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Gorenstein Fano threefolds with base points in the anticanonical system

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Abstract

We classify all Gorenstein Fano threefolds with at worst canonical singularities for which the anticanonical system $|-K|$ has a nonempty base locus.

1. Introduction

In the classification of Fano varieties, those which are not 'Gino Fano', i.e. for which $-K_X$ is ample but not very ample, are usually annoying. In the beginning of his classification of Fano threefolds Iskovskikh listed those for which $|-K_X|$ is not free (see [Isk78]). The purpose of this article is to see how his result extends to the canonical Gorenstein case.

If X is a Gorenstein Fano threefold with at worst canonical singularities and $\mathrm{Bs}\,|-K_X| \neq \emptyset$, then the rational map defined by $|-K_X|$ goes to a surface W, which is a rational ruled surface Σ_e with $e \geqslant 0$ or $\widehat{C}_d$, the cone over a rational normal curve of degree d. The following theorem lists the possible pairs (X, W).

Theorem 1.1. *Let X be a Gorenstein Fano threefold with at worst canonical singularities and $\mathrm{Bs}\,|-K_X| \neq \emptyset$. Then we are in one of the following cases.*

(i) $\dim \mathrm{Bs}\,|-K_X| = 0$. *In this case X is a complete intersection in $\mathbb{P}(1^4, 2, 3)$ of a quadric Q, defined in the first four linear variables, and a sextic F_6; $(-K_X)^3 = 2$ and W is the quadric Q in $\mathbb{P}_3$.*

(ii) $\dim \mathrm{Bs}\,|-K_X| = 1$. *Then $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$ and:*
   - (a) *X is the blowup of a sextic in $\mathbb{P}(1^3, 2, 3)$ along a complete intersection curve of arithmetic genus 1, $(-K_X)^3 = 4$ and $W \simeq \Sigma_1$; or*
   - (b) *$X \simeq S_1 \times \mathbb{P}_1$, where S_1 is a del Pezzo surface of degree 1 with at worst Du Val singularities, $(-K_X)^3 = 6$ and $W \simeq \mathbb{P}_1 \times \mathbb{P}_1$; or*
   - (c) *$X = X_{2m-2}$ is an anticanonical model of the blowup of the variety U_m (see below) along a smooth, rational complete intersection curve $\Gamma_0 \subset U_{m,\mathrm{reg}}$ for $3 \leqslant m \leqslant 12$, $(-K_X)^3 = 2m-2$ and $W \simeq \widehat{C}_m$.*

Here U_m denotes a double cover of $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(m) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4) \oplus \mathcal{O}_{\mathbb{P}_1})$ with at worst canonical singularities, such that $-K_{U_m}$ is the pullback of the tautological line bundle $\mathcal{O}(1)$. For $m \geqslant 4$, this is a hyperelliptic Gorenstein almost Fano threefold of degree $4m-8$. The curve Γ_0 lies over the complete intersection of some general element in $|\mathcal{O}(1)|$ and the 'minimal surface' $B \in |\mathcal{O}(1) - mF|$, where $|F|$ denotes the pencil (note that Γ_0 is always contained in the ramification locus). If $m = 3$, then Γ_0 is the only curve, on which $-K_{U_3}$ is not nef. For details of the construction, see § 5.

The cases (a) and (b) are as in Iskovskikh's list. In a different context, case (i) appears in [Mel99] and [IT01], and apparently also in [Mor88].

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2. Preliminaries

We recall the following fundamental results.

Theorem 2.1 [Sho80, Rei83]. *Let X be a Gorenstein Fano threefold with at worst canonical singularities. Then $|-K_X|$ contains an irreducible surface S with at worst Du Val singularities, called* general elephant.

The birational contraction $h\colon Y \to X$ in the following theorem is called a *partial crepant resolution* or *terminal modification* of X.

Theorem 2.2 [Rei79, Kaw88]. *Let X be a threefold with only canonical singularities. Then there exists a $\mathbb{Q}$-factorial threefold Y with only terminal singularities and a birational contraction $h\colon Y \to X$ such that $K_Y = h^*K_X$.*

If X is Gorenstein, then Y is, in fact, factorial (for example, [Kaw88, Lemma 5.1]).

A Gorenstein threefold X for which $-K_X$ is big and nef is called *almost Fano*. It is called *hyperelliptic*, if $|-K_X|$ is free, but the associated map φ fails to be injective at the generic point. In that case $\varphi\colon X \longrightarrow W \subset \mathbb{P}_N$ is generically two-to-one and W is a so-called variety of minimal degree, i.e. $\deg W = \operatorname{codim} W + 1$. Varieties of minimal degree have been classified by del Pezzo [delP85] in dimension 2 and by Bertini in arbitrary dimension n (see [Ber07]). The list (with some repetitions) is as follows:

(i) $\mathbb{P}_n$;

(ii) the n-dimensional quadric $Q_n \subset \mathbb{P}_{n+1}$;

(iii) (a cone over) the Veronese surface;

(iv) (a cone over) a rational scroll.

The *cone over a (rational) scroll*, denoted by $\overline{\mathbb{F}(d_1,\dots,d_n)}$, is the image of

$$\mathbb{F}(d_1,\dots,d_n) = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(d_n)), \quad d_1 \geqslant \cdots \geqslant d_n \geqslant 0$$

in $\mathbb{P}_{d_1+\cdots+d_n+n-1}$ under the map associated with the tautological line bundle which will be denoted by $\mathcal{O}(1)$. Note that for $d_n \geqslant 1$, $\overline{\mathbb{F}(d_1,\dots,d_n)}$ and $\mathbb{F}(d_1,\dots,d_n)$ are isomorphic. The pencil on $\mathbb{F}(d_1,\dots,d_n)$ will be denoted by $|F|$.

