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NONLINEAR EVOLUTION OF SINGULAR DISTURBANCES TO A $\tanh^3 y$ MIXING LAYER

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Abstract

We consider the nonlinear evolution of a disturbance to a mixing layer, with the base profile given by $u_0(y) = \tanh^3 y$ rather than the more usual $\tanh y$, so that the first two derivatives of u_0 vanish at $y = 0$. This flow admits three neutral modes, each of which is singular at the critical layer. Using a non-equilibrium nonlinear critical layer analysis, equations governing the evolution of the disturbance are derived and discussed. We find that the disturbance cannot exist on a linear basis, but that nonlinear effects inside the critical layer do permit the disturbance to exist. We also present results of a direct numerical simulation of this flow and briefly discuss the connection between the theory and the simulation.

1. Introduction

One of the most fundamental flows in science and engineering is the mixing layer. An example of this is the flow downstream of a splitter plate. In this situation, two streams of fluid with differing velocities merge and downstream this flow will diffuse to a smooth profile. One of the more common models used to study the stability of this smooth profile is the hyperbolic tangent mixing layer, $u_0 = \tanh y$. A necessary condition for instability of such a flow is Rayleigh's inflection point criterion [8], as a result of which mixing layer instability is sometimes called inflection point instability. This condition says that in order to be unstable, the base velocity profile must have an inflection point. A stronger form of this is Fjortoft's theorem [8], which says that a necessary condition for instability is that $u_0''(u_0 - u_s) < 0$ somewhere in the flow, where u_s is that value of u_0 at the inflection point. The sinusoidal base flow $u_0 = \sin y$ is a well-known counter-example that shows that neither of these two conditions is sufficient for stability [8]. The hyperbolic tangent profile satisfies the inflection point criterion at $y = 0$, where $u_0'' = 0$, and also the conditions for Fjortoft's theorem, and

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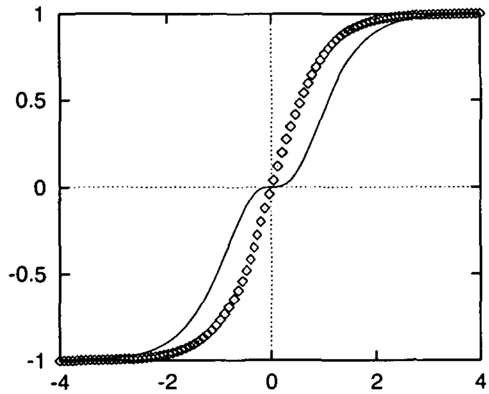

FIGURE 1. The base flow $u_0 = \tanh^3 y$ (solid) together with $\tanh y$ (diamonds).

has also been shown to be absolutely unstable [16]. The stability of mixing layers for two-dimensional disturbances is now reasonably well understood, thanks to numerous experimental and numerical studies of these flows. More recently, nonlinear stability theory has been able to provide a reasonably adequate theoretical description of the nonlinear vortex roll-up and subsequent equilibration and nutation which occurs in two-dimensional transition. Goldstein and Leib [12] considered the case of a homogeneous incompressible mixing layer, and used a non-equilibrium, nonlinear critical layer analysis to derive an equation for the amplitude of the disturbance. This evolution equation is given in Section 3, and consists of a nonlinear PDE together with an integral jump condition. This work was later extended [10] to include viscosity, and to a more general flow including comparisons with experimental data [17]. Other authors later added additional effects, such as hypersonic flow [13] and stratified flow [19], or studied different base flows, such as the Bickley jet [18] for which there were not one but two neutral modes. In the present study, rather than the traditional hyperbolic tangent, we take as our base flow

$$u_0(y) = \tanh^3 y.$$

In Figure 1, we plot this profile, together with the hyperbolic tangent. The principal difference visible in this figure is that the $\tanh^3 y$ profile has a pronounced kink near the origin. This is because for this profile, both the first and second derivatives vanish at $y = 0$. As we shall see when we come to our analysis, this changes the flow in a number of ways. For example, there are three neutral modes rather than just one, and these neutral modes are singular at the critical layer.

Part of the original motivation for studying this problem arose out of work on helical transition in axisymmetric jets [20]. If one considers the velocity profile (in cylindrical coordinates (r, θ, z))

$$\underline{u_0} = (0, 0, u_0(r)),$$

Rayleigh found a necessary but not sufficient condition for instability of a profile of this form to a disturbance proportional to $e^{i\alpha(z-ct)+im\theta}$ (see for example [8]). Rayleigh found that a particular mode can be unstable only if

$$u_0' = \frac{r u_0''(\alpha^2 r^2 + m^2)}{\alpha^2 r^2 - m^2} \tag{1.1}$$

at some value of r. This is somewhat analogous to Rayleigh's inflection point criterion mentioned above. For plane parallel flows, recent theoretical studies (see for example [11, 22, 29]) have indicated that resonant triad interactions can play an important role in three-dimensional transition. For a plane parallel flow, a resonant triad consists of a plane wave together with a pair of oblique waves inclined at $\pm 60°$ to the mean flow, with the oblique waves having a streamwise wavenumber one half that of the plane wave. The counterpart for an axisymmetric flow would be an axisymmetric wave together with a pair of helical waves with azimuthal wavenumber ± 1 and again a streamwise wavenumber one half that of the axisymmetric mode. It is still a little unclear from the experimental literature whether a resonant triad can exist for an axisymmetric flow. Some experimental studies have indicated that such interaction does not take place, and further that only one or the other but not both of the two modes (the axisymmetric mode and the helical mode) are present at once. On the other hand, some studies have claimed that a resonant triad interaction can indeed occur. For example, Corke *et al.* [6] reported that the $m = 0$ and $m = \pm 1$ modes did not exist at the same time or space, and postulated that (in their words) each mode was a basin of attraction which suppresses the existence of the other. Corke *et al.* found that this lack of coexistence was accompanied by an apparently non-deterministic switching between the modes, and that the dominant mode at any instant was likely to be the one that had the highest initial amplitude forcing. However, Corke *et al.* considered their results to be something of a quandary, in that they found that the two modes seemingly could not coexist but they also reported the presence of a mode representing the difference between the modes which they could not fully explain: it seemed to pose something of a paradox that two modes could not coexist but yet could interact to produce the difference mode. By contrast to this, in later work, Corke and Kusek [5] did indeed find a resonant interaction between these modes, so as mentioned above, the issue is a little cloudy. Our interest in this matter arises from the criterion we get by applying the stability condition above (1.1) to the axisymmetric and helical modes, since the two modes yield different results. Applying Rayleigh's criterion to the axisymmetric wave tells us that we require

$$u_0' = r u_0'',$$

while applying it to the helical wave yields

$$u_0' = \frac{r u_0''(\alpha^2 r^2 + 1)}{\alpha^2 r^2 - 1}.$$

In order for both of these results to hold at the same location, we require

$$u_0' = u_0' = 0$$

at some point in the flow. It is this last condition, which seems to impose a rather severe restriction on the base flow, that suggests itself as interesting and it is in order to gain insight into this sort of flow that we originally chose to study the $\tanh^3 y$ mixing layer.

