GENERATING FUNCTIONS FOR ULTRASPHERICAL FUNCTIONS

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1. Introduction. The ultraspherical function

$$P_n^{(\lambda)}(x) = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+1)} F[-n, n+2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-x)] \tag{1.1}$$

for $|1 - x| < 2$ is a solution of the differential equation

$$(1-x^2)\frac{d^2v}{dx^2} - (2\lambda+1)x\frac{dv}{dx} + n(n+2\lambda)v = 0. \tag{1.2}$$

This equation has two independent solutions; of the two, only $P_n^{(\lambda)}(x)$ is analytic at $x = 1$, aside for some special values of λ, which we shall not consider. The expression (1.1) vanishes identically when n is a negative integer. Hence we choose, when n is a positive integer, the ultraspherical polynomial as

$$P_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!} F[-n, n+2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-x)];$$

otherwise we choose the ultraspherical function as

$$F[-n, n+2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-x)].$$

Replacing the parameter n in (1.2) by $y\partial/\partial y$, we construct the partial differential equation $Lv = 0$ where

$$L = (1-x^2)\frac{\partial^2}{\partial x^2} - (2\lambda+1)x\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y^2} + (2\lambda+1)y\frac{\partial}{\partial y}. \tag{1.3}$$

This operator L annuls $u(x, y) = v(x)y^n$ if and only if $v(x)$ satisfies (1.2).

We show in § 2 that the partial differential equation $Lu = 0$ admits a three-parameter Lie group. Following the methods of Weisner **(11)**, we use this group to obtain generating functions for ultraspherical functions.

2. Operators. We define the following operators:

$$A = y\partial/\partial y, \qquad B = y^{-1}\left\{(1-x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right\}, \tag{2.1}$$

$$C = y\left\{(1-x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - 2\lambda x\right\},$$

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and a linear operator T which satisfies $Tf(x, y) = y^{-2\lambda}f(x, y^{-1})$, where f is an arbitrary function.

The operators A, B, and C satisfy the commutation relations

$$(2.2)\qquad [A, B] = -B, \qquad [A, C] = C, \qquad \text{and} \quad [B, C] = -2A - 2\lambda,$$

where $[A, B] = AB - BA$, and therefore generate a three-parameter Lie group G.

From the relations (2.1) we obtain the relation

$$(2.3)\qquad CB + A^2 + (2\lambda - 1)A = (1 - x^2)L.$$

Hence it follows that A, B, and C each commute with $(1 - x^2)L$ and therefore convert each solution of $Lu = 0$ into another solution. Also we have that the operator T converts every solution of $Lu = 0$ into a solution. In particular,

$$\begin{aligned} &AF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = nF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n, \\ (2.4)\quad &BF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = nF[-n + 1, n + 2\lambda - 1; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n-1}, \\ &CF[-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n \\ &\qquad = -(n + 2\lambda)F[-n - 1, n + 2\lambda + 1; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n+1}, \end{aligned}$$

where n is an arbitrary complex number.

The operator A generates a trivial group; $x' = x$ and $y' = ty$ $(t \neq 0)$. The extended form of the group generated by A, B, and C is described by

$$(2.5)\qquad e^{cC}e^{bB}f(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}f(X, Y),$$

where

$$X = \frac{b + (1 + 2bc)xy + c(1 + bc)y^2}{[(1 + 2cxy + c^2y^2)\{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2\}]^{\frac{1}{2}}},$$

$$Y = \left[\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right]^{\frac{1}{2}}.$$

b and c are arbitrary constants and $f(x, y)$ is an arbitrary function. The signs of the surds being so chosen that X and Y reduce to x and y, respectively, when $b = 0$ and $c = 0$.

3. Conjugate sets. First we want to examine the functions annulled by L and $R = r_1 A + r_2 B + r_3 C + r_4$, where the r's are arbitrary constants, other than $r_1 = r_2 = r_3 = r_4 = 0$. It is sufficient to consider one operator from each of the conjugate sets into which the operators R fall with respect to the group G.

As in (**11**, p. 1035), we have

(3.1) $e^{aA}Be^{-aA} = e^{-a}B, \qquad e^{aA}Ce^{-aA} = e^{a}C,$

(3.2) $e^{bB}Ae^{-bB} = A + bB, \qquad e^{bB}Ce^{-bB} = -2bA - b^2B + C - 2\lambda b,$

(3.3) $e^{cC}Ae^{-cC} = A - cC, \qquad e^{cC}Be^{-cC} = 2cA + B - c^2C + 2\lambda c,$

(3.4) $SAS^{-1} = (1 + 2bc)A + bB - c(1 + bc)C + 2\lambda bc,$

where $S = e^{cC}e^{bB}$.

It follows that R is conjugate to $mA + n$ for suitable choices of a, b, c, m, and n, except when $r_1^2 + 4r_2 r_3 = 0$, in which case it may be inferred that R is conjugate to $mB + n$ from (3.3).