Any effective divisor D on $\mathbb{F}(d_1,\dots,d_n)$ is in a system $D \in |\mathcal{O}(k) - lF|$, $k \geqslant 0$ and $l \in \mathbb{Z}$. Fiberwise, $D \cap F$ is a hypersurface of degree k in $\mathbb{P}_{n-1}$. If $x_1,\dots,x_n$ denote homogeneous coordinates of $\mathbb{P}_{n-1}$ corresponding to the summands of our vector bundle, then the monomial $x_1^{e_1}\cdots x_n^{e_n}$ with $e_1+\cdots+e_n = k$ has as coefficient a function taken from $H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(e_1d_1+\cdots+e_nd_n - l))$. We will use this in the following form. Consider $\mathbb{F}(m, m-4) \simeq \Sigma_4$. Denote by ξ_4 the minimal section. Any divisor

$$D \in |\mathcal{O}(k) - lF|, \quad k \geqslant 0 \text{ and } l > k(m-4)$$

contains ξ_4 as a component. Indeed, using the above notation, ξ_4 corresponds fiberwise to $x_1 = 0$. It therefore suffices to prove that the coefficient function of x_2^k vanishes. This is a section of $\mathcal{O}_{\mathbb{P}_1}(k(m-4) - l)$, so the claim follows.

3. The general elephant in the case $\mathbf{Bs}|-K_X| \neq \emptyset$

Let X be a canonical Gorenstein Fano threefold with $\operatorname{Bs}|-K_X| \neq \emptyset$. Choose a general elephant $\bar{S} \in |-K_X|$. By the Kawamata–Viehweg vanishing theorem $H^0(X, -K_X) \longrightarrow H^0(\bar{S}, -K_X|_{\bar{S}})$ is surjective, implying $\operatorname{Bs}|-K_X| = \operatorname{Bs}|-K_X|_{\bar{S}}| \neq \emptyset$. Let $\nu\colon S \to \bar{S}$ be a minimal desingularization

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of $\bar{S}$. By Saint-Donat's results on linear systems on smooth K3 surfaces [Sai74, Shi89],

$$\nu^*|-K_X|_{\bar{S}}| = |\Gamma + mf|,$$

where $m \geqslant 2$ and:

(i) $|f|$ is an elliptic pencil; and

(ii) $\Gamma = \mathrm{Bs}\,|\Gamma + mf| \simeq \mathbb{P}_1$ is a section.

Let $\Gamma' \subset S$ be an irreducible curve contracted by ν. Then $(\Gamma + mf)\cdot\Gamma' = 0$, implying $\Gamma \cap \Gamma' = \emptyset$ or $\Gamma = \Gamma'$. In the first case S and $\bar{S}$ are isomorphic near Γ and $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1 \subset \bar{S}_{\mathrm{reg}}$. In the second case, Γ is contracted to a point, $\mathrm{Bs}\,|-K_X| = \{p\}$ and $p \in X_{\mathrm{sing}}$. This is part of the following result of Shin.

Theorem 3.1 [Shi89]. *Let X be a Gorenstein almost Fano threefold with at worst canonical singularities and assume $\mathrm{Bs}\,|-K_X| \neq \emptyset$. With $\bar{S} \in |-K_X|$ a general member we have:*

(i) *if $\dim \mathrm{Bs}\,|-K_X| = 1$, then scheme-theoretically $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$ is contained in X_{reg} and $\mathrm{Bs}\,|-K_X| \cap \mathrm{Sing}(\bar{S}) = \emptyset$;*

(ii) *if $\dim \mathrm{Bs}\,|-K_X| = 0$, then $\mathrm{Bs}\,|-K_X|$ consists of exactly one point and $\bar{S}$ has an ordinary double point at $\mathrm{Bs}\,|-K_X|$; in this case $\mathrm{Bs}\,|-K_X| \subset \mathrm{Sing}(X)$.*

Note that in the case $\mathrm{Bs}\,|-K_X| = \{p\}$ we have $(\Gamma + mf).\Gamma = 0$ on S, implying $m = 2$ and hence $(-K_X)^3 = 2$.

4. The case dim Bs $|-K_X| = 0$

Let X be the complete intersection of a quadric Q in the linear variables and a sextic F_6 in $\mathbb{P}(1^4, 2, 3)$. If we choose F_6 general enough, then (see [Mel99])

$$X \cap \{x_0 = x_1 = x_2 = x_3 = 0\} = [0:0:0:0:-1:1] = p$$

and X does not meet the singular locus of $\mathbb{P}(1^4, 2, 3)$. Then Q and F_6 are Cartier near X and by adjunction, $-K_X \simeq \mathcal{O}_{\mathbb{P}}(1)|_X$ and therefore $\mathrm{Bs}\,|-K_X| = \{p\}$. The rational map defined by $|-K_X|$ sends X to the quadric in $\mathbb{P}_3$ defined by Q.

Proposition 4.1. *If $\dim \mathrm{Bs}\,|-K_X| = 0$, then X is as above a complete intersection in $\mathbb{P}(1^4, 2, 3)$ of a quadric Q, defined in the first four linear variables, and a sextic F_6.*

Proof. (See [Mor82, Mel99, IT01].) We know $(-K_X)^3 = 2$ (see § 3). By the Riemann–Roch theorem we get $h^0(-K_X) = 4$. Let $x_0, \dots, x_3 \in H^0(-K_X)$ be generating sections. We have $h^0(-2K_X) = 10 = \dim S^2H^0(-K_X)$. However, $|-2K_X|$ is base point free, so there exists some

$$y \in H^0(-2K_X), \quad y \notin S^2H^0(-K_X).$$

Then we must have a nontrivial relation Q in $S^2H^0(-K_X)$. The x_i and y then define a 20-dimensional subspace of $H^0(-3K_X)$. By the theorem of Riemann–Roch $h^0(-3K_X) = 21$. Denote the missing function by $z \in H^0(-3K_X)$. Continuing in this way, we see that there must be a nontrivial relation F_6 in $H^0(-6K_X)$. In the end X is the complete intersection of Q and F_6 in $\mathbb{P}(1^4, 2, 3)$. □

Remark 4.2. As Q is singular at p, any $S \in |-K_X|$ is singular at p. If we choose Q and F_6 general, p will be a terminal point of X. If we take for Q the quadric cone, X will have canonical singularities along a curve.