We now turn to nonlinear critical layer theory. A few words on the origins of this field are in order. As we shall see in Section 2, the outer expansion, representing the flow in the main body of the fluid, becomes singular at the critical layer y_c where the phase velocity c of the disturbance is equal to the fluid velocity of the flow u_0. These singularities occur due to the expansion becoming disordered there, and so it is necessary to pose an inner expansion in the vicinity of the critical layer and match the inner and outer expansions together. Traditionally, viscous effects were reintroduced inside the critical layer to remedy the singularities, with this approach being known as a viscous critical layer. Benney and Bergeron [1] and Davis [7] were the first to employ a nonlinear critical layer, wherein they retained nonlinear effects rather than viscosity inside the critical layer to deal with the singularities.

While the early studies only considered steady state solutions, later authors employed a nonlinear critical layer to study how a disturbance would develop, with notable contributions including (amongst many others) Warn and Warn [28], Stewartson [25], both of which involved forced waves, Hickernell [14] and Goldstein and co-workers (for example [12, 18]), with this approach termed a non-equilibrium nonlinear critical layer by some. A review of some of the early work on nonlinear critical layers is contained in [23]. As mentioned above, Goldstein and Leib [12] studied the evolution of a plane (two-dimensional) disturbance to a shear layer, and found that if nonlinear effects were taken into account inside the critical layer, the evolution of the disturbance was governed by a nonlinear PDE together with a jump condition across the critical layer, rather than the Landau equation assumed by Stuart-Watson type nonlinear stability theory or the Hickernell-type integro-differential equation found in some nonlinear critical layer problems (for example [14, 9]). Although [12] concerned eigenfunctions which are regular rather than singular, there are some similarities between that study and the present one, and because of this, frequent reference will be made to that work. We shall see in Section 3 that for our problem the evolution of a disturbance is given by a system of nonlinear PDEs together with jump conditions across the critical layer, as in [12]. Another note-worthy feature of [12] is that all of the harmonics became important inside the critical layer, and that is also true in our analysis.

As mentioned above, in the present study, we consider disturbances to the $\tanh^3 y$ mixing layer whose critical layer lies at $y = 0$, and the expansion of the mean velocity about this point is of the form

$$u_0 \sim c_0 + \frac{u'''_{0c} y^3}{3!} + \frac{u'''''_{0c} y^5}{5!} + \cdots .$$

The crucial difference between the problem considered here and the more usual type of stability problem is that the coefficients of both y and y^2 in the above expansion are zero, that is $u'_{0c} = u''_{0c} = 0$, and we shall see that this changes the nature of the flow significantly, with the singularities at the critical layer being far worse than those in the problem considered by Goldstein and Leib, where a logarithmic singularity entered at second order in the outer expansion. Several authors have considered flows where $u'_{0c} = 0$ but $u''_{0c} \neq 0$. These flows are more singular than the more usual type of stability problem but not as singular as the problem considered here. Brunet and Warn [3] studied the parabolic jet, $u_0(y) = \gamma y^2$, taking an approach similar to that used in [28], and arrived at a system of equations which were studied in more detail in [2]. Mallier and Davis [21] studied the stability of the Bickley jet, $u_0(y) = \operatorname{sech}^2 y$ for modes with a critical layer at $y = 0$. This latter study employed very similar techniques to the present one. The flow studied there had two neutral modes with wave numbers 1 and 3. Swaters [26] also studied the Bickley jet, but took a much more numerical approach than [21].

In Section 2, we shall see that the leading order disturbance behaves like y^{-2} near the critical layer, while [12] dealt only with a logarithmic singularity at second order and the modes in [21] behaved like y^{-1}. The severity of this singularity at the critical layer means that nonlinear effects enter into the inner expansion at an earlier order than in [12], with the leading order disturbance inside the critical layer containing all the higher harmonics. This also happened in [21]. Indeed, we shall see when we come to the inner expansion in Section 3 that disturbances of the form considered here with a critical layer centered on $y = 0$ cannot exist without the aid of nonlinear effects.

The remainder of the paper is as follows. In the following section, we formulate the problem and present the basic perturbation expansion which describes the motion outside the critical layer. We consider a disturbance which consists of three modes with wavenumbers equal to 1, 3 and 5 respectively. Since these modes share the same critical layer, an interaction between them is possible, and therefore all three modes are included for completeness.

In Section 3, the analysis of the flow inside the critical layer is considered. Initially, we ignore the higher harmonics and show that, without them, the disturbance cannot exist. When these higher harmonics are included in the inner expansion, where they enter at the same order as the disturbance itself, we are able to derive a system of equations governing the evolution of the disturbance. This system consists of several nonlinear PDEs together with jump conditions across the critical layer. In both Sections 2 and 3, many of the details are omitted, but are available in [24].

In Section 4, we present some temporally evolving numerical simulations of the $\tanh^3 y$ mixing layer, with the disturbance consisting of spatially periodic monochromatic waves. These simulations are included to confirm that the waves studied here do indeed exist. Finally, in Section 5, we make some concluding remarks.

2. Formulation and outer expansion

We consider the stability of the $\tanh^3 y$ mixing layer, $u_0 = \psi_0' = \tanh^3 y$, to an $\mathcal{O}(\varepsilon)$ disturbance which is assumed to vary on the slow time-scale

$$T = \mu t. \tag{2.1}$$

Here ε and μ are small parameters. We are interested in wave-like disturbances proportional to $e^{i\alpha(x-ct)}$ whose critical layers lie at $y = 0$, where the mean profile $u_0 \sim y^3 - y^5 + \cdots$, and therefore we shall assume that the disturbance consists of modes with a zero phase velocity, $c = 0$. As we will see shortly, there are three such modes which satisfy the homogeneous boundary conditions as $y \to \pm\infty$, with wave numbers 1, 3 and 5 respectively.

