diam}(K_\sigma^k \cup \sigma) < 1/k$.*
(9) *If τ is a vertex of P^k, $\bigcup_{i=k}^{\infty} E_\tau^i$ is a noncompact surface E_τ with one boundary component in ∂N^k, and whose closure is $E_\tau \cup \tau$.*

Proof. The proof of Theorem 3′ is analogous to the proof of Theorem 3.

6. Constructing the monotone map which tames J.

LEMMA 1. *Let K be an orientable compact 3-manifold with connected boundary, and let S be a disk with orientable handles properly embedded in K so that ∂S does not separate ∂K. Let H be a solid torus, and let F be a handle for H (i.e., F is a non-separating properly embedded disk in H). Let f_0 be a monotone map of ∂K onto ∂H and f_1 be a monotone map of S onto F where each of the finite number of nondegenerate point inverses of f_0 and f_1 is a finite 1-complex missing ∂S, and where $f_0|\partial S = f_1|\partial S$. Then f_0 and f_1 can be extended to a monotone map f from K onto H such that $f(\text{Int } K) = \text{Int } H$. Furthermore, suppose X is a compact set in $\text{Int } K - S$ with the following property: For each open set $U \subset \text{Int } K$, either $U - (U \cap X)$ is connected or $(\text{Bd } U) \cap X \neq \emptyset$. Then f can be constructed so that each component of X is a point inverse.*

Remark. A similar result could be proved for any cube-with-handles H. This lemma will be used to construct a monotone mapping from each K_i^k constructed in Theorem 3 onto a solid torus.

Proof. Let $R(S)$ be an embedding of $S \times [-1, 1]$ in K with $S \times 0$ identified with S and lying so close to S that it is disjoint from the non-degenerate point inverses of f_0. Let $R(F)$ be an embedding of $F \times [-1, 1]$ in H such that $f_0(\partial K \cap R(S)) = \partial H \cap R(F)$. By using the product structures of $R(S)$ and $R(F)$, we extend f_0 and f_1 to a "level preserving" monotone map

$$f : \partial K \cup R(S) \to \partial H \cup R(F).$$

Let $K_1 = \text{Cl}(K - R(S))$ and $H_1 = \text{Cl}(H - R(F))$. Then $f\,|\partial K_1$ is a monotone map onto ∂H_1.

Finitely many point inverses of f lie in ∂K_1 and each is a finite 1-complex. Using [**4**, Lemma 4], we can extend f to take a collar (missing X) of ∂K_1 in K_1 onto a collar of ∂H_1 in H_1 so that f has precisely one point inverse on the inside of this collar in K_1 and each point inverse of f is a connected finite 1-complex. As in the proofs of [**2**, Theorems 6.2 and 7.6], f can be extended to carry K_1 minus this collar onto the 3-cell H_1 minus the collar of ∂H_1 so that f has each component of X as a point inverse. Thus f is the required monotone map of K onto H extending f_0 and f_1.

THEOREM 4. *Let M^3 be a closed orientable 3-manifold, and let J be a simple closed curve topologically embedded in M^3. If J is homologous to zero in M^3, then there is a monotone map f of M^3 onto S^3 such that:*

(1) *$f(J)$ is a tame unknotted simple closed curve in S^3.*

(2) *$f|J$ is a homeomorphism.*

(3) *$f(M^3 - J) = S^3 - f(J)$.*

Furthermore, suppose X is a compact set in $M^3 - J$ so that if U is any connected open set in M^3, either $(\text{Bd } U) \cap X = \emptyset$ *or* $U - (X \cap U)$ *is connected. If J is homologous to zero in $M^3 - X$, then the map f can be chosen so that each component of X is a point inverse.*

Remark. The point inverse of f form an upper semi-continuous decomposition of M^3 whose decomposition space is S^3 and whose natural quotient map is f.

Proof. Regard S^3 as E^3 union a point at infinity. Let $f|J$ be a homeomorphism of J onto the unit simple closed curve $\{(x, y, z) \in E^3 : z = 0$ and $x^2 + y^2 = 1\}$ in the xy-plane. Let $A = \{p \in E^3 : (p, f(J)) \leqq 1/2\}$ be a solid torus with centreline $f(J)$. If we write the torus ∂A as $J \times S^1$, then we can regard the solid torus A as the quotient space of $J \times S^1 \times [0, 1]$ obtained by collapsing the circles $\{p\} \times S^1 \times \{0\}$ to the points $f(p)$ of the centreline $f(J)$ of A. Then we have a quotient map $h : J \times S^1 \times [0, 1] \rightarrow A$ such that

(1) $h(J \times S^1 \times \{1\}) = \partial A$,

(2) $h|J \times S^1 \times (0, 1]$ is a homeomorphism onto $A - f(J)$,

(3) if $p \in J, h(\{p\} \times S^1 \times \{0\}) = f(p) \in f(J)$.