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5. The examples for the case dim Bs $|-K_X| = 1$

Let U be a canonical Gorenstein threefold. Assume that $|-K_U|$ contains a smooth K3 surface S such that $-K_U|_S = 2\Gamma_0 + mf$ for some $m \geqslant 3$. Here $\mathbb{P}_1 \simeq \Gamma_0 \subset U_{\text{reg}}$ and $|f|$ is an elliptic pencil as in § 3. Note that U is a hyperelliptic almost Fano threefold for $m \geqslant 4$.

Let $Y = Bl_{\Gamma_0}(U)$ be the blowup of U in Γ_0. The strict transform of S is a smooth K3 surface in $|-K_Y|$ which we also denote by S. We have $-K_Y|_S = \Gamma_0 + mf$, implying $\text{Bs}\,|-K_Y| = \Gamma_0 \simeq \mathbb{P}_1$. An anticanonical model X of Y is a canonical Gorenstein Fano threefold for which $\text{Bs}\,|-K_X| \simeq \mathbb{P}_1$.

Examples for U as above are constructed as follows. For $m \geqslant 4$, U is almost Fano and the anticanonical map associated with $-K_U$ sends U to a variety of minimal degree $U \longrightarrow W \subset \mathbb{P}_{2m-2}$. Here S is sent to Σ_4, the fourth Hirzebruch surface. The idea is therefore to construct U as a ramified twosheeted covering of some variety of minimal degree, for which a general hyperplane section is isomorphic to Σ_4.

We now come to the examples in Theorem 1.1(ii) in reverse order.

Example 1 (Theorem 1.1(ii)(c)). The projective bundle

$$W = \mathbb{F}(m, m-4, 0), \quad m \geqslant 3$$

is a resolution of a cone over Σ_4. The projection of the underlying bundle onto the first two summands gives a split exact sequence and a smooth surface in $|\mathcal{O}_W(1)|$ isomorphic to Σ_4. For simplicity, we denote it by $\Sigma_4 \in |\mathcal{O}_W(1)|$. There exists a unique section $B \in |\mathcal{O}_W(1) - mF|$ meeting Σ_4 in its minimal section ξ_4. Below we prove that for $m \leqslant 12$ we may choose $D \in |\mathcal{O}_W(4) - (4m-12)F|$, such that the square root of D yields a threefold U_m with at worst canonical singularities. We have

$$\mu\colon U_m \xrightarrow{2:1} \mathbb{F}(m, m-4, 0) \quad \text{and} \quad -K_{U_m} = \mu^*\mathcal{O}_W(1).$$

The section $\xi_4 = B \cap \Sigma_4 \subset D_{\text{reg}}$. Its reduced inverse image in U_m will be denoted by Γ_0. As in Theorem 1.1(ii)(c), we denote by X_{2m-2} an anticanonical model of $Bl_{\Gamma_0}(U_m)$ for $3 \leqslant m \leqslant 12$. We claim that X_{2m-2} are canonical Gorenstein Fano threefolds with base locus $\text{Bs}\,|-K_{X_{2m-2}}| \simeq \mathbb{P}_1$.

In order to prove this it suffices to show that for D general enough each U_m is a canonical Gorenstein threefold as in the beginning of this section. As Σ_4 comes from a splitting sequence, $D \cap \Sigma_4$ is a general member of $|4\xi_4 + 12\mathfrak{f}|$, with $\mathfrak{f} \simeq \mathbb{P}_1$ a fiber of Σ_4. A general member of $|4\xi_4 + 12\mathfrak{f}|$ splits as $\xi_4 + C$ with $C \in |3\xi_4 + 12\mathfrak{f}|$ smooth and disjoint from ξ_4 (cf. § 2). The double covering of Σ_4 yields a smooth K3 surface $S \in |-K_{U_m}| = |\mu^*\mathcal{O}_W(1)|$ with $\mu_S\colon S \longrightarrow \Sigma_4$ ramified along ξ_4 and C. The pullback of $\mathfrak{f}$ gives an elliptic pencil $|f|$ on S with the section Γ_0 lying over ξ_4 and $-K_{U_m}|_S = \mu_S^*\mathcal{O}(1) = 2\Gamma_0 + mf$. It remains to show that U_m has at worst canonical singularities for $3 \leqslant m \leqslant 12$ and $\Gamma_0 \subset U_{m,\text{reg}}$.

For $m = 3$ we can choose D and hence U_m smooth and there is nothing to prove. For $m \geqslant 4$, we always have $D = B + R$ with $R \in |\mathcal{O}_W(3) - (3m-12)F|$. Fiberwise $D \cap F$ consists of a line together with some cubic.

For $4 \leqslant m \leqslant 12$ we can take R to be irreducible, i.e. $D \cap F$ consists of a line and an irreducible cubic. For $m = 4$, the cubic is smooth, meeting the line transversally in three points. For $m \geqslant 5$, the line and the cubic intersect in one point, i.e. in a flex if the cubic is smooth. This gives an A–D–E singularity in the fiber, implying that U_m indeed has at worst canonical singularities for $3 \leqslant m \leqslant 12$. As $R.\xi_4 = 0$ we can choose R disjoint from ξ_4. Hence, $\Gamma_0 \subset U_{m,\text{reg}}$.

For $m \geqslant 13$, on the other hand, $R = R_1 + R_2 + R_3$ with $R_i \in |\mathcal{O}_W(1) - (m-4)F|$, so $D \cap F$ consists of four lines through a point. This means that over F we will not have Du Val singularities, implying that U_m is not canonical for $m \geqslant 13$.

Remark 5.1. The construction also works for $m = 2$. Here $\text{Bs}\,|-K_{X_2}| = \{p\}$ and we get a special case of the threefold X in § 4 with Q the quadric cone (see Remark 4.2).

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Example 2 (Theorem 1.1(ii)(b)). The product of S_1, a del Pezzo surface with canonical singularities of degree 1, and $\mathbb{P}_1$ is a classical example [Isk80]. Choose eight points on $\mathbb{P}_2$ general enough, such that the blowup $\hat{\mathbb{P}}_2$ of $\mathbb{P}_2$ in these points still has a nef anticanonical system, and denote by S_1 an anticanonical model of $\hat{\mathbb{P}}_2$. Then $|-K_{S_1}|$ is one dimensional by the Riemann–Roch theorem, its members corresponding to elliptic curves passing through the eight points. These curves will meet in a ninth point, implying $\mathrm{Bs}\,|-K_{S_1}| = \{p\}$. Then the product $X = S_1 \times \mathbb{P}_1$ is a canonical Gorenstein Fano threefold with $\mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1$.