The outer expansion in the limit as $y \to 0$ will be of the form

$$\begin{aligned}\psi \sim \psi_0 &+ \varepsilon\left[A_1\hat{\phi}_{101}(y)e^{ix} + A_3\hat{\phi}_{103}(y)e^{3ix} + A_5\hat{\phi}_{105}(y)e^{5ix} + c.c.\right]\\ &+ \varepsilon\mu\left[A_1'\phi_{111}(y)e^{ix} + A_3'\phi_{113}(y)e^{3ix} + A_5'\phi_{115}(y)e^{5ix} + c.c.\right]\\ &+ \varepsilon\mu^2\left[A_1''\phi_{131}(y)e^{ix} + A_3''\phi_{133}(y)e^{3ix} + A_5''\phi_{125}(y)e^{5ix} + c.c.\right]\\ &+ \mathcal{O}\left(\varepsilon\mu^3\right) + \mathcal{O}\left(\varepsilon^2\right),\end{aligned} \tag{2.2}$$

where ψ is the stream-function and A_1, A_3 and A_5, which represent the disturbance amplitudes, are functions of the slow time-scale T. Thus the leading order disturbance in the outer expansion consists of components with wavenumbers 1, 3 and 5. Here, we assume that $\mu \ll 1$ and $\varepsilon \ll 1$ and keep the μ and ε expansions separate. As noted above, μ comes from the slow time-scale (2.1) while ε represents the size of the disturbance. This expansion (2.2) is substituted into the governing equations which are inviscid and incompressible,

$$\nabla^2\psi_t - J(\psi, \nabla^2\psi) = 0,$$

where $J(a, b) = a_x b_y - a_y b_x$ is a Jacobian. This substitution leads to a hierarchy of equations at various powers of ε and μ.

At leading order ($\mathcal{O}(\varepsilon^1\mu^0)$), we find that the perturbation obeys the Rayleigh equation,

$$\mathcal{L}_n\phi \equiv \phi'' + \left(18 - 12\tanh^2 y - \frac{6}{\tanh^2 y} - n^2\right)\phi = 0, \tag{2.3}$$

which has three neutral modes with critical layers at $y = 0$, namely modes with wavenumbers 1, 3 and 5,

$$\hat{\phi}_{101} = \hat{\phi}_{1A} = \frac{\operatorname{sech} y}{\tanh^2 y}\left(5\tanh^4 y + 2\tanh^2 y + 1\right),$$

$$\hat{\phi}_{103} = \hat{\phi}_{3A} = \frac{\operatorname{sech}^3 y}{3\tanh^2 y}\left(5\tanh^2 y + 3\right), \quad \hat{\phi}_{105} = \hat{\phi}_{5A} = \frac{1}{\sinh^2 y\cosh^3 y}.$$

To the best of our knowledge, none of these modes have previously appeared in the literature. All of these neutral modes vanish as $y \to \pm\infty$, and each behaves like y^{-2} as $y \to 0$. By contrast, the $\tanh y$ mixing layer has a single neutral mode with wavenumber 1 which is regular at the origin, while the Bickley jet, for which the first but not the second derivative of the base velocity vanishes at $y = 0$, has two neutral modes with a critical layer there, ([15, 21]), with wavenumbers 1 and 3, each of which behaves like y^{-1} as $y \to 0$.

Returning to the $\tanh^3 y$ mixing layer, because the singularity in this expansion is much stronger than the logarithmic singularity encountered in more usual shear flows (see for example [12]), we shall see in Section 3 that the nonlinear terms inside the critical layer enter at an earlier order than they did in [12]. At the next linear order in the outer expansion, $\mathcal{O}\left(\varepsilon^1\mu^1\right)$, we find that

$$\mathcal{L}_1\phi_{111} = -\frac{6i\left(5 - 12\cosh^2 y + 8\cosh^4 y\right)\left(\cosh^2 y - 2\right)}{\cosh^2 y\sinh^7 y},$$

$$\mathcal{L}_3\phi_{113} = -\frac{2i\left(8\cosh^2 y - 5\right)\left(\cosh^2 y - 2\right)}{3\cosh^2 y\sinh^7 y}, \quad \mathcal{L}_5\phi_{115} = -\frac{6i\left(\cosh^2 y - 2\right)}{5\cosh^2 y\sinh^7 y},$$

with solutions

$$\phi_{111} = \left(C_{111} + \frac{63i}{32}\int^y \frac{y_1 dy_1}{\sinh 2y_1}\right)\hat{\phi}_{1A} - \frac{42i}{5}\hat{\phi}_{1B}\log\tanh|y| + \frac{i\left(172\cosh^6 y - 627\cosh^4 y + 786\cosh^2 y - 315\right)}{64\cosh^2 y\sinh^5 y}$$

$$\phi_{113} = \left(C_{113} + \frac{33i}{64}\int^y \frac{y_1 dy_1}{\sinh 2y_1}\right)\hat{\phi}_{3A} + \frac{22i}{15}\hat{\phi}_{3B}\log\tanh|y| + \frac{i\left(88\cosh^8 y - 176\cosh^6 y + 189\cosh^4 y - 234\cosh^2 y + 165\right)}{384\cosh^2 y\sinh^5 y}$$

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$$\phi_{115} = \left(C_{115} - \frac{45i}{64}\int^{y}\frac{y_1\,dy_1}{\sinh 2y_1}\right)\hat{\phi}_{5A} + 6i\hat{\phi}_{5B}\log\tanh|y| + \frac{i(960\cosh^{10}y - 2760\cosh^{8}y + 2360\cosh^{6}y)}{640\cosh^{2}y\sinh^{5}y} - \frac{i(375\cosh^{4}y + 378\cosh^{2}y - 225)}{640\cosh^{2}y\sinh^{5}y}.$$