4. Generating functions annulled by operators of the first order. We observe that

$$u_1 = F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^\nu \qquad \text{for} \quad |1 - x| < 2$$

and

$$u_2 = F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + x)]y^\nu \qquad \text{for} \quad |1 + x| < 2,$$

where ν is an arbitrary constant, are both annulled by L and $A - \nu$.

Hence from (2.6) and (3.4) it follows that

(4.1) $G_1(x, y) = M^\nu(1 + 2cxy + c^2y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-X)]$

and $G_2(x, y) = M^\nu(1 + 2cxy + c^2y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + X)]$

where

$$M = [b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2]^{\frac{1}{2}}$$

and

$$X = \frac{b^2 + (1 + 2bc)xy + c(1 + bc)y^2}{M(1 + 2cxy + c^2y^2)^{\frac{1}{2}}}$$

are both annulled by L and

$$R = (1 + 2bc)A + bB - c(1 + bc)C + 2\lambda bc - \nu.$$

In the following work, we shall be examining G_1 or G_2 depending on which is analytic at $x = 1$.

Case 1. In (4.1) putting $b = -1$ and $c = 0$, we obtain $R = A - B - \nu$ and

$$G_2(x, y) = \rho^\nu F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 + X)]$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$ and $X = (-1 + xy)/\rho$.

This function has an expansion of the form

$$\sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n.$$

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The constant c_n is determined by putting $x = 1$.

Thus

$$\rho^{\nu}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n \tag{4.2}$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$ and $X = (1 - xy)/\rho$ for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$. This is equivalent to that of Brafman (**2**, p. 945, eq. 18).

Special cases. When $\nu = -2\lambda$, we obtain

$$(1 - 2xy + y^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)y^n, \tag{4.3}$$

which is sometimes taken as a definition for ultraspherical polynomials.

When $\nu = -(\lambda + \frac{1}{2})$, we obtain

$$\rho^{-1}\left(\frac{1 + \rho - xy}{2}\right)^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda + \frac{1}{2})_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n; \tag{4.4}$$

cf. (**7**, p. 82, eq. 4.7.16).

When $\nu = -(\lambda - \frac{1}{2})$, we obtain

$$\left(\frac{1 - xy + \rho}{2}\right)^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda - \frac{1}{2})_n}{(2\lambda)_n} P_n^{(\lambda)}(x)y^n; \tag{4.5}$$

Carlitz (**5**, p. 151, eq. 9) has given an equivalent result for the Jacobi polynomials.

When $\nu = n$, a positive integer, (4.2) reduces to a polynomial identity:

$$\rho^n P_n^{(\lambda)}\left(\frac{1 - xy}{\rho}\right) = \frac{(2\lambda)_n}{n!} \sum_{m=0}^{n} \frac{(-n)_m}{(2\lambda)_m} P_m^{(\lambda)}(x)y^m; \tag{4.6}$$

cf. (**2**, p. 946, eq. 22).

The above expansion (4.2) is valid only in $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$; $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, not being single valued in the region

$$|x - (x^2 - 1)^{\frac{1}{2}}| < |y| < |x + (x^2 - 1)^{\frac{1}{2}}|,$$

cannot have an expansion in the annular region, whereas for the outer region an expansion can be obtained by the application of the operator T of (2.1) to the next result.

Unless otherwise mentioned the above remark holds good for all subsequent expansions.

Case 2. In (4.1), putting $b = 0$ and $c = -1$, we obtain $R = A + C - \nu$ and

$$G_1(x, y) = y^{\nu}(1 - 2xy + y^2)^{-\lambda-\nu/2}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)],$$

where $X = (x - y)/\rho$ and $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$.

This function has an expansion of the form

$$\sum_{n=0}^{\infty} c_n F[-n - \nu, n + \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^{n+\nu}.$$

The constant is determined by putting $x = 1$. Thus

$$(4.7)\quad \rho^{-(2\lambda+\nu)}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} \frac{(2\lambda + \nu)_n}{n!} F[-n - \nu, n + \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n,$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2 - 1)^{\frac{1}{2}}|$ and $x \neq -1$. Truesdell (**9**, p. 85, eq. 13) has an equivalent result for Associated Legendre functions.

Special cases. When $\nu = -(\lambda + \frac{1}{2})$, we obtain

$$(4.8)\quad \left(\frac{x - y + \rho}{2}\right)^{-\lambda+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\lambda - \frac{1}{2})_n}{n!} \times F[-n + \lambda + \tfrac{1}{2}, n + \lambda - \tfrac{1}{2}; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n.$$

When $\nu = -(\lambda - \frac{1}{2})$, we have

$$(4.9)\quad \rho^{-1}\left(\frac{x - y + \rho}{2}\right)^{-\lambda+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\lambda + \frac{1}{2})_n}{n!} \times F[-n + \lambda - \tfrac{1}{2}, n + \lambda + \tfrac{1}{2}; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - x)]y^n$$

When $\nu = n$, a positive integer, (4.7) reduces to

$$(4.10)\quad \rho^{-2\lambda-n}P_n^{(\lambda)}\left(\frac{x - y}{\rho}\right) = \sum_{k=0}^{\infty} \frac{(n + k)!}{n!\,k!} P_{n+k}^{(\lambda)}(x)y^k;$$

cf. (**6**, p. 280, eq. 23).