Example 3 (Theorem 1.1(ii)(a)). The blowup X in the intersection of two members of $|-\frac{1}{2}K_U|$ of the double cover U of the Veronese cone W, ramified along a cubic, is a classical example [Isk80]. We give some details to show the connection to the above description.

The blowup of the Veronese cone in its vertex O yields $\mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(2)) \longrightarrow W$. The strict transform of a special hyperplane section through O gives a $\mathbb{P}_1$-bundle over a conic. It either decomposes into two copies of Σ_2 or gives one irreducible surface Σ_4.

The image of Σ_4 in W gives $\widehat{C}_4$, the cone over the rational normal curve of degree 4. In U, lying over $\widehat{C}_4$ we find a singular K3 surface $\bar{S} \in |-K_U|$ with a double point over O. In the reducible case, the two copies of Σ_2 induce $H_i \in |-\frac{1}{2}K_U|$ for $i = 1, 2$, and their intersection with $\bar{S}$ is the singular point.

In the blowup X of U along $H_1 \cap H_2$ the singularity of $\bar{S}$ is resolved, i.e. we get a smooth K3 surface $S \in |-K_X|$. The same formulas as above show $-K_X|_S = \Gamma + 2f$ with Γ the -2-curve over the singularity and $|f|$ the induced elliptic pencil. If we choose H_1, H_2 general enough, then X will be a canonical Gorenstein Fano threefold with $\mathrm{Bs}\,|-K_X| \simeq \Gamma \simeq \mathbb{P}_1$.

6. The general setting in the case dim Bs $|-K_X| = 1$

We now look at the general setting in the case $\dim \mathrm{Bs}\,|-K_X| = 1$ (cf. [Isk80, IP99]). By Shin's Theorem, $\Gamma = \mathrm{Bs}\,|-K_X| \simeq \mathbb{P}_1 \subset X_{\mathrm{reg}}$. We can write

$$N_{\Gamma/X} = \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b), \quad a \geqslant b, \tag{6.0.1}$$

for some $a, b \in \mathbb{Z}$. A general elephant $\bar{S} \in |-K_X|$ may have double points, but $\Gamma \subset \bar{S}_{\mathrm{reg}}$. If $\nu\colon S \to \bar{S}$ denotes a resolution of the singular locus, then $\nu^*(-K_X) = \Gamma + mf$, $m \geqslant 3$, with $|f|$ an elliptic pencil and Γ a section (§ 3). The numbers are related as follows:

$$-K_X \cdot \Gamma = m - 2 = a + b + 2.$$

Let $\sigma\colon X_\Gamma \to X$ be the blowup of X along Γ with exceptional divisor $E_\Gamma = \mathbb{P}(N^*_{\Gamma/X}) = \Sigma_{a-b}$. Then $|-K_{X_\Gamma}| = |\sigma^*(-K_X) - E_\Gamma|$ is free, defining a map onto some surface W (see [Rei83]).

$$\begin{array}{ccc} X_\Gamma & \xrightarrow{\ \varphi\ } & W \subset \mathbb{P}_{m+1} \\ {\scriptstyle\sigma}\big\downarrow & \nearrow & \\ X & & \end{array} \tag{6.0.2}$$

The surface W is of minimal degree, i.e. $m = \deg(W) = \mathrm{codim}(W) + 1$. Again by del Pezzo's theorem, in our situation W is one of the following:

(i) $\widehat{C}_m$, the cone over a rational normal curve of degree $m = a + b + 4 \geqslant 2$; or

(ii) Σ_{a-b}, $a \geqslant b$.

The map $E_\Gamma \to W$ is either an isomorphism or the contraction of the minimal section. The map $X_\Gamma \to W$ is (generically) an elliptic fibration, and as $-K_X$ is ample, any fiber over a point in W_{reg} is an irreducible, generically reduced curve of arithmetic genus one. We distinguish two cases.

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The case W a smooth ruled surface

Here we denote by F_Γ the pullback to X_Γ of a fiber of W, and by $Z_{\Gamma,X}$ the pullback of the minimal section (or the second ruling in the case $W = \mathbb{P}_1 \times \mathbb{P}_1$). Note that $|F_\Gamma|$ descends to a pencil $|F|$ on X. Adjunction on E_Γ shows $-K_{X_\Gamma} = Z_{\Gamma,X} + (a+2)F_\Gamma$. As $\Gamma \subset X_{\mathrm{reg}}$ and $Z_{\Gamma,X}$ meets E_Γ transversally near the minimal section ξ_{a-b} of E_Γ, $Z_{\Gamma,X}$ is smooth near $Z_{\Gamma,X} \cap E_\Gamma$, and $\sigma(Z_{\Gamma,X}) \simeq Z_{\Gamma,X}$ is smooth near Γ.

The case W a cone

Here we denote by F_Γ the strict transform in X_Γ of a line in W through the vertex O. Note that this is just a Weil divisor. Let

$$h'\colon X'_\Gamma \longrightarrow X_\Gamma \tag{6.0.3}$$

be a $\mathbb{Q}$-factorialization of X_Γ with respect to F_Γ (see [Kaw88]). The map h' is small, X'_Γ is again Gorenstein with at worst canonical singularities, and the strict transform F'_Γ of F_Γ is $\mathbb{Q}$-Cartier. We can choose X'_Γ such that F'_Γ is h'-ample [Kaw88]. As $\Gamma \subset X_{\mathrm{reg}}$, both X'_Γ and X_Γ are isomorphic near E_Γ. We denote the pullback of E_Γ to X'_Γ by E'_Γ. We claim (cf. [Che99]) the following.