At the $\mathscr{O}(\varepsilon\mu^2)$ level we obtain the non-homogeneous Rayleigh equations

$$\begin{aligned}\mathscr{L}_1\phi_{121} &= \frac{6\cosh y\left(\cosh^2 y - 2\right)}{\sinh^5 y}\left(\coth^3 y\hat{\phi}_{1A} - i\phi_{111}\right)\\ \mathscr{L}_3\phi_{123} &= \frac{2\cosh y\left(\cosh^2 y - 2\right)}{\sinh^5 y}\left(\frac{1}{3}\coth^3 y\hat{\phi}_{3A} - i\phi_{113}\right)\\ \mathscr{L}_5\phi_{125} &= \frac{6\cosh y\left(\cosh^2 y - 2\right)}{5\sinh^5 y}\left(\frac{1}{5}\coth^3 y\hat{\phi}_{5A} - i\phi_{115}\right).\end{aligned} \tag{2.4}$$

The solutions to each of these will behave like $\mathscr{O}(y^{-8})$ as $y \to 0$, and in order to satisfy the boundary conditions as $y \to \pm\infty$, the solution for ϕ_{121} will include a term $D^{(\pm)}_{121}\phi_{1B}$, where the constants $D^{(\pm)}_{121}$ will take different values above and below the critical layer. Here, ϕ_{1B}, which is given in [24], is the second linearly independent solution to the Rayleigh equation (2.3) which behaves like y^3 at the origin but does not satisfy the homogeneous boundary conditions as $y \to \pm\infty$. Similarly, the solution for ϕ_{123} will include a term $D^{(\pm)}_{123}\phi_{3B}$ and the solution for ϕ_{125} will include a term $D^{(\pm)}_{125}\phi_{5B}$. We note here that since $\phi_{1B} \sim y^3$ as $y \to 0$, there will be a jump in the third derivative of the term $D^{(\pm)}_{121}\phi_{1B}$ across the critical layer,

$$\lim_{\delta\to 0+}\left(D^{(+)}_{121}\phi'''_{1B}|_{y=+\delta} - D^{(-)}_{121}\phi'''_{1B}|_{y=-\delta}\right) = 6\left(D^{(+)}_{121} - D^{(-)}_{121}\right), \tag{2.5}$$

with similar jumps in the wavenumber 3 and 5 modes, and we will find in Section 3 that these jumps will match onto jumps in the third derivative of the stream-function in the inner expansion. We can obtain an explicit expression for the jump $D^{(+)}_{121} - D^{(-)}_{121}$ from the outer expansion by multiplying the equation for $\mathscr{L}_1\phi_{121}$ (that is, (2.4)) by the homogeneous solution ϕ_{1A} and integrating with respect to y from $-\infty$ to ∞, but excluding the interval around $y = 0$ where the integrand is singular. Thus we take the Cauchy principal value

$$\lim_{\delta\to 0+}\left(\int_{-\infty}^{-\delta} + \int_{+\delta}^{\infty}\right)\phi_{1A}\mathscr{L}\phi_{121}. \tag{2.6}$$

We can integrate the term arising from the left-hand side of (2.4) by parts, and then use the Frobenius solution for ϕ_{121} to evaluate the resulting expression. This leads to the condition

$$D_{121}^{(+)} - D_{121}^{(-)} = -\frac{1}{5}\int_{-\infty}^{\infty} \frac{6\cosh y\left(\cosh^2 y - 2\right)}{\sinh^5 y}\left(\coth^3 y\hat{\phi}_{1A} - i\phi_{111}\right)\hat{\phi}_{1A}\, dy, \quad (2.7)$$

where the integral in (2.7) is between $-\infty$ to ∞. This integral can be evaluated by employing complex residue theory, using both the contour $-\infty \to \infty \to \infty + i\pi \to -\infty + i\pi \to -\infty$ and the contour $-\infty \to \infty \to \infty + i\pi/2 \to -\infty + i\pi/2 \to -\infty$. Using similar techniques for the other modes, we find that this leads to the solvability conditions

$$\begin{aligned} D_{121}^{(+)} - D_{121}^{(-)} &= \frac{1185071}{61600} - \frac{9261}{320}\zeta(3) \\ D_{123}^{(+)} - D_{123}^{(-)} &= -\frac{2526541}{1108300} + \frac{847}{640}\zeta(3) \\ D_{125}^{(+)} - D_{125}^{(-)} &= \frac{1074247}{123200} - \frac{945}{128}\zeta(3), \end{aligned} \quad (2.8)$$

where $\zeta(z)$ is the Riemann Zeta function. Thus (2.5) together with (2.8) gives the jumps in the outer expansion, and matching these jumps to the critical layer solution derived in the following section will lead to our amplitude equations. It is worth noting that these outer jumps came from the $\mathcal{O}(\varepsilon\mu^2)$ terms and contain a second derivative with respect to the slow time T. It is more usual in shear layer problems (see for example [19]) for the outer jump to come from the $\mathcal{O}(\varepsilon\mu)$ terms and to contain a first derivative with respect to T. Finally in this section, we note that nonlinear terms enter into the outer expansion at $\mathcal{O}(\varepsilon^2)$, with these terms arising from nonlinear interactions between the $\mathcal{O}(\varepsilon)$ terms, and all of the $\mathcal{O}(\varepsilon^2)$ nonlinear terms behave like $\mathcal{O}(y^{-8})$ as $y \to 0$; this behaviour will help us to determine the balance between ε and μ when we include nonlinear terms in the inner expansion in Section 3.

3. Inner expansion

We note that the outer expansion in Section 2 becomes disordered as we approach the critical layer, and because of this it is necessary to use another expansion near $y = 0$, where we shall introduce the stretched variable $Y = \mu^{-1/3}y$. This scaling is chosen so that the unsteady term and nonlinear interactions between the disturbance and the mean flow will enter into the equations inside the critical layer at the same order. Initially, we shall examine what happens in the purely linear case, that is, we shall neglect terms that are $\mathcal{O}(\varepsilon^2)$. We shall see that this leads to jumps in the inner expansion across the critical layer that cannot be matched to the outer expansion, which leads us to conclude that these waves cannot exist on a purely linear basis, although of course linear travelling waves can exist with non-zero phase velocities, since the critical layer for such modes would not be located at $y = 0$. Subsequently, we show how nonlinear effects may allow the existence of a disturbance of the form assumed. Many of the details have been omitted from this section, but are available in [24].