Furthermore, we can choose h such that, for $s_0 \in S^1, h(J \times \{s_0\} \times [0, 1])$ is an annulus lying to the inside of J in the xy-plane.

Now we suppose we have the neighbourhoods $N^1, N^2, N^3, \ldots$ of J constructed in Theorem 3, and we suppose $X \subset M^3 - (N^1 \cup S^0)$. Since each S_i^k is a disk with handles (see (7) and (8) of Theorem 3), f can be extended to a map, also called f, from $S \cup J$ onto the disk $\{(x, y, 0) : x^2 + y^2 \leqq 1\}$ so that S^0 goes to the disk $\{(x, y, 0) : x^2 + y^2 \leqq 1/2\}$ and S_i^k goes onto the disk $h(J_i^k \times \{s_0\} \times [1/k, 1/k + 1])$. Furthermore, f can be chosen so that each nondegenerate point inverse of f is a 1-complex lying either in the interior of an S_i^k or in the interior of S^0.

Since $\partial N^k = \bigcup_{i=1}^{n_k} \alpha_i^k$, and each α_i^k is an annulus with handles (see (4) of Theorem 3), f can be extended to take ∂N^k onto $h(J \times S^1 \times \{1/k\})$ such that $f(\alpha_i^k) = h(J_i^k \times S^1 \times \{1/k\})$ and such that each nondegenerate point inverse of $f|\partial N^k$ is a finite 1-complex in the interior of some α_i^k missing S_i^k.

Each E_i^k is an annulus with handles and f has been defined on ∂E_j^k and on the spanning arc $S^k \cap E_i^k$. Thus f can be extended to take E_i^k onto $h(\{p_i^k\} \times S^1 \times [1/k, 1/k + 1])$ so that $f|E_i^k$ has at most one nondegenerate point inverse, which is a 1-complex in $\text{Int } E_i^k$.

Since f has already been defined on the boundary of each K_i^k and on the spanning surface S_i^k, then f can be extended to take K_i^k monotonically onto

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the solid torus $h(J_i^k \times S^1 \times [1/k, 1/k + 1])$ as in Lemma 1. Thus we have defined f to take N^1 onto A as well as to take the spanning surface S^0 of $M^3 - \text{Int}\, N^1$ onto the spanning disk $\{(x, y, 0) : x^2 + y^2 \leqq 1/2\}$ of the solid torus $S^3 - \text{Int}\, A$. By Lemma 1, f can now be extended to $M^3 - \text{Int}\, N^1$ to give the required map of M^3 onto S^3. This completes the proof of Theorem 4.

It follows from our use of Bing's results [**2**], that each nondegenerate point inverse of the map f constructed in Theorem 4 is either a component of X or is a finite 1-complex in $M^3 - J$. Using results of Armentrout [**1**] as restated in [**12**, Lemma 5], one can see that there is no such map $f : M^3 \to S^3$ which tames a wild simple closed curve J if each point inverse of f in some neighbourhood of J is cellular. If each point inverse of f in a neighbourhood of J is strongly acyclic over Z or Z_2, or has trivial Čech cohomology with coefficients Z or Z_2, it follows from [**12**, Corollaries 1 and 3] that each point inverse of f in some neighbourhood of J is cellular. Also, if the image of the nondegenerate point inverses is 0-dimensional in S^3, then it follows from [**12**, Theorem 7] that each point inverse of f in some neighbourhood of J is cellular.

COROLLARY 6. *Let J be a simple closed curve which is topologically embedded in the interior of a 3-manifold M^3. Suppose that J has a solid torus neighbourhood N in M^3 so that J is homologous to a centreline of N. Then there is a monotone map f from M^3 onto itself such that:*

(1) *$f|J$ is a homeomorphism.*
(2) *$f|M^3 - \text{Int}\, N$ is a homeomorphism.*
(3) *$f(M^3 - J) = M^3 - f(J)$.*
(4) *$f(J)$ is tame in M^3.*

Proof. There is a neighbourhood N^1 of J in Int N satisfying the requirements of Theorem 1. By the techniques of Steps 2 and 3 of § 3, there is an annulus with orientable handles E properly embedded in $N - \text{Int}\, N^1$ so that $E \cap \partial N$ is a simple closed curve in ∂E, and $E \cap \partial N^1$ is a simple closed curve in ∂E which bounds a disk in N^1. Using the techniques of Step 4, there is an annulus with orientable handles S properly embedded in $N - \text{Int}\, N^1$ so that S has one boundary component in each of ∂N and ∂N^1, and so that $S \cap E$ is a spanning arc of both S and E. Thus the proof of Theorem 4 can be carried through to produce a map $f : N \to N$ with $f|\partial N = $ identity.