Lemma 6.1. *On X'_Γ, two general members of $|F'_\Gamma|$ do not intersect.*

Proof. Assume $F'_{\Gamma,1} \cap F'_{\Gamma,2} \neq \emptyset$. The intersection is clearly in the fiber over the vertex O of W. Choose an irreducible curve $C \subset F'_{\Gamma,1} \cap F'_{\Gamma,2}$. On the one hand, the restriction of some multiple of $F'_{\Gamma,2}$, which is Cartier, gives an effective Cartier divisor on $F'_{\Gamma,1}$ supported in the fiber over O, implying $F'_{\Gamma,2} \cdot C \leqslant 0$. On the other hand, as $F'_{\Gamma,1}$ and $F'_{\Gamma,2}$ do not meet on E'_Γ, we have $C \cap E'_\Gamma = \emptyset$. As $-K_{X'_\Gamma} \cdot C = 0$ and $E'_\Gamma \cdot C = 0$ imply $h'^*\sigma^*(-K_X) \cdot C = 0$, the curve C must be h'-exceptional. Then, by our choice of X'_Γ, we have $F'_{\Gamma,2} \cdot C > 0$. Hence, $F'_{\Gamma,1} \cap F'_{\Gamma,2} = \emptyset$. □

Denote by Y_Γ a terminal modification of X'_Γ. The pullback of F'_Γ to Y_Γ defines a pencil on Y_Γ, showing that the map to W factors over the blowup Σ_{a-b} of W in O. Near E'_Γ, Y_Γ and X'_Γ are isomorphic, and we can blow the divisor down to obtain Y, a terminal modification $h\colon Y \to X$ of X. We call the map $Y_\Gamma \to Y$ again σ and end up with the following diagram.

$$\begin{array}{ccccc} & & & & \Sigma_{a-b} \\ & & \overset{\psi}{\nearrow} & & \downarrow \\ Y_\Gamma & \longrightarrow & X_\Gamma & \longrightarrow & W \\ \downarrow{\scriptstyle\sigma} & & \downarrow & & \\ Y & \xrightarrow{h} & X & & \end{array} \tag{6.1.1}$$

Below, we study Y instead of X and think of X as an anticanonical model. Note that we have chosen Y as a terminal modification of a particular $\mathbb{Q}$-factorialization of X.

For simplicity, denote divisors on Y_Γ and X_Γ by the same letters: the exceptional divisor of $Y_\Gamma \to Y$ is again E_Γ, the curve $\mathrm{Bs}\,|-K_Y| = \Gamma$. The pullback of a general fiber of Σ_{a-b} to Y_Γ is F_Γ. By $Z_\Gamma + B_\Gamma$ we denote the pullback of the minimal section of Σ_{a-b} to Y_Γ, where Z_Γ denotes here the unique irreducible component that meets E_Γ in its minimal section, and B_Γ consists of the remaining components, disjoint from E_Γ. As above we get

$$-K_{Y_\Gamma} = Z_\Gamma + B_\Gamma + (a+2)F_\Gamma. \tag{6.1.2}$$

The pencil $|F_\Gamma|$ again descends to the pencil $|F|$ on Y. The surface Z_Γ is smooth near $E_\Gamma \cap Z_\Gamma$; we will denote the isomorphic images of Z_Γ and B_Γ in Y by Z and B.

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Remark 6.2. The general member of the pencil $|F_\Gamma|$ is a smooth surface with a relatively minimal elliptic pencil. The intersection $F_\Gamma \cap (Z_\Gamma + B_\Gamma)$ is hence either smooth or one of Kodaira's exceptional fibers.

7. The case W a cone

Proposition 7.1. *If W is a cone, then $3 \leqslant m \leqslant 12$ and $X = X_{2m-2}$ is one of the threefolds constructed in Example 1. Here $W = \widehat{C}_m$.*

Proof. We use the notation from the last section. As $-K_{X_\Gamma}$ is not ample on E_Γ, $b = -2$ and $a \geqslant 1$ in (6.0.1). We can hence use $a + b = m - 4$ to eliminate a and b and write everything in terms of m:

$$N_{\Gamma/X} = \mathcal{O}_{\mathbb{P}_1}(m-2) \oplus \mathcal{O}_{\mathbb{P}_1}(-2), \quad m \geqslant 3,$$

and $W = \widehat{C}_m$. In diagram (6.1.1), the map from Y_Γ to $\widehat{C}_m$ now factors over Σ_m.

We first assume that Z is h-nef and show that in this case Y is obtained by blowing up some Gorenstein threefold V along some smooth curve $\Gamma_0 \simeq \mathbb{P}_1 \subset V_{\mathrm{reg}}$, such that Z is the exceptional divisor. We compute $Z \cdot \Gamma = -2$ and $-K_Y \cdot \Gamma = m - 2 > 0$. Hence $[\Gamma]$ is contained in the K_Y-negative part of $\overline{NE}(Y)$. This part is polyhedral, spanned by K_Y-negative extremal rays. The divisor Z is negative on $[\Gamma]$ and nonnegative on any K_Y-trivial curve by assumption. We conclude that Z must be negative on at least one extremal ray. Let

$$\phi\colon Y \longrightarrow V \tag{7.1.1}$$

be the contraction of this ray. By [Ben85], the contraction is divisorial, contracting Z either to a curve or to a point. We claim the following.

Lemma 7.2. *The map $\phi\colon Y \to V$ in (7.1.1) is the blowup of a smooth rational curve $\Gamma_0 \subset V_{\mathrm{reg}}$ with normal bundle $N_{\Gamma_0/V} = \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4)$. The contraction is in direction of $|F|$. There exists a smooth K3 surface $S \in |-K_V|$, such that $-K_V|_S = 2\Gamma_0 + mf$ with $|f|$ an elliptic pencil induced by $|F|$, and $\Gamma_0 \simeq \mathbb{P}_1$ a smooth section.*

Remark 7.3. The threefold V is a hyperelliptic Gorenstein almost Fano threefold of degree $(-K_V)^3 = 4m - 8$ for $m \geqslant 4$. For $m = 3$, the anticanonical system is nef on any curve $\neq \Gamma_0$, while $-K_V \cdot \Gamma_0 = m - 4 = -1$. For the case $m = 3$ (as well as $m = 2$), see also [DPS93].