3.1. Linear case From the outer expansion, we know that the inner expansion will be of the form

$$\psi \sim \frac{\mu^{4/3}Y^4}{4} - \frac{\mu^{6/3}Y^6}{6} + \frac{11\mu^{8/3}Y^8}{120} + \varepsilon\mu^{-2/3}\left(\Psi_{111}e^{ix} + \Psi_{113}e^{3ix} + \Psi_{115}e^{5ix} + c.c.\right) + \mathcal{O}\left(\varepsilon\mu^0\right) + \mathcal{O}\left(\varepsilon^2\mu^{-8/3}\right).$$

If we assume that $\varepsilon \ll \mu^2$, so that we can neglect the nonlinear terms in the expansion, then the leading order disturbance in the inner expansion will consist of components with wavenumbers 1, 3 and 5 (together with their complex conjugates) which obey the following equations for $n = 1, 3$ and 5,

$$\left(\frac{\partial^3}{\partial Y^2 \partial T} + inY^3\frac{\partial^2}{\partial Y^2} - 6inY\right)\Psi_{11n} = 0,$$

which we can integrate once using the outer expansion,

$$\left(\frac{\partial^2}{\partial Y \partial T} + iY^3\frac{\partial}{\partial Y} - 3inY^2\right)\Psi_{11n} = -5niA_n.$$

We can solve these equations in terms of integrals of the confluent hypergeometric function [27]

$$\Psi_{11n} = -5inY\int_{-\infty}^{T} {}_1F_1\left(-\frac{2}{3}, \frac{4}{3}, inY^3(T_0 - T)\right)A_n(T_0)\,dT_0, \tag{3.1}$$

where

$${}_1F_1\,(a, c; z) = \sum_{n=0}^{\infty}\frac{(a)^{\overline{n}}}{n!(c)^{\overline{n}}}z^n \quad (c \neq 0, 1, 2\dots)$$

is the confluent hypergeometric function, and the Pochhammer symbols are defined by:

$$\lambda^{\overline{0}} = 1, \quad \lambda^{\overline{n}} = (\lambda)(\lambda+1)\cdots(\lambda+n-1), \quad \lambda^{\underline{n}} = (\lambda)(\lambda-1)\cdots(\lambda-n+1).$$

It should be pointed out that this is a little different from the usual nonlinear non-equilibrium critical layer equation of the form (see for example [9])

$$\left(\frac{\partial}{\partial\tau} + i\alpha Y\right)\gamma = \chi(Y, T)$$

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which has a solution

$$\gamma = \int_{-\infty}^{T} \chi\,(Y, T_0)\, e^{i\alpha(T_0-T)} d T_0,$$

with this difference arising because of the severity of the singularity at the critical layer.

At this point there is a jump in Ψ_{111YY} across the critical layer, where this jump is defined to be

$$\int_{-\infty}^{\infty} \Psi_{111YYY} d Y = \lim_{Z\to\infty} (\Psi_{1111YY}|_{Y=Z} - \Psi_{111YY}|_{Y=-Z})\,. \tag{3.2}$$

This jump can be evaluated from the solution (3.1) using recurrence relations for the confluent hypergeometric function and its asymptotic behaviour [27]. In particular, we make use of the identity

$$\frac{d}{dz}\, {}_1F_1\,(a, c; z) = \frac{a}{c}\, {}_1F_1\,(a+1, c+1; z)\,,$$

to write

$$\Psi_{111YY} = Y^{-2} \int_{-\infty}^{T} \kappa_{111} \left(Y^3\,(T_0 - T)\right) A_1(T_0) d T_0,$$

where we have defined

$$\kappa_{111}(\eta) = -60\eta\, {}_1F_1\left(\frac{1}{3}, \frac{7}{3}, i\eta\right) - \frac{405 i\eta^2}{14}\, {}_1F_1\left(\frac{4}{3}, \frac{10}{3}, i\eta\right) + \frac{27\eta^3}{7}\, {}_1F_1\left(\frac{7}{3}, \frac{3}{3}, i\eta\right).$$

We then use the asymptotic expansion valid in the limit $|z| \to \infty$,

$$\begin{aligned} {}_1F_1(a, c; z) \sim{}& \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm a\pi i} z^{-a} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n a^{\overline{n}} (1-c+a)^{\overline{n}}}{n!\, z^n}\right) \\ &+ \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \left(1 + \sum_{n=1}^{\infty} \frac{(c-a)^{\overline{n}} (1-a)^{\overline{n}}}{n!\, z^n}\right), \end{aligned}$$

where we take the positive sign in the first term when $-\pi/2 < \arg(z) < 3\pi/2$ and the negative sign when $-3\pi/2 < \arg(z) < \pi/2$. This lets us evaluate the jump defined in (3.2),

$$\int_{-\infty}^{\infty} \Psi_{111YYY}\, d Y = -\frac{45\sqrt{3}}{2} \Gamma\left(\frac{7}{3}\right) \int_{-\infty}^{T} (T - T_0)^{2/3} A_1(T_0)\, d T_0. \tag{3.3}$$

Unfortunately, there is no corresponding jump in the outer expansion to match onto this jump, since the first jump in the outer expansion ((2.5), (2.8)) will enter into the

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inner expansion at a later order. Therefore it follows that this jump in Ψ_{111YYY} must be zero, and we shall show now that as a result A_1 must be zero if we neglect nonlinear terms. From the jump (3.3) it follows that we require

$$\int_{-\infty}^{T} (T - T_0)^{2/3} A_1(T_0)\, dT_0 = 0.$$

By use of Laplace transforms, it can be shown that the solution of

$$\int_{-\infty}^{T} (T_0 - T)^{2/3} A_1(T_0)\, dT_0 = R(T)$$

is

$$A_1(T) = \frac{3}{2\,(\Gamma(2/3))^2} \frac{d^2}{dT^2} \int_{-\infty}^{T} \frac{R(T_0)}{(T - T_0)^{2/3}}\, dT_0.$$

Since $R(T) \equiv 0$ in the linear case, it follows that $A_1(T) \equiv 0$. A similar approach for the other modes leads to $A_3(T) \equiv A_5(T) \equiv 0$. A similar result was found by Mallier and Davis [21] for the Bickley jet nose modes, and for that problem it was found that even the addition of weak viscosity could not eliminate these jumps, and we believe that the same holds true for the $\tanh^3 y$ mixing layer. Thus linear theory tells us that these modes cannot exist. However, direct numerical simulations of this problem presented in Section 4 indicate that these modes can and do exist, and we believe that, as in Mallier and Davis, it is necessary to include nonlinear effects at leading order within the critical layer in order to derive a system of evolution equations for these modes.