Case 3. In (4.1) substituting $b = w^{-1}$ and $c = -1$, we obtain

$$R = (2 - w)A - B + (1 - w)C + 2\lambda + w\nu$$

and

$$(4.11)\quad \rho^{-2\lambda-\nu}\mu^{\nu}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] = \sum_{n=0}^{\infty} F(-n, -\nu; 2\lambda; w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1 - 2xy + y^2)^{\frac{1}{2}}$, $\mu = \{1 - 2(1 - w)xy + (1 - w)^2y^2\}^{\frac{1}{2}}$, and

$$X = \frac{\rho^2 + wy(x - y)}{\mu\rho},$$

for $|y| < \min\{|x \pm (x^2 - 1)^{\frac{1}{2}}|, |\{x \pm (x^2 - 1)^{\frac{1}{2}}\}/(1 - w)|\}$.

Special cases.

$$(4.12)\quad \mu^{-1}\left\{\frac{\mu\rho + \rho^2 + wy(x - y)}{2}\right\}^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} F[-n, \lambda + \tfrac{1}{2}; 2\lambda; w]P_n^{(\lambda)}(x)y^n.$$

$$(4.13)\quad \rho^{-1}\left\{\frac{\mu\rho + \rho^2 + wy(x - y)}{2}\right\}^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} F[-n, \lambda - \tfrac{1}{2}; 2\lambda; w]P_n^{(\lambda)}(x)y^n.$$

$$(4.14)\quad \rho^{-2\lambda-n}\mu^nP_n^{(\lambda)}\left[\frac{\rho^2 + wy(x - y)}{\mu\rho}\right] = \frac{(2\lambda)_n}{n!}\sum_{m=0}^{\infty} F[-n, -m; 2\lambda; w]P_m^{(\lambda)}(x)y^m.$$

After replacing $(1-w)$ by w^{-1} for the annular region,

$$|w\{x \pm (x^2-1)^{\frac{1}{2}}\}| < |y| < |x \pm (x^2-1)^{\frac{1}{2}}|,$$

we obtain

$$(4.15)\quad (1-2wxy^{-1}+w^2y^{-2})^{\nu/2}(1-2xy+y^2)^{-\lambda-\nu/2} \times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1+X)]$$
$$= \sum_{n=0}^{\infty} \frac{(2\lambda+\nu)_n}{n!} F[-\nu, \nu+2\lambda+n; n+1; w] \times F[-n-\nu, n+\nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]y^n$$
$$+ \sum_{n=1}^{\infty} \frac{(-\nu)_n}{n!} F[\nu+2\lambda, -\nu+n; n+1; w] \times F[n-\nu, -n+\nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]w^n y^{-n},$$

where

$$X = \frac{y\{1-(1+w)xy^{-1}+wy^{-2}\}}{\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}}}$$

for $|w\{x \pm (x^2-1)^{\frac{1}{2}}\}| < |y| < |x \pm (x^2-1)^{\frac{1}{2}}|$ and $x \neq -1$.

Special cases.

$$(4.16)\quad \{1-2wxy^{-1}+w^2y^{-2}\}^{-\frac{1}{2}} \cdot [\tfrac{1}{2}\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}} - \tfrac{1}{2}y\{1-(1+w)xy^{-1}+wy^{-2}\}]^{\frac{1}{2}-\lambda}$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda-\frac{1}{2})_n}{n!} F[\lambda+\tfrac{1}{2}, \lambda+n-\tfrac{1}{2}; n+1; w] \times F[-n+\lambda+\tfrac{1}{2}, n+\lambda-\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]y^n$$
$$+ \sum_{n=1}^{\infty} \frac{(\lambda+\frac{1}{2})_n}{n!} F[\lambda-\tfrac{1}{2}, \lambda+n+\tfrac{1}{2}; n+1; w] \times F[+n+\lambda+\tfrac{1}{2}, -n+\lambda-\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)]w^n y^{-n}.$$

$$(4.17)\quad (1-2wxy^{-1}+w^2y^{-2})^{n/2}(1-2xy+y^2)^{-\lambda-n/2}P_n^{(\lambda)}(X)$$
$$= \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\,m!} F[-n, 2\lambda+n+m; m+1; w]P_{n+m}^{(\lambda)}(x)y^m$$
$$+ \sum_{m=1}^{n} \frac{(1-2\lambda-n)_m}{m!} F[n+2\lambda, -n+m; m+1; w]P_{n-m}^{(\lambda)}(x)w^m y^{-m},$$

where

$$X = \frac{y\{1-(1+w)xy^{-1}+wy^{-2}\}}{\{(1-2xy+y^2)(1-2wxy^{-1}+w^2y^{-2})\}^{\frac{1}{2}}}.$$

5. Generating functions annulled by $2A - B + C + 2\lambda - w$. We next examine the simultaneous equations $Lu = 0$ and $Bu = -u$; the general solution of the latter equation is $u = e^{-xy}f(y(1-x^2)^{\frac{1}{2}})$, where f is an arbitrary function.