Proof of Lemma 7.2 and Remark 7.3. As Z_Γ meets E_Γ transversally in the minimal section, we have $\Gamma \subset Z_{\mathrm{reg}}$. We compute

$$\deg N_{\Gamma/Z} = Z_\Gamma \cdot_{Y_\Gamma} E_\Gamma^2 = m - 2 > 0. \tag{7.3.1}$$

Let us first show $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$. If $Z \simeq \mathbb{P}_1 \times \mathbb{P}_1$, then $B \neq 0$, implying that B meets Z in some curve. By (7.3.1) Γ is ample on Z. Then $\Gamma \cap B \neq \emptyset$, which is impossible as B maps to X_{sing}, while $\Gamma \subset X_{\mathrm{reg}}$.

If Z is mapped to a point, then by [Cut88], $Z \simeq \mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ or the quadric cone. As Z comes with a pencil and $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$, all of these cases are impossible. By [Cut88], $Y = Bl_{\Gamma_0}(V)$ the blowup of V in some curve $\Gamma_0 \subset V_{\mathrm{reg}}$, which is locally a complete intersection. From $\deg N_{\Gamma/Z} = m - 2 > 0$ we conclude that Γ maps surjectively onto Γ_0, and from $\Gamma \subset Z_{\mathrm{reg}}$ we infer that Γ_0 must be smooth. Then $Z = \mathbb{P}(N^*_{\Gamma_0/V}) \simeq \Sigma_e$ for some $e > 0$, where $e > 0$ follows from $Z \not\simeq \mathbb{P}_1 \times \mathbb{P}_1$. It is now clear that ϕ is in the direction of $|F|$, i.e. fiberwise ϕ contracts a -1-curve in F. Denote the induced pencil on V by $|F_V|$. Note that $Z \simeq \Sigma_e$ implies $B \neq 0$.

(1) Any curve in $Z_\Gamma \cap B_\Gamma$ is contracted by $Y_\Gamma \to X_\Gamma$, and therefore B intersects Z set theoretically in the minimal section ξ_e of $Z = \Sigma_e$. As Γ does not meet ξ_e, we conclude $\Gamma = \xi_e + (m-2)\mathfrak{f}_e$, where $\mathfrak{f}_e$

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is a fiber of Σ_e. From $\Gamma \cdot_Z \Gamma = m-2$ (see (7.3.1)) we infer $e = m-2$. Moreover, $-K_Y \cdot \xi_e = 0$ implies $N_{\Gamma_0/V} = \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(m-4)$. By the adjunction formula, $-K_V \cdot \Gamma_0 = m-4$, hence $(-K_V)^3 = 4m-8$.

(2) Let $S \in |-K_Y|$ be general. As S meets Z transversally in Γ, its image in V is a special member of $|-K_V|$. Identifying S with its image in V we find $-K_V|_S = 2\Gamma_0 + mf$, where $|f|$ is an elliptic pencil and Γ_0 is a section (see § 5). If $C \subset V$ is an irreducible curve such that $-K_V \cdot C < 0$, then $S \cdot C < 0$ and $C \subset S$. Then $-K_V \cdot C = (2\Gamma_0 + mf) \cdot C < 0$ so that $\Gamma_0 \cdot C < 0$ and hence $C = \Gamma_0$, $m = 3$. □

The argument before Lemma 7.2 showing the contractibility of Z in Y requires Z being h-nef. In order to achieve this we might have to change the terminal modification by running the relative $(K_Y + \epsilon Z)$-program, $\epsilon \in \mathbb{Q}^+$, $\epsilon \ll 1$, with respect to $h\colon Y \to X$.

The contraction of any $(K_Y + \epsilon Z)$-negative extremal ray in $\overline{NE}(Y/X)$ is small; the curves contracted are K_Y-trivial and contained in Z. After finitely many flops, we end up with the following diagram [KM98, Theorem 6.14 and Corollary 6.19].

$$\begin{array}{ccc} Y & \overset{\chi}{\dashrightarrow} & Y^+ \\ & {}_{h}\searrow \quad \swarrow_{h^+} & \\ & X & \end{array} \tag{7.3.2}$$

Here Y^+ is again a terminal Gorenstein threefold with $-K_{Y^+}$ big and nef, having X as an anticanonical model. The map χ is rational and an isomorphism in codimension one. We superscribe any strict transform under χ with a '+' sign. As $K_{Y^+} + \epsilon Z^+$ is h^+-nef, Z^+ is h^+-nef. As above we conclude that Z^+ is contractible in Y^+.

Lemma 7.2 holds for Y^+ instead of Y as long as $|F^+|$ is still spanned on Y^+. This need not be the case. Recall that we have chosen Y as a terminal modification of some $\mathbb{Q}$-factorialization X' of X; in the above program we might flop some horizontal curves in Z, thereby producing a base locus.

LEMMA 7.4. *The system $|F^+|$ is spanned unless $m = 3$ and $(Z^+, \mathcal{O}_{Z^+}(Z^+)) = (\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(-2))$.*

Here $|F^+|$ restricted to Z^+ corresponds to lines through a given point.

Proof. Assume that $|F^+|$ is not spanned. Let $\phi^+\colon Y^+ \longrightarrow V^+$ be the divisorial contraction as in (7.1.1), contracting Z^+. In order to decide what Z^+ is, we again use the classification from [Cut88]. If Z^+ maps to a curve and $\mathfrak{f}$ denotes the general fiber, then $Z^+ \cdot \mathfrak{f} = -1$ and $-K_{Y^+} \cdot \mathfrak{f} = 1$. On Y^+ we have

$$-K_{Y^+} = Z^+ + B^+ + mF^+. \tag{7.4.1}$$

As $\operatorname{Bs}|F^+| \cap Z^+ \neq \emptyset$ we must have $F^+ \cdot \mathfrak{f} > 0$. From $B^+ \cdot \mathfrak{f} \geqslant 0$ we conclude that $0 < m\mathfrak{f} \cdot F^+ \leqslant 2$, which is impossible as $m > 2$.