3.2. Nonlinear case One possible way around the problem discussed in the linear case above is to set $\varepsilon = \mu^2$, so that all the harmonics enter into the inner expansion at leading order, so that there is a nonlinear jump in addition to the linear one. This scaling arises because the $\mathcal{O}(\varepsilon^2\mu^0)$ terms in the outer expansion behave like y^{-8}, the $\mathcal{O}(\varepsilon^3\mu^0)$ terms behave like y^{-14}, and so on, and choosing this scaling causes each of these terms to enter into the inner expansion at the same order. With this choice, we have in the inner region

$$\begin{aligned}\psi = {}& \mu^{4/3}\left(\frac{1}{4}Y^4 + Q^{(4)}\right) + \mu^{6/3}\left(-\frac{1}{2}Y^6 + Q^{(6)}\right) + \mu^{7/3}Q^{(7)} \\ & + \mu^{8/3}\left(\frac{11}{120}Y^8 + Q^{(8)}\right) + \mu^{9/3}Q^{(9)} + \mu^{10/3}\left(-\frac{44}{943}Y^{10} + Q^{(10)}\right) \\ & + \mu^{11/3}Q^{(11)} + \mu^{12/3}\left(\frac{641}{28350}Y^{12} + Q^{(12)}\right) + \mu^{13/3}Q^{(13)} \\ & + \mu^{14/3}\left(-\frac{19}{1782}Y^{14} + Q^{(14)}\right) + \mu^{15/3}Q^{(15)} + \cdots . \qquad (3.4)\end{aligned}$$

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The jump discontinuity in the third derivative of the outer solution, which occurs at $\mathcal{O}(\varepsilon\mu^2)$ in the outer, will match onto the term at $\mathcal{O}(\mu^{15/3})$ in the inner, that is, $Q^{(15)}$. This is because the jump comes from a term proportional to y^3 and $\varepsilon\mu^2 y^3 = \mu^{15/3}Y^3$. Thus we must derive equations for the terms $Q^{(4)}, \dots, Q^{(15)}$ in the expansion (3.4). (The term $Q^{(5)}$ has been omitted from this expansion since matching to the outer solution reveals it to be identically zero.) Because the nonlinear terms now enter at leading order, $Q^{(4)}$ will contain all the harmonics,

$$Q^{(4)}(x, Y, T) = \sum_{n=-\infty}^{\infty} Q_n^{(4)}(x, Y, T)e^{inx},$$

and we find that it obeys the following equation,

$$\left(\frac{\partial^2}{\partial Y \partial T} - 3Y^2\frac{\partial}{\partial x} + Y^3\frac{\partial^2}{\partial Y \partial x}\right) Q^{(4)} - Q^{(4)}_{YYYY}Q^{(4)}_x + Q^{(4)}_{YYYx}Q^{(4)}_Y$$
$$= -25iA_5e^{5ix} - 15iA_3e^{3ix} - 5iA_1e^{ix} + c.c.,$$

subject to the condition that as $Y \to \pm\infty$,

$$Q^{(4)} \sim Y^{-2}\left(A_5e^{5ix} + A_3e^{3ix} + A_1e^{ix} + c.c.\right)$$
$$+ Y^{-5}\left(\frac{iA_5'}{20}e^{5ix} + \frac{iA_3'}{12}e^{3ix} + \frac{iA_1'}{4}e^{ix} + c.c.\right) + \mathcal{O}\left(Y^{-8}\right).$$

This is much more nonlinear than the corresponding equation for the evolution of a monochromatic wave in the $\tanh y$ mixing layer which is of the form (see for example [12]),

$$\left[\frac{\partial}{\partial T} + Y\frac{\partial}{\partial \theta} - i\left(Ae^{i\theta} - A^*e^{-i\theta}\right)\frac{\partial}{\partial Y}\right] Q_{YY} = 2A'e^{i\theta} + 2A^{*\prime}e^{-i\theta}, \tag{3.5}$$

since (3.2) involves nonlinear terms in $Q^{(4)}$. The jump conditions across the critical layer for the wave number 1, 3 and 5 modes tell us that

$$\int_0^{2\pi}\int_{-\infty}^{\infty} Q^{(4)}_{YYY}e^{-ix}\,dY dx = \int_0^{2\pi}\int_{-\infty}^{\infty} Q^{(4)}_{YYY}e^{-3ix}\,dY dx$$
$$= \int_0^{2\pi}\int_{-\infty}^{\infty} Q^{(4)}_{YYY}e^{-5ix}\,dY dx = 0. \tag{3.6}$$

At higher orders, we get a set of equations for $Q^{(6)}, Q^{(7)}, \dots$, namely

$$\left[\frac{\partial^3}{\partial Y^2 \partial T} - \left(6Y + Q^{(4)}_{YYY}\right)\frac{\partial}{\partial x} + Q^{(4)}_{xYY}\frac{\partial}{\partial Y}\right.$$
$$\left. + \left(Y^3 + Q^{(4)}_Y\right)\frac{\partial^3}{\partial x \partial Y^2} - Q^{(4)}_x\frac{\partial^3}{\partial Y^3}\right] Q^{(n)} = \mathcal{F}^{(n)} \tag{3.7}$$

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where $n = 6, \dots, 15$ and the forcing terms $\mathscr{F}^{(n)}$ are given in [24] together with the asymptotic behaviour of the $Q^{(n)}$ as $Y \to \pm\infty$ and the jump conditions on the $Q^{(n)}$ across the critical layer. The jumps in $Q^{(15)}_{YY}$ match onto the jumps in the D_{12n} in the outer layer ((2.5), (2.8))

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\left(Q^{(15)}_{YYYY}e^{-ix} - \frac{382433A_1'C_{111}Y^2}{240}\right)dYdx = \left[\frac{3555213}{15400} - \frac{27783}{80}\zeta(3)\right]\pi A_1''$$
$$\int_0^{2\pi}\int_{-\infty}^{\infty}\left(Q^{(15)}_{YYYY}e^{-3ix} + \frac{137407A_3'C_{113}Y^2}{240}\right)dYdx = \left[\frac{2541}{160}\zeta(3) - \frac{7579623}{277075}\right]\pi A_3'' \tag{3.8}$$
$$\int_0^{2\pi}\int_{-\infty}^{\infty}\left(Q^{(15)}_{YYYY}e^{-5ix} - \frac{93953A_5'C_{115}Y^2}{240}\right)dYdx = \left[\frac{3222741}{30800} - \frac{2835}{32}\zeta(3)\right]\pi A_5''.$$

Thus we have the coupled nonlinear evolution equations (3.2), (3.7) together with the jump conditions (3.6), (3.8) and the behaviour of the $Q^{(n)}$ as $Y \to \pm\infty$ (3.2).