If this is to be annulled by L, then $f(X)$ must satisfy the equation

$$X\cdot\frac{d^2f}{dX^2}+2\lambda\frac{df}{dX}+Xf=0,$$

where $X = y(1-x^2)^{\frac{1}{2}}$. Two linearly independent solutions of this are

$$F[-;\lambda+\tfrac{1}{2};-\tfrac{1}{4}X^2]$$

and

$$(-\tfrac{1}{4}X^2)^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;-\tfrac{1}{4}X^2].$$

Hence the solutions of $Lu = 0$ and $(B+1)u = 0$ are

$$(5.1)\qquad e^{-xy}F[-;\lambda+\tfrac{1}{2};-\tfrac{1}{4}y^2(1-x^2)],$$
$$e^{-xy}\{\tfrac{1}{4}y^2(1-x^2)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;-\tfrac{1}{4}y^2(1-x^2)].$$

The first of these is analytic at $x = 1$ and we obtain

$$(5.2)\qquad e^{-xy}F\left[-;\lambda+\tfrac{1}{2};-\frac{y^2(1-x^2)}{4}\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\lambda)_n}P_n^{(\lambda)}(x)y^n;$$

(1) gives an equivalent result for Associated Legendre polynomials.

Equations (2.5), (3.3), and (5.1) show that

$$\rho^{-2\lambda}\exp\{-w(x-y)y/\rho^2\}\;F[-;\lambda+\tfrac{1}{2};-w^2y^2(1-x^2)/4\rho^4]$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, is annulled by L and

$$R=-2A+B-C-2\lambda+w.$$

Using the generating function for Laguerre polynomials **(7**, p. 100**)**, we obtain

$$(5.3)\qquad \rho^{-2\lambda}\exp\{-w(x-y)y/\rho^2\}F[-;\lambda+\tfrac{1}{2};-w^2y^2(1-x^2)/4\rho^4]$$
$$=\sum_{n=0}^{\infty}\frac{n!}{(2\lambda)_n}L_n^{(2\lambda-1)}(w)P_n^{(\lambda)}(x)y^n;$$

cf. **(8)**.

We have thus obtained in the normalized form functions which are annulled by L and $R = r_1 A + r_2 B + r_3 C + r_4$, where the r's are constants.

6. Generating functions annulled by second-order operators. In some cases by suitable choice of a new set of variables, the equation $Lu = 0$ may be transformed into one solvable by the method of separation of variables.

Taking $X = \frac{1}{2}y(x+1)$ and $Y = \frac{1}{2}y(x-1)$ the equation $Lu = 0$ is transformed into

$$X\frac{\partial^2u}{\partial X^2}-Y\frac{\partial^2u}{\partial Y^2}+(\lambda+\tfrac{1}{2})\frac{\partial u}{\partial X}-(\lambda+\tfrac{1}{2})\frac{\partial u}{\partial Y}=0.$$

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Without loss of generality, the separation constant can be taken as 1. Four linearly independent solutions are

$$(6.1)\quad \begin{cases} u_1 = F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)],\\ u_2 = \{y(x+1)\}^{\frac{1}{2}-\lambda}F[-;-\lambda+\tfrac{3}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)],\\ u_3 = \{y(x-1)\}^{\frac{1}{2}-\lambda}F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x-1)],\\ u_4 = \{y^2(x^2-1)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x+1)]F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x-1)]. \end{cases}$$

These functions are also annulled by

$$X\cdot\frac{\partial^2}{\partial X^2}+(\lambda+\tfrac{1}{2})\frac{\partial}{\partial X}-1 = -Y(X-Y)^{-2}L+(A+\lambda+\tfrac{1}{2})B-1$$

and hence by $AB+(\lambda+\frac{1}{2})B-1$.

Of these four solutions, only the first two are analytic at $x=1$ and hence we shall be considering only these two cases. We obtain

$$(6.2)\quad F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)] = \sum_{n=0}^{\infty}\frac{1}{(2\lambda)_n(\lambda+\frac{1}{2})_n}P_n^{(\lambda)}(x)y^n.$$

Similarly,

$$(6.3)\quad \{\tfrac{1}{2}(x+1)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}y(x+1)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}y(x-1)] = \sum_{n=0}^{\infty}\frac{1}{(\frac{3}{2}-\lambda)_n\,n!}F[-n+\lambda-\tfrac{1}{2},n+\lambda+\tfrac{1}{2};\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-x)]y^n$$

for $x\neq -1$. Both of these equations can be obtained from (**10**, p. 148, eq. 2).