If Z^+ goes to a point, then $(Z^+, \mathcal{O}_{Z^+}(Z^+))$ is either $(\mathbb{P}_2, \mathcal{O}(-1))$ or $(\mathbb{P}_2, \mathcal{O}(-2))$ or $(Q_2 \subset \mathbb{P}_3, \mathcal{O}(-1))$. Near Γ the two surfaces Z and Z^+ are isomorphic. With the original pencil on Z we conclude that Z^+ contains a smooth rational curve that meets another irreducible curve in a single point. From $Z^+ \cdot \Gamma = -2$ we infer $(Z^+, \mathcal{O}_{Z^+}(Z^+)) = (\mathbb{P}_2, \mathcal{O}(-2))$. Then $|F^+|$ restricted to Z^+ is a family of lines. Using $-K_{Y^+} \cdot \Gamma = m-2$ and the adjunction formula, we find $m = 3$. The proof of the lemma is complete. □

Lemma 7.2 also holds in the exceptional case $(Z^+, \mathcal{O}_{Z^+}(Z^+)) = (\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(-2))$ for some terminal modification of X', we only cannot argue as above. Instead, we proceed as follows.

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We first run the relative $(K_Y + \epsilon Z)$-program with respect to $Y \to X'$, where X' is the above $\mathbb{Q}$-factorialization of X. In the end we may assume that Z is at least nef on every K_Y-trivial curve contained in a fiber of the pencil $Z \to \mathbb{P}_1$. Omitting some details, we conclude that a single flop of a K_Y-trivial section of Z transforms Y into Y^+ in (7.3.2) and Z into $Z^+ = \mathbb{P}_2$ as above. Then $Z \simeq \Sigma_1$ and $Z \cdot \mathfrak{f} = -1$ for the general fiber $\mathfrak{f} \simeq \mathbb{P}_1$. We conclude that Z must be negative on at least one extremal ray in $\overline{NE}(Y/\mathbb{P}_1)$ and conclude Lemma 7.2 as above.

For the proof of Proposition 7.1 it remains to show that V in Lemma 7.2 is a terminal modification of U_m in § 5. In order to prove this, we consider the system $|-K_V + \lambda F_V|$, $\lambda \geqslant 0$, and choose λ such that $m+\lambda \geqslant 4$. Restricted to S we get $2\Gamma_0 + (m+\lambda)f$, which is now big and nef. Then $-K_V + \lambda F_V$ is big and nef and by the Kawamata–Viehweg vanishing theorem $H^1(\mathcal{O}_V(\lambda F_V)) = H^1(\mathcal{O}_V(K_V + (-K_V + \lambda F_V))) = 0$ implying surjectivity of

$$H^0(V, \mathcal{O}_V(-K_V + \lambda F_V)) \longrightarrow H^0(S, \mathcal{O}_S(2\Gamma_0 + (m+\lambda)f)).$$

Then, as $|F_V|$ is free and $|2\Gamma_0 + (m+\lambda)f|$ is free, $|-K_V + \lambda F_V|$ is free. For $\lambda \geqslant 1$ and $m + \lambda \geqslant 5$, any irreducible curve having zero intersection with $-K_V + \lambda F_V$ must lie in a member of $|F_V|$. This follows immediately from $-K_V + \lambda F_V = (-K_V + (\lambda-1)F_V) + F_V =$ nef + nef. The system is free, for example, if we choose $\lambda = 1$, for $m \geqslant 4$, and $\lambda = 2$, for $m = 3$.

Fix this choice from now on. The map associated with $|-K_V + \lambda F_V|$ is generically two-to-one sending V to a variety of minimal degree $\nu\colon V \longrightarrow W \subset \mathbb{P}_{2m+3\lambda-2}$. As W comes with a pencil $|F_W|$, it must be a scroll. We may rescale the entries such that $-K_V \simeq \nu^*\mathcal{O}_W(1)$. Then $W \simeq \mathbb{F}(d_1, d_2, d_3)$, $d_1 \geqslant d_2 \geqslant d_3 \geqslant -1$, where $d_3 = -1$ in the case $m = 3$, while $d_3 \geqslant 0$ for $m \geqslant 4$. Stein factorization of $V \to W$ leads to a canonical Gorenstein threefold U and a double cover $\mu\colon U \longrightarrow W \simeq \mathbb{F}(d_1, d_2, d_3)$, such that $-K_U = \mu^*\mathcal{O}_W(1)$. Hence, μ is ramified along a reduced divisor $D \in |\mathcal{O}_W(4) - 2(d_1 + d_2 + d_3 - 2)F_W|$. From $(\mathcal{O}_W(1))^3 = \frac{1}{2}(-K_V)^3 = 2m - 4$ we infer $d_1 + d_2 + d_3 = 2m - 4$. The only section of $H^0(V, -K_V - mF_V)$ is that corresponding to the image of B in V (cf. (6.1.2)). As μ is fiberwise ramified along a quartic, we also have $h^0(W, \mathcal{O}_W(1) - mF_W) = 1$, implying $d_1 = m$, $d_2 < m$. In the special case $m = 3$ we have $d_3 = -1$ and $W \simeq \mathbb{F}(3, 0, -1)$. It remains to consider the case $m \geqslant 4$.

Denote the image of B in W by B_W. If $d_3 > 0$, then $2B_W$ is a component of D. However D is reduced, hence we must have $d_3 = 0$. Then $d_1 = m$, $d_2 = m-4$, i.e. $V \longrightarrow U \longrightarrow W \simeq \mathbb{F}(m, m-4, 0)$. We have seen in § 5, that $U = U_m$ can never have canonical singularities for $m \geqslant 13$, hence $m \leqslant 12$.

Back on the surface $S \in |-K_V|$ in Lemma 7.2, we see that S is generically a double cover of some member $H \in |\mathcal{O}_W(1)|$. The map ν sends S to $\mathbb{F}(m, m-4)$ and Γ_0 lies over the minimal section, which is the restriction of the above divisor B_W. In particular, Γ_0 is not contracted by $V \to U_m$ and does not meet any curve contracted, i.e. $\Gamma_0 \subset U_{m,\mathrm{reg}}$ and V is isomorphic to U_m near Γ_0. This completes the proof of Proposition 7.1. ☐

8. The case W a ruled surface

This case is as in [Isk80]. Instead of Y and Y_Γ we focus on X and X_Γ, and diagram (6.0.2). We use the notation introduced in § 6.