4. Numerical simulations

To corroborate the analysis in this study, included in this section are direct numerical simulations of the temporal evolution of two dimensional disturbances to a $\tanh^3 y$ mixing layer. The purpose of this section is not a full-scale numerical investigation of this problem, but rather to confirm that the disturbances studied in Sections 2 and 3 can indeed exist.

The computations were performed using a standard pseudospectral (Fourier) method [4], and we studied the temporal evolution of two-dimensional, spatially periodic, monochromatic disturbances to the base flow $u_0 = \tanh^3 y$. The results presented here are numerical solutions of the governing equations using the streamfunction-vorticity formulation

$$\nabla^2\psi_t - J(\psi, \nabla^2\psi) = \frac{1}{\mathrm{Re}}\nabla^4\psi. \tag{4.1}$$

The initial flow for each run consisted of the base flow together with a small disturbance, so that it was of a form similar to the first few terms of (2.2), namely

$$\psi = \psi_0 + \varepsilon\left[A_1\hat{\phi}_{101}(y)e^{ix} + A_3\hat{\phi}_{103}(y)e^{3ix} + A_5\hat{\phi}_{105}(y)e^{5ix} + c.c.\right].$$

This initial flow was then advanced in time using (4.1). At each time-step, we first advanced the vorticity $\omega = -\nabla^2\psi$ using

$$\omega_t = J(\psi, \omega) + \frac{1}{\mathrm{Re}}\nabla^2\omega,$$

and then recovered the streamfunction using Poisson's equation $\nabla^2\psi = -\omega$. Because of the singular nature of the flow, a large number of Fourier modes were used to ensure adequate resolution, and we took 1024 modes in the y direction and 128 in the x direction. It was also necessary for the initial disturbance to be fairly small, and the runs shown here correspond to taking ε to be approximately 0.01. This was because there was very little spectral decay for larger disturbances. A small amount of viscosity was necessary for numerical stability, and an initial Reynolds number of 5000 was taken. We simulated the flow between boundaries at $y = \pm y_0$, where we imposed free slip conditions, rather than between $-\infty < y < \infty$. Here y_0 was taken to be sufficiently large that the presence of the boundaries did not affect the simulation. Spatial derivatives were calculated in Fourier space, and switching between physical and Fourier spaces was accomplished *via* fast Fourier transforms. The viscous and nonlinear terms were advanced in time separately, using an implicit Adams-Moulton (Crank-Nicholson) scheme for the viscous terms and a Adams-Bashforth scheme for the nonlinear terms. In both cases, we used second order accurate schemes. Our results are shown in Figure 2, where we plot the stream function of the total flow (base flow plus perturbation) at a series of times. The initial condition consisted of a continuously differentiable function; other runs not presented here with slightly different initial conditions exhibited similar long term behaviour.

In the run shown, the initial disturbance consisted of a row of vortices, with only the wavenumber 1 mode present in the initial disturbance. However, the other two modes (with wavenumbers 3 and 5) were generated by nonlinear interaction of the wavenumber 1 mode with itself. Throughout the run, there was always a single vortex in the box, but the evolution of the vortex appears to be nonlinear.

A number of additional runs not presented here were also performed in which various combinations of the three different modes were present in the initial disturbance. When the initial disturbance consisted solely of either of the wavenumber 3 or 5 modes, the vortices which were initially present became tilted and elongated with increasing time. However, although the vortices became tilted, their amplitude did not appear to change much during the run. In several other runs, the initial disturbance consisted of at least two of the three modes, and sometimes all three, and for long enough times the behaviour in these multiple mode runs resembled that in Figure 2. This is to be expected as, due to nonlinear interactions, the run shown in Figure 2 will contain all three modes. In these multiple mode runs, once again, the initial disturbance consisted of a row of vortices and these vortices again became tilted and elongated, but they also began to rotate about each other and display nonlinear behaviour. In each of these runs, we ended up with a single vortex in the box after a long enough time. By contrast, in the runs where only the wavenumber 3 mode was present, there were always three vortices, and likewise in the runs where only the wavenumber 5 mode was present, there were always five vortices.