Equations (2.6), (3.3), and (6.1) show that

$$(6.4)\quad \begin{cases} (1+2cxy+c^2y^2)^{-\lambda}F[-;\lambda+\tfrac{1}{2};X]F[-;\lambda+\tfrac{1}{2};Y],\\ (1+2cxy+c^2y^2)^{-\frac{1}{2}}\{b+(1+2bc)xy+c(1+bc)y^2+M\}^{\frac{1}{2}-\lambda}\\ \qquad\times F[-;\tfrac{3}{2}-\lambda;X]F[-;\lambda+\tfrac{1}{2};Y], \end{cases}$$

where

$$X = -\frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{1+2cxy+c^2y^2}\right\},$$

$$Y = -\frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2-M}{1+2cxy+c^2y^2}\right\},$$

with $M = [(1+2cxy+c^2y^2)\{b^2+2b(1+bc)xy+(1+bc)^2y^2\}]^{\frac{1}{2}}$, are both annulled by L and R;

$$\begin{aligned} R = {} & 3c(1+bc)A^2+bB^2+c^3(1+bc)C^2+(1+4bc)AB\\ & -c^2(3+4bc)AC+6\lambda c(1+2bc)A+(\lambda+\tfrac{1}{2})(1+4bc)B\\ & -c^2(\lambda-\tfrac{1}{2})(3+4bc)C+\lambda c(2\lambda+1)(1+2bc)+w. \end{aligned}$$

Case 1. Putting $b=-1$ and $c=0$, we have

$$R = B^2-AB-(\lambda+\tfrac{1}{2})B-w.$$

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Thus

$$(6.5)\quad F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy+\rho)]F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy-\rho)] = \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{(2\lambda)_n(\lambda+\frac{1}{2})_n}F[-;\lambda+n+\tfrac{1}{2};w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$.

This is equivalent to the result of Weisner (**13**, p. 154, eq. 6.1) for Bessel functions.

Similarly

$$(6.6)\quad \{\tfrac{1}{2}(1-xy+\rho)\}^{\frac{1}{2}-\lambda}F[-;\tfrac{3}{2}-\lambda;\tfrac{1}{2}w(1-xy+\rho)] \times F[-;\lambda+\tfrac{1}{2};\tfrac{1}{2}w(1-xy-\rho)] = \sum_{n=0}^{\infty}\frac{(\lambda-\frac{1}{2})_n}{(2\lambda)_n}F[-;\tfrac{3}{2}-\lambda-n;w]P_n^{(\lambda)}(x)y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$. An equivalent result for Bessel function is given by Weisner (**13**, p. 155, eq. 6.2).

Case 2. Putting $b = 0$ and $c = -1$, we have

$$R = 3A^2 + C^2 - AB + 3AC + 6\lambda A - (\lambda+\tfrac{1}{2})B + 3(\lambda-\tfrac{1}{2})C + \lambda(2\lambda+1) - w.$$

Thus

$$(6.7)\quad \rho^{-2\lambda}F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y+\rho}{\rho^2}\right]F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y-\rho}{\rho^2}\right] = \sum_{n=0}^{\infty}{}_1F_2[-n;\lambda+\tfrac{1}{2},2\lambda;w]P_n^{(\lambda)}(x)y^n$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$; cf. (**3**, p. 1321, eq. 15). Similarly,

$$(6.8)\quad \rho^{-1}\left(\frac{x-y+\rho}{2}\right)^{\frac{1}{2}-\lambda}F\left[-;\frac{3}{2}-\lambda;-\frac{wy}{2}\frac{x-y+\rho}{\rho^2}\right] \times F\left[-;\lambda+\tfrac{1}{2};-\frac{wy}{2}\frac{x-y-\rho}{\rho^2}\right] = \sum_{n=0}^{\infty}\frac{(\lambda+\frac{1}{2})_n}{n!}\,{}_1F_2[-n;\lambda+\tfrac{1}{2},\tfrac{3}{2}-\lambda;w] \times {}_2F_1[-n+\lambda-\tfrac{1}{2},n+\lambda+\tfrac{1}{2};\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-x)]y^n,$$

where $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$ and $x \neq -1$. An equivalent result for Associated Legendre polynomials is given by Yadao (**14**, p. 120, eq. 1.3).