Proposition 8.1. *In the case $W \simeq \Sigma_{a-b}$, $a > b$, X is the blowup of a sextic in $\mathbb{P}(1^3, 2, 3)$ along an irreducible curve of arithmetic genus one (and $a = 0$, $b = -1$, $m = 3$).*

Proof. As $-K_{X_\Gamma}$ is ample on E_Γ, we have $b \geqslant -1$ and $a \geqslant 0$. Hence,

$$Z_{\Gamma,X} \cdot \xi_{a-b} = b - a < 0 \quad \text{and} \quad -K_{X_\Gamma} \cdot \xi_{a-b} = b + 2 > 0,$$

where $\xi_{a-b} = E_\Gamma \cap Z_{\Gamma,X}$ is the minimal section of E_Γ. As $Z_{\Gamma,X}$ is trivial on any K_{X_Γ}-trivial curve, we conclude that $Z_{\Gamma,X}$ must be negative on at least one extremal ray in $\{K_{X_\Gamma} < 0\}$.

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Denote by $\phi_X \colon X_\Gamma \longrightarrow V_X$ the contraction of this ray. It is a birational map with exceptional set $Z_{\Gamma,X}$ by [Ben85]. As $Z_{\Gamma,X}$ contains K_{X_Γ}-trivial curves, it is contracted to a curve.

If $Z_{\Gamma,X}$ is singular along a curve, then its normalization is a smooth ruled surface. The second map implies that it is $\mathbb{P}_1 \times \mathbb{P}_1$. As $\xi_{a-b} \subset Z_{\Gamma,X,\mathrm{reg}}$ does not meet the singular locus, we must have $\deg N_{\xi_{a-b}/Z_{\Gamma,X}} = a = 0$, implying $b = -1$. If $Z_{\Gamma,X}$ is smooth in codimension one, then $h^1(Z_{\Gamma,X}, \mathcal{O}_{Z_{\Gamma,X}}) \leqslant 1$ by [Rei83] and Iskovskikh's original argument applies: using the ideal sequence of $Z_{\Gamma,X}$ and the identity $-K_{X_\Gamma} = Z_{\Gamma,X} + (a+2)F_\Gamma$ (cf. § 6), we see

$$h^1(Z_{\Gamma,X}, \mathcal{O}_{Z_{\Gamma,X}}) = h^2(X_\Gamma, \mathcal{O}_{X_\Gamma}(-Z_{\Gamma,X})) = h^1(X_\Gamma, \mathcal{O}_{X_\Gamma}(-(a+2)F_\Gamma)).$$

Then the ideal sequence of $(a+2)$ general members of $|F_\Gamma|$

$$0 \longrightarrow \mathcal{O}_{X_\Gamma}(-(a+2)F_\Gamma) \longrightarrow \mathcal{O}_{X_\Gamma} \longrightarrow \mathcal{O}_{(a+2)F_\Gamma} \longrightarrow 0$$

yields $h^0(\mathcal{O}_{(a+2)F_\Gamma}) - 1 \leqslant 1$, hence $a \leqslant 0$.

As $E_\Gamma \cdot \xi_{a-b} = a = 0$, the image Z_X of $Z_{\Gamma,X}$ is still contractible. We can even explicitly give the supporting divisor: denote the image of F_Γ in X by F. They are Cartier, as $\Gamma \subset X_{\mathrm{reg}}$. The supporting divisor is $H = Z_X + F \in \mathrm{Pic}(X)$, which is big and nef. Indeed, $\sigma^* H = Z_{\Gamma,X} + F_\Gamma + E_\Gamma$. As $Z_{\Gamma,X} + F_\Gamma = \varphi^*(\xi_1 + \mathfrak{f})$ is nef, and $\sigma^* H$ restricted to E_Γ is trivial, H is nef. A direct computation shows $H^3 = 1$. By the base point free theorem, $|kH|$ is free for $k \gg 0$, defining a birational contraction $\phi\colon X \longrightarrow V$, contracting Z_X to a curve. The base locus $\Gamma \subset Z_X$ is contracted to a point, the general fiber of the elliptic pencil on Z_X is a section. The variety V is again a Gorenstein Fano threefold with canonical singularities and $K_X = \phi^* K_V + Z_X$. From $\phi^* K_V = K_X - Z_X = -2H$ we conclude that $-K_V$ is divisible by 2 in $\mathrm{Pic}(V)$. From $H^0(X, kH) = 1 + \frac{k}{6}(8 + 3k + k^2)$ we see that V is a sextic in $\mathbb{P}(1^3, 2, 3)$. □

Proposition 8.2. *If $W \simeq \mathbb{P}_1 \times \mathbb{P}_1$, then $X \simeq \mathbb{P}_1 \times S_1$, where S_1 denotes a normal del Pezzo surface of degree* 1 *(and $a = b = 0$, $m = 4$).*

Proof. In this case, $Z_{\Gamma,X}$ is the pullback of one ruling of $W = \mathbb{P}_1 \times \mathbb{P}_1$. The general fiber of $Z_{\Gamma,X}$ is a smooth elliptic curve, and $Z_{\Gamma,X}$ meets the singular locus of X_Γ at most in points. Going from X_Γ to Y_Γ, we see $a \leqslant 0$. As $E_\Gamma \simeq W$, we have $a = b$, and X Fano implies that $a = b = 0$. As φ followed by the natural projection $W \to \mathbb{P}_1$ contracts all the fibers of $\sigma\colon X_\Gamma \to X$ to points, we obtain an induced map $X \longrightarrow \mathbb{P}_1$ with general fiber $F = \sigma(F_\Gamma)$ and section Γ, where F is a normal del Pezzo surface of degree one. We have $-K_{X_\Gamma} = Z_{\Gamma,X} + 2F_\Gamma$. As above, $-K_X = Z_X + 2F$, and we see that Z_X is nef, so $|kZ_X|$ is free for $k \gg 0$. The map defined by $|kZ_X|$ is a $\mathbb{P}_1$-bundle with section F and fiber Γ. As in [Isk80] we conclude that $X \simeq F \times \mathbb{P}_1$ is a product. □

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