Canad. Math. Bull. Vol. **23** (2), 1980

ISOMORPHIC GROUP RINGS WITH NON-ISOMORPHIC COEFFICIENT RINGS*

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1. **Introduction.** The following question has been floating around for some time now and is also stated as Research Problem 26 in [4]:

Let R, S be unital rings and let $\langle x\rangle$ be an infinite cyclic group. Does $R\langle x\rangle \simeq S\langle x\rangle$ imply $R \simeq S$?

In this note, we present a collection of examples which answer the question in the negative. However, all of these examples consist of non-commutative rings, and the problem is still open in the case where R and S are assumed to be commutative.

For an excellent survey of known results in this area, the reader should consult [4].

2. **The examples.** The basic idea is very simple. Let us assume that we have found groups G and H with the property that $G \times \mathbf{Z} \simeq H \times \mathbf{Z}$. Then

$$\mathbf{Z}G\langle x\rangle \simeq \mathbf{Z}(G \times \mathbf{Z}) \simeq \mathbf{Z}(H \times \mathbf{Z}) \simeq \mathbf{Z}H\langle x\rangle,$$

where $\langle x\rangle$ is infinite cyclic. If we know that $\mathbf{Z}G \not\simeq \mathbf{Z}H$, then we have found an example of the type required. Hence, what we need are examples of groups G and H such that $G \times \mathbf{Z} \simeq H \times \mathbf{Z}$ but $\mathbf{Z}G \not\simeq \mathbf{Z}H$.

In [5], Walker shows that the groups $G = \langle y, z \mid y^{11} = 1, z^{-1}yz = y^2\rangle$ and $H = \langle y, z \mid y^{11} = 1, z^{-1}yz = y^8\rangle$ are not isomorphic but have the property that $G \times \mathbf{Z} \simeq H \times \mathbf{Z}$. Since G and H are metacyclic with infinite cyclic abelianization, the following result of Sehgal [3] implies that $\mathbf{Z}G \not\simeq \mathbf{Z}H$.

PROPOSITION. *If G is a metabelian group and G/G' is torsionfree, then $\mathbf{Z}G \simeq \mathbf{Z}H$ implies $G \simeq H$.*

In Walker's example, the groups are metacyclic but not nilpotent. Mislin [1] gives a number of examples of non-isomorphic finitely generated nilpotent class two groups G and H with the property that $G \times \mathbf{Z} \simeq H \times \mathbf{Z}$. It is known that $\mathbf{Z}G \not\simeq \mathbf{Z}H$ for such groups (essentially, this is in [2]). But since Mislin's examples have torsionfree abelianizations, it also follows from the proposition above.

* Research supported in part by the National Research Council of Canada, grants no. A-8741 and A-8775.

Received by the editors May 22, 1979.

https://doi.org/10.4153/CMB-1980-034-4 Published online by Cambridge University Press

Perhaps it is worth noting that all these examples obviously have the property that $\mathbf{Z}\langle x\rangle G \simeq \mathbf{Z}\langle x\rangle H$, thus providing non-isomorphic groups G and H whose group rings over $\mathbf{Z}\langle x\rangle$ are isomorphic.

Finally, we remark again that this procedure only yields non-commutative examples. This is because if G and H are abelian, then $G\times\mathbf{Z}\simeq H\times\mathbf{Z}$ implies $G\simeq H$.

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https://doi.org/10.4153/CMB-1980-034-4 Published online by Cambridge University Press