Note. There is a computational error in Yadao's result. The correct version is

$$\frac{\rho^{-1}}{\Gamma(1-m)}\left(\frac{x-t+\rho}{x-t-\rho}\right)^{m/2} F\left[-;1-m;-\frac{ty(x-t-\rho)}{2\rho^2}\right]$$
$$\times F\left[-;1+m;-\frac{ty(x-t+\rho)}{2\rho^2}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1-m)_n}{n!}F[-n;1-m,1+m;y]P_n^{m}(x)t^n.$$

In the general case, from (6.4) we have

$$G_1(x,y) = (1+2cxy+c^2y^2)^{-\lambda}F[-;\lambda+\tfrac{1}{2};X]F[-;\lambda+\tfrac{1}{2};Y]$$

and

$$G_2(x,y) = (1+2cxy+c^2y^2)^{-\frac{1}{2}}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{2b}\right\}^{\frac{1}{2}-\lambda}$$
$$\times F[-;\tfrac{3}{2}-\lambda;X]F[-;\lambda+\tfrac{1}{2};Y],$$

where

$$X = \frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2+M}{1+2cxy+c^2y^2}\right\},$$
$$Y = \frac{w}{2}\left\{\frac{b+(1+2bc)xy+c(1+bc)y^2-M}{1+2cxy+c^2y^2}\right\},$$

and $M = [(1+2cxy+c^2y^2)\{b^2+2b(1+bc)xy+(1+bc)^2y^2\}]^{\frac{1}{2}}$. These give

(6.9) $$G_1(x,y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n,$$

where

$$c_n = \sum_{m=0}^{n}\frac{(-1)^n(-n)_m}{(2\lambda)_m(\lambda+\frac{1}{2})_m}\frac{w^m c^{n-m}}{m!}F[-;\lambda+m+\tfrac{1}{2};wb]$$

and

(6.10) $$G_2(x,y) = \sum_{m=0}^{\infty} c_n P_n^{(\lambda)}(x)y^n,$$

where

$$c_n = \sum_{m=0}^{n}\frac{(-1)^{m+n}(\lambda-\frac{1}{2})_m(-n)_m}{(2\lambda)_m\, m!}b^{-m}c^{n-m}F[-;\tfrac{3}{2}-\lambda-m;wb].$$

7. Functions annulled by $AB+(\lambda+\frac{1}{2})B-A^2-2\lambda A+\nu(2\lambda+\nu)$. If we choose the new variables as $X=\rho-y$ and $Y=\rho+y$, where

$$\rho = (1-2xy+y^2)^{\frac{1}{2}},$$

the equation $Lu = 0$ is transformed into

$$(1-X^2)\frac{\partial^2 u}{\partial x^2}-(1-Y^2)\frac{\partial^2 u}{\partial Y^2}-(2\lambda+1)X\frac{\partial u}{\partial X}+(2\lambda+1)Y\frac{\partial u}{\partial Y}=0.$$

Selecting $\nu(2\lambda+\nu)$ for the separation constant, the above equation has four linearly independent solutions:

$$
\begin{aligned}
&F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-X)]F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-Y)],\\
&(1-X)^{\frac{1}{2}-\lambda}F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-Y)]\\
&\qquad\times F[-\nu-\lambda+\tfrac{1}{2},\nu+\lambda+\tfrac{1}{2};\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-X)],\\
&(1-Y)^{\frac{1}{2}-\lambda}F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-X)]\\
&\qquad\times F[-\nu-\lambda+\tfrac{1}{2};\nu+\lambda+\tfrac{1}{2};\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-Y)],\\
&\{(1-X)(1-Y)\}^{\frac{1}{2}-\lambda}F[-\nu-\lambda+\tfrac{1}{2},\nu+\lambda+\tfrac{1}{2};\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-X)]\\
&\qquad\times F[-\nu-\lambda+\tfrac{1}{2},\nu+\lambda+\tfrac{1}{2};\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-Y)].
\end{aligned}
\tag{7.1}
$$

These functions are also annulled by

$$
\begin{aligned}
&(1-X^2)\frac{\partial^2}{\partial X^2}-(2\lambda+1)X\frac{\partial}{\partial X}+\nu(2\lambda+\nu)\\
&\qquad=\frac{XY+2Y-1}{2(Y-X)}L+AB+(\lambda+\tfrac{1}{2})B-A^2-2\lambda A+\nu(2\lambda+\nu)
\end{aligned}
$$

and hence by $AB+(\lambda+\frac{1}{2})B-A^2-2\lambda A+\nu(2\lambda+\nu)$.

We shall be considering the first two cases only. We obtain

$$
\begin{aligned}
&F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-\rho+y)]F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-\rho-y)]\\
&\qquad=\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+2\lambda)_n}{(\lambda+\frac{1}{2})_n(2\lambda)_n}P_n^{(\lambda)}(x)y^n,
\end{aligned}
\tag{7.2}
$$

where $\rho=(1-2xy+y^2)^{\frac{1}{2}}$, for $|y|<|x\pm(x^2-1)^{\frac{1}{2}}|$; cf. (**2**, p. 945, eq. **17**).

