Canad. Math. Bull.Vol. **34** (3), 1991 pp. 412–416

A NOTE ON MINIMAL USCO MAPS

ANDREI VERONA AND MARIA ELENA VERONA

ABSTRACT. We prove that the composition of a minimal usco map, defined on a Baire space, with a lower semicontinuous function is single valued and usco at each point of a dense G_δ subset of its domain. This extends earlier results of Kenderov and Fitzpatrick. As a first consequence, we prove that a Banach space, with the property that there exists a strictly convex, weak* lower semicontinuous function on its dual, is a weak Asplund space. As a second consequence, we present a short proof of the fact that a Banach space with separable dual is an Asplund space.

Throughout this paper we shall consider only Hausdorff topological spaces. Let A and Z be two such spaces. Recall that a multivalued map $F: A \longrightarrow 2^Z$ is called *upper semicontinuous at* $a \in A$ if for any open set $U \subseteq Z$ such that $F(a) \subseteq U$, the set $F^{-1}(U) = \{x \in A : F(x) \subseteq U\}$ is a neighborhood of a in A; if in addition $F(a)$ is nonempty and compact, then F is called *usco at* a. F is called *usco on* $A_0 \subseteq A$ if it is usco at each $a \in A_0$. The following lemma gives a useful necessary and sufficient condition for a multivalued map to be usco at a point of its domain. Its proof is straightforward and we omit it.

LEMMA 1. *Let $F: A \longrightarrow 2^Z$ be a multivalued map and $a \in A$. The following assertions are equivalent:*

(i) F is usco at a;

(ii) If (a_α) is a net in A converging to a and (z_α) is a net in Z such that $z_\alpha \in F(a_\alpha)$ for every α, then the set consisting of all cluster points of the net (z_α) is nonempty and is contained in $F(a)$.

If Z is a topological vector space, the multivalued map $F: A \longrightarrow 2^Z$ is called *convex* if $F(a)$ is a convex subset of Z for every $a \in A$.

Recall that the *graph* of a multivalued map $F: A \longrightarrow 2^Z$ is the set

$$\mathcal{G}(F) = \{(a, z) \in A \times Z : z \in F(a)\}.$$

A usco (resp. convex usco) map is called *minimal* if its graph does not properly contain the graph of any other usco (resp. convex usco) with the same domain and co-domain.

We shall now recall some known, useful facts concerning usco maps. Details and proofs can be found in [3], [4], or [7]; some of them follow easily from the above lemma. Let $F: A \longrightarrow 2^Z$ and $H: A \longrightarrow 2^Z$.

(1) If F is usco, then its graph is a closed subset of $A \times Z$.

(2) If F is usco and $\mathcal{G}(H)$ is closed and contained in $\mathcal{G}(F)$, then H is usco.

Received by the editors September 12, 1989; revised: October 4, 1990.
AMS subject classification: 54C60, 46B22.

(3) Let Z be a topological vector space satisfying the following property:

(∗) the closed, convex hull of every compact, convex subset of Z is compact.

Define $\overline{\text{co}}\, F: A \longrightarrow 2^Z$ as follows: $(\overline{\text{co}}\, F)(a)$ is the closed convex hull of $F(a)$. Then, if F is usco, $\overline{\text{co}}\, F$ is a convex usco map, which is minimal convex if F is minimal ([4], Corollary 2.3 and Proposition 2.5; see also [7], Lemma 7.12).

(4) Assume that F is usco (resp. convex usco and Z satisfies (∗)) and let A_0 be a dense subset of A. Let $F|A_0: A_0 \longrightarrow 2^Z$ be the restriction of F and $F^0: A \longrightarrow 2^Z$ be the multivalued map whose graph is the closure in $A \times Z$ of the graph of $F|A_0$. If F is minimal (resp. minimal convex), then $F^0 = F$ (resp. $\overline{\text{co}}\, F^0 = F$) and $F|A_0$ is a minimal (resp. minimal convex) usco map on A_0 (see [3], Theorem 4.7).