Question. Let G be a graph which is embedded in the interior of a 3-manifold M^3, and let N be a neighbourhood of G in M^3. Is there a monotone mapping f from M^3 onto itself with the following properties:

(1) $f|G$ is a homeomorphism,
(2) $f|M^3 - N$ is a homeomorphism,
(3) $f(M^3 - G) = M^3 - f(G)$,
(4) $f(G)$ is tame?

7. In this section, we give an alternative proof of Smythe's result [**11**] that any knot, link, or wedge of circles G, which is homologous to zero in an orientable 3-manifold M^3, is homologous to zero in a cube-with-handles $K \subset M^3$. We do not require, as Smythe does, that G be polyhedrally embedded. Smythe can obtain that if G bounds a singular surface of genus g in M^3, then it bounds a singular surface of the same genus in K. Our proof, however, gives no such bound on the genus of a surface S bounded by G in K.

COROLLARY 7. *Let G be a finite* 1-*complex topologically embedded in the interior of an orientable* 3-*manifold M^3. Suppose that each* 1-*simplex of G is oriented, and that G is a* 1-*cycle if each* 1-*simplex has coefficient* ± 1 *according to orientation. (Thus, for each vertex v of G, the number of edges of G pointing into v is the same as the number of edges pointing out from v.) If G, considered as a* 1-*cycle, is homologous to zero in M^3, then there is a compact* 3-*manifold* $K \subset \text{Int } M^3$, *where each component of K is a cube-with-handles, such that G is homologous to zero in* $\text{Int } K$.

Remark. As special cases, G can be taken to be a simple closed curve, an oriented link, or an oriented wedge of simple closed curves.

Proof. We apply Theorem 1′ to each component of G to obtain a neighbourhood N of G where each component of N is a cube-with-handles. There is a collection $J_1, J_2, \ldots, J_m$ of oriented polyhedral simple closed curves in N so that $J = J_1 \cup J_2 \cup \ldots \cup J_m$ is homologous to G in N. Then J bounds a compact, orientable surface S (not necessarily connected). Recall from Theorem 1′ that for each vertex τ of some special decomposition P of G, there is a spanning disk D_τ of N. Using the techniques of Step 1 of § 4, we can assume that the surface S intersects the boundary of each disk D_τ exactly once. For each 1-element σ of P, there is a corresponding section $N_\sigma = T_\sigma \cup H_\sigma$ of N. Using the techniques of Step 1 again, we can assume that $S \cap (\partial N_\sigma \cap \partial N)$ contains no simple closed curve.

Let U be a regular neighbourhood of the surface S in $\text{Int } M^3 - \text{Int } N$. Then each component of U is a cube-with-handles. For each vertex τ of the special decomposition P, let V_τ be a regular neighbourhood of the disk D_τ in N which is so close to D_τ that $U \cap V_\tau$ is a disk. Then $U' = U \cup (\bigcup \{V_\tau : \tau \text{ is a vertex of } P\})$ is homeomorphic to U.

Each component of $N - \bigcup \{V_\tau : \tau \text{ is a vertex of } P\}$ is a cube-with-handles whose intersection with U' is a finite number of disks. Thus each component of $K = N \cup U$ is a cube-with-handles, and G is homologous to zero in $\text{Int } K$.

COROLLARY 8. *Let G be a finite* 1-*complex topologically embedded in the interior of a* 3-*manifold M^3. If G is inessential in M^3, then there is a compact* 3-*manifold* $K \subset \text{Int } M^3$, *where each component of K is a cube-with-handles, so that G is inessential in* $\text{Int } K$.

Proof. Since G is an ANR, there is a neighbourhood N of G which is inessential in M^3. By Theorem 1′, N can be chosen so that it is compact and each component of N is a cube-with-handles. The required 3-manifold K can now be produced by the Corollary of [**11**] or the techniques of [**6**, § 2].

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Western Michigan University,
Kalamazoo, Michigan