Special case.

$$
P_n^{(\lambda)}(\rho-y)P_n^{(\lambda)}(\rho+y)=\left\{\frac{(2\lambda)_n}{n!}\right\}^2\times\sum_{m=0}^{n}\frac{(-n)_m(n+2\lambda)_m}{(\lambda+\frac{1}{2})_m(2\lambda)_m}P_m^{(\lambda)}(x)y^m.
\tag{7.3}
$$

Next we obtain

$$
\begin{aligned}
&\left(\frac{1-\rho+y}{2y}\right)^{\frac{1}{2}-\lambda}F[-\nu-\lambda+\tfrac{1}{2},\nu+\lambda+\tfrac{1}{2};\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-\rho+y)]\\
&\qquad\times F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-\rho-y)]\\
&=\sum_{n=0}^{\infty}\frac{(-\nu-\lambda+\frac{1}{2})_n(\nu+\lambda+\frac{1}{2})_n}{(\frac{3}{2}-\lambda)_n\,n!}\\
&\qquad\times F[-n-\tfrac{1}{2}+\lambda,n+\tfrac{1}{2}+\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-x)]y^n,
\end{aligned}
\tag{7.4}
$$

where $\rho=(1-2xy+y^2)^{\frac{1}{2}}$, for $|y|<|x\pm(x^2-1)^{\frac{1}{2}}|$ and $x\neq-1$.

From (2.5), (3.4), and (7.1) we obtain

$$
\begin{cases}
(1+2cxy+c^2y^2)^{-\lambda}F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-X)]\\
\qquad\times F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-Y)],\\
(1+2cxy+c^2y^2)^{-\lambda}(1-X)^{\frac{1}{2}-\lambda}F[-\nu-\lambda+\tfrac{1}{2},\nu+\lambda+\tfrac{1}{2};\\
\qquad\tfrac{3}{2}-\lambda;\tfrac{1}{2}(1-X)]F[-\nu,\nu+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1}{2}(1-Y)],
\end{cases}
\tag{7.5}
$$

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where

$$X = \left\{\frac{(1+wb)^2 + 2(1+wb)(c+w+wbc)xy + (c+w+wbc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}} + w\left\{\frac{b^2 + 2b(1+bc)xy + (1+bc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}}$$

and

$$Y = \left\{\frac{(1+wb)^2 + 2(1+wb)(c+w+wbc)xy + (c+w+wbc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}} - w\left\{\frac{b^2 + 2b(1+bc)xy + (1+bc)^2y^2}{1+2cxy+c^2y^2}\right\}^{\frac{1}{2}},$$

and that these are annulled by L and R;

$$\begin{aligned} R = {} & \{w + 3c(1+bc)(1+2wbc)\}A^2 + b(1+bw)B^2 \\ & + c^2(1+bc)(c+w+wbc)C^2 - \{1 - 2(1+wb)(1+2bc)\}AB \\ & - c\{c + 2(1+bc)(c+w+wbc)\}AC + 2\lambda\{w + 3c(1+bc) \\ & + 6wbc(1+bc)\}A + (\lambda + \tfrac{1}{2})\{1 + 2b(2c+w+2wbc)\}B \\ & + c(\lambda - \tfrac{1}{2})\{c - 2(1+2bc)(2c+w+2wbc)\}C \\ & + \lambda c(2\lambda+1)\{1 + 2b(c+w+wbc)\} - \nu w(2\lambda+\nu). \end{aligned}$$

Case 1. Putting $b = -1$ and $c = 0$, we have

$$R = wA^2 - (1-w)B^2 + (1-2w)AB + 2\lambda wA + (\lambda + \tfrac{1}{2})(1-2w)B - \nu w(2\lambda+\nu).$$

We obtain, after replacing wy by $-y$,

$$\text{(7.6)} \quad F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-X)]F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)] = \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu+2\lambda)_n}{(\lambda+\frac{1}{2})_n(2\lambda)_n} F[-\nu+n, \nu+2\lambda+n; \lambda+\tfrac{1}{2}+n; w]P_n^{(\lambda)}(x)y^n,$$

where

$$X = \{(1-w)^2 - 2(1-w)xy + y^2\}^{\frac{1}{2}} - \{w^2 + 2wxy + y^2\}^{\frac{1}{2}}$$

and

$$Y = \{(1-w)^2 - 2(1-w)xy + y^2\}^{\frac{1}{2}} + \{w^2 + 2wxy + y^2\}^{\frac{1}{2}},$$

for $|y| < \min\{|(1-w)[x \pm (x^2-1)^{\frac{1}{2}}]|, |w[x \pm (x^2-1)^{\frac{1}{2}}]|\}$.

Special case.

$$\text{(7.7)} \quad P_n^{(\lambda)}(X)P_n^{(\lambda)}(Y) = \left(\frac{(2\lambda)_n}{n!}\right)^2 \sum_{m=0}^{n} \frac{(-n)_m(n+2\lambda)_m}{(\lambda+\frac{1}{2})_m(2\lambda)_m} \times F[-n+m, n+2\lambda+m; \lambda+\tfrac{1}{2}+m; w]P_m^{(\lambda)}(x)y^m.$$

Similarly

$$(7.8)\quad \left(\frac{1-X}{2w}\right)^{\frac{1}{2}-\lambda} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda; \tfrac{1}{2}(1-X)] \times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)] = \sum_{n=0}^{\infty} \frac{(\lambda-\frac{1}{2})_n}{(2\lambda)_n} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda-n; w] P_n^{(\lambda)}(x) y^n.$$