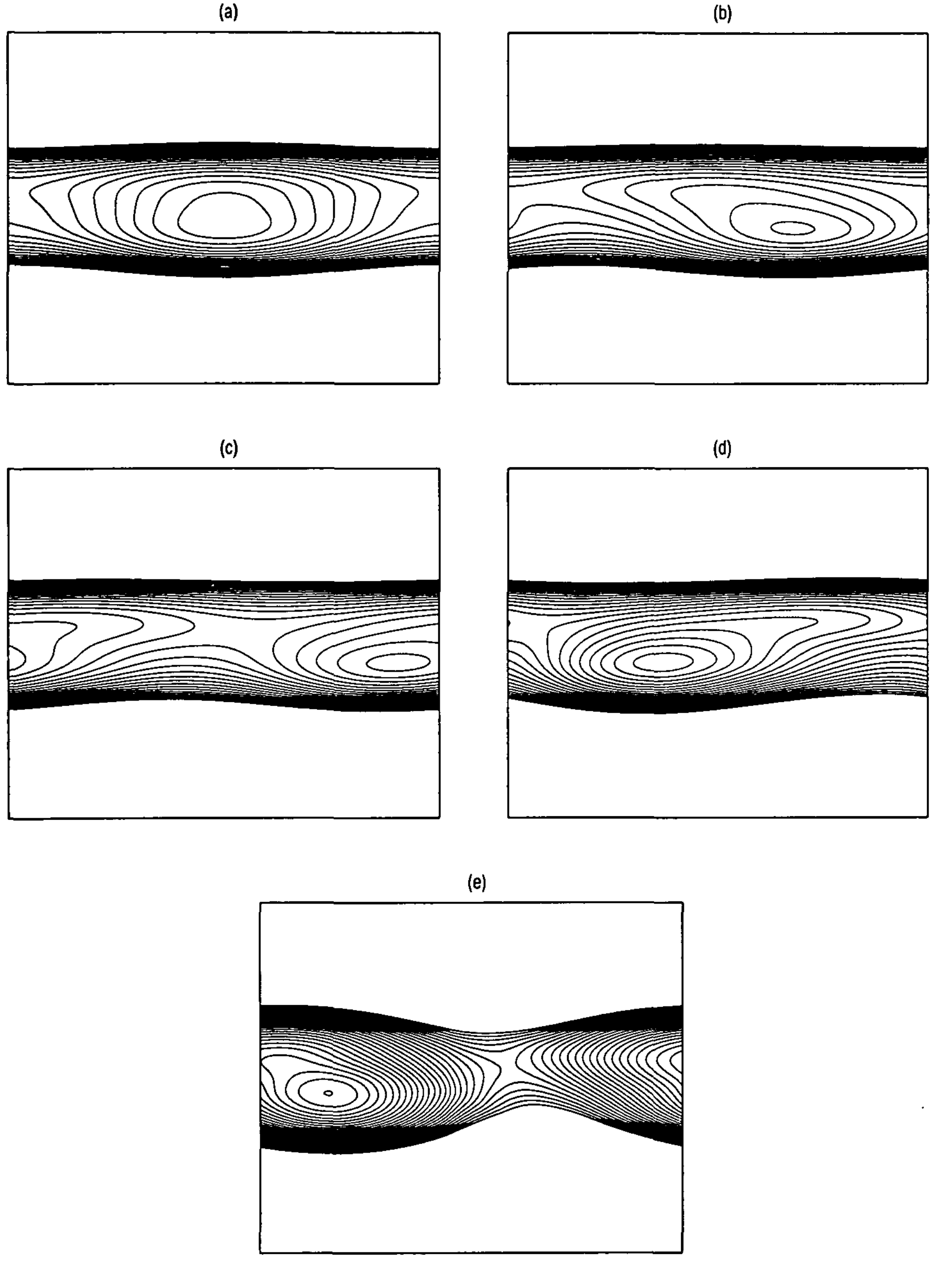

FIGURE 2. Evolution of the stream function in the critical layer region for an initial disturbance consisting only of the wave number 1 mode. Horizontal range $0 \le x \le 2\pi$ and vertical range $-3 \le y \le 3$: (a) $t = 0.0$; (b) $t = 2.5$; (c) $t = 5.0$; (d) $t = 12.5$; (e) $t = 25.0$.

5. Concluding remarks

In the previous sections, we considered the nonlinear evolution of a disturbance to the $\tanh^3 y$ mixing layer, with the disturbance consisting of modes with a critical layer at $y = 0$ where the first two derivatives of the base flow vanished. We found that there were three different neutral modes for this flow, with wavenumbers 1, 3 and 5, each of which was singular at the critical layer. Our inner expansion tells us that these modes cannot exist on a purely linear basis. This is also true for the Bickley jet when the disturbance consists of modes with a critical layer at $y = 0$ which are also singular [21]. However, the direct numerical simulations presented in Section 4 indicate that these modes can indeed exist, and in Section 3, we employed a non-equilibrium, nonlinear critical layer analysis where we brought in nonlinear terms consisting of the higher harmonics at leading order. The principal result of this paper is the set of coupled nonlinear PDEs derived in that section together with the jump conditions across the critical layer. A similar approach was used in [21] for the Bickley jet, where Mallier and Davis found a similar set of nonlinear PDEs, but they had fewer equations because the singularity in their problem was not as strong as that here. The authors of [12] arrived at a single PDE for a more normal shear layer. This is because in their problem, nonlinear effects entered into the inner expansion later than the original disturbance, and consequently they were able to write down explicit expressions for the first few terms in the inner expansion, while in the problem considered here, nonlinear effects enter at the same order as the disturbance itself inside the critical layer. In addition, for the problem considered here, we saw in the outer expansion that solutions to the Rayleigh equation (2.3) that satisfy the homogeneous boundary conditions as $y \to \pm\infty$ behave like y^{-2} near the critical layer but the second Frobenius solution behaves like y^3, and consequently the jump across the critical layer will enter at a later order in the inner expansion. In addition, our system of PDEs is more nonlinear than the PDE found in [12], which was of the form (3.5). The nonlinearity in (3.5) involves the product of the amplitude A with the function Q, while (3.2) involves products of $Q^{(4)}$ with itself. One difference between our problem and that of the Bickley jet is that in the present study, it appears from our numerical simulations that both the wavenumber 3 and 5 modes can exist separately without the presence of the other modes. (Of course, whenever the wavenumber 1 mode is present, the other two modes will be generated by nonlinear interactions.)

For the Bickley jet, which had 2 neutral modes with a critical layer at $y = 0$, we found that if both modes were present, the flow developed with a single critical layer. However, if only the wave number 3 mode was present, the critical layer actually split in two, so there were critical layers at $y = \pm y_c$ where $0 < y_c \ll 1$. In their study, Mallier and Davis had an integral constraint similar to (3.6), and they concluded that these constraints could only have a non-zero solution if both modes are present. Since the $\tanh^3 y$ profile is an odd profile, while the Bickley jet was an even profile, such a splitting cannot happen here, so we can conclude that the integral constraints arrived at in Section 3 can have a solution even if only a single mode is present.

Finally we touch on suggestions for further work. We are unaware of any laboratory experiments on this sort of singular flow, and we suggest that they might be useful. We note that it may be possible to use a combination of shear layer to arrive at a flow similar to ours where the first two derivatives of the base flow vanish at some point. For example, the flow

$$u_0 = \operatorname{erf}\left(y - \sqrt{\log 2}\right) + \operatorname{erf}\left(y + \sqrt{\log 2}\right) - \operatorname{erf}(y) \tag{5.1}$$

would fit this bill. In an experiment, if one started off with a initial profile of the form

$$u_0 = \begin{cases} 2, & y > y_1 \text{ and } -y_1 < y < 0, \\ 0, & 0 < y < y_1 \text{ and } y < -y_1, \end{cases}$$

one would expect this to diffuse to a series of error functions (together with a uniform flow) similar to that given in (5.1), and one would expect that the first two derivatives of this base flow would vanish at some distance downstream. It would be interesting to see if such a flow can indeed be created in a laboratory.

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