Case 2. Putting $b = 0$ and $c = -1$, we have

$$R = (3-w)A^2 + (1-w)C^2 - AB + (3-2w)AC + 2\lambda(3-w)A - (\lambda+\tfrac{1}{2})B + (\lambda-\tfrac{1}{2})(3-2w)C + \lambda(2\lambda+1) + \nu w(2\lambda+\nu).$$

We obtain

$$(7.9)\quad \rho^{-2\lambda} F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-X)] F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)] = \sum_{n=0}^{\infty} {}_3F_2[-n, -\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}, 2\lambda; w] P_n^{(\lambda)}(x) y^n$$

where

$$X = \frac{[1-2(1-w)xy + (1-w)^2 y^2]^{\frac{1}{2}} + wy}{\rho},$$

$$Y = \frac{[1-2(1-w)xy + (1-w)^2 y^2]^{\frac{1}{2}} - wy}{\rho}$$

and $\rho = (1-2xy+y^2)^{\frac{1}{2}}$, for $|y| < |x \pm (x^2-1)^{\frac{1}{2}}|$; cf. (**3**, p. 1319, eq. 2).

Special cases.

$$(7.10)\quad \rho^{-1}\left\{\frac{\rho(\rho^2+2wxy)^{\frac{1}{2}} + wy(x-y) + \rho^2}{2}\right\}^{\frac{1}{2}-\lambda} = \sum_{n=0}^{\infty} F[-n, \lambda-\tfrac{1}{2}; 2\lambda; w] P_n^{(\lambda)}(x) y^n,$$

$$(7.11)\quad \rho^{-2\lambda} P_n^{(\lambda)}(X) P_n^{(\lambda)}(Y) = \left\{\frac{(2\lambda)_n}{n!}\right\}^2 \sum_{m=0}^{\infty} {}_3F_2[-m, -n, n+2\lambda; \lambda+\tfrac{1}{2}; 2\lambda, w] P_m^{(\lambda)}(x) y^m,$$

and from the second equation of (7.5)

$$(7.12)\quad \rho^{-2\lambda}\left(\frac{X-1}{2wy}\right)^{\frac{1}{2}-\lambda} F[-\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda; \tfrac{1}{2}(1-X)] \times F[-\nu, \nu+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-Y)] = \sum_{n=0}^{\infty} \frac{(\lambda+\frac{1}{2})_n}{n!} {}_3F_2[-n, -\nu-\lambda+\tfrac{1}{2}, \nu+\lambda+\tfrac{1}{2}; \tfrac{3}{2}-\lambda, \lambda+\tfrac{1}{2}; w] \times F[-n+\lambda-\tfrac{1}{2}, n+\lambda+\tfrac{1}{2}; \lambda+\tfrac{1}{2}; \tfrac{1}{2}(1-x)] y^n;$$

cf. (**4**, p. 81, eq. 5).

In the general case, from (7.5) we have

$$G_1(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - X)] \times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],$$

$$G_2(x, y) = (1 + 2cxy + c^2y^2)^{-\lambda}\left(\frac{X - 1}{2bw}\right)^{\frac{1}{2}-\lambda} \times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda; \tfrac{1}{2}(1 - X)] \times F[-\nu, \nu + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1 - Y)],$$

where

$$X = \left\{\frac{(1 - wb)^2 + 2(1 - wb)(c - w - wbc)xy + (c - w - wbc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}} - w\left\{\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}}$$

and

$$Y = \left\{\frac{(1 - wb)^2 + 2(1 - wb)(c - w - wbc)xy + (c - w - wbc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}} + w\left\{\frac{b^2 + 2b(1 + bc)xy + (1 + bc)^2y^2}{1 + 2cxy + c^2y^2}\right\}^{\frac{1}{2}}.$$

In these cases we have

$$G_1(x, y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x) y^n, \tag{7.13}$$

where

$$c_n = \sum_{m=0}^{n} \frac{(-1)^n(-n)_m}{(2\lambda)_m} \frac{(-\nu)_m(\nu + 2\lambda)_m}{(\lambda + \frac{1}{2})_m} \frac{w^m c^{n-m}}{m!} \times F[-\nu + m, 2\lambda + \nu + m, \lambda + \tfrac{1}{2} + m; wb$$

and

$$G_2(x, y) = \sum_{n=0}^{\infty} c_n P_n^{(\lambda)}(x) y^n, \tag{7.14}$$

where

$$c_n = \sum_{m=0}^{n} (-1)^{n+m} \frac{(\lambda - \frac{1}{2})_m(-n)_m}{(2\lambda)_m} \frac{b^{-m} c^{n-m}}{m!} \times F[-\nu - \lambda + \tfrac{1}{2}, \nu + \lambda + \tfrac{1}{2}; \tfrac{3}{2} - \lambda - m; wb]$$

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