THEOREM 2. *Assume that A is a Baire space and let $F: A \longrightarrow 2^Z$ be a minimal (convex) usco map. Let also $f: Z \longrightarrow R$ be a (convex) lower semicontinuous function. Then there exists a dense G_δ subset A_0 of A such that $f \circ F: A \longrightarrow 2^R$, $(f \circ F)(a) = f\big(F(a)\big)$, is single valued and usco at each point of A_0.*

PROOF. Step I. Define $\psi: A \longrightarrow R$ by

$$\psi(a) = \min\{f(z) : z \in F(a)\}.$$

The definition is correct since $F(a)$ is compact and f is lower semicontinuous. We shall prove that ψ is lower semicontinuous on A. To this end, let (a_α) be a net in A converging to a. For every α choose $z_\alpha \in F(a_\alpha)$ such that $\psi(a_\alpha) = f(z_\alpha)$. Since F is usco, the set consisting of the cluster points of the net (z_α) is nonempty and contained in $F(a)$; let $z \in F(a)$ be such a point. Since f is lower semicontinuous, we have

$$\liminf \psi(a_\alpha) = \liminf f(z_\alpha) \geq f(z) \geq \psi(a),$$

which proves our assertion.

Step II. Define a (convex) multivalued map $F_0: A \longrightarrow 2^Z$ by

$$F_0(a) = \big\{z \in F(a) : f(z) = \psi(a)\big\}.$$

Clearly F_0 is compact (and convex) valued. We shall prove next that F_0 is upper semicontinuous at each point at which ψ is continuous. To this end, let $a \in A$ be such a point and consider a net (a_α) in A converging to a and a net (z_α) in Z such that $z_\alpha \in F_0(a_\alpha)$. Since $F_0(a_\alpha) \subseteq F(a_\alpha)$ and F is usco, the set of all cluster points of the net (z_α) is nonempty and contained in $F(a)$. Let z be such a point. Using the continuity of ψ at a and the lower semicontinuity of f, we have

$$\psi(a) = \lim \psi(a_\alpha) = \liminf f(z_\alpha) \geq f(z) \geqq \psi(a).$$

It follows that $f(z) = \psi(a)$ and therefore $z \in F_0(a)$. By Lemma 1, we can conclude that F_0 is upper semicontinuous at a.

Step III. Since ψ is defined on a Baire space and is lower semicontinuous, there exists a dense G_δ subset A_0 of A such that ψ is continuous at each point of A_0. Thus F_0 is usco at each point of A_0. Since $\mathcal{G}(F_0|A_0) \subseteq \mathcal{G}(F|A_0)$ and $F|A_0$ is a minimal (convex) map (see (4) above), it follows that $F|A_0 = F_0|A_0$. This proves that $f \circ F$ is single valued at each point of A_0.

Step IV. It remains to show that $H = f \circ F$ is upper semicontinuous at each point of A_0. Let $a \in A_0$, $t = \psi(a)$ and $\epsilon > 0$. Since ψ is continuous at a, there exists an open neighborhood U of a such that $\psi(U) \subseteq (t - \epsilon/2, t + \epsilon/2)$. Let $C = f^{-1}((-\infty, t + \epsilon/2])$; it is a closed (convex) subset of Z. Clearly $F(A_0 \cap U) \subseteq C$ and, by the minimality of F, it follows that $F(U) \subseteq C$ (see e.g. [3], Proposition 4.5 and [4], Proposition 4.1). This implies that $H(U) \subseteq (-\infty, t + \epsilon)$. On the other hand, since F is usco and f is lower semicontinuous, there exists an open neighborhood V of a such that $H(V) \subseteq (t - \epsilon, \infty)$. Clearly $H(U \cap V) \subseteq (t - \epsilon, t + \epsilon)$ and this proves that H is upper semicontinuous at a.

REMARK 3. (i) The first assertion of the previous theorem can be reformulated as follows: $F(a)$ is contained in a level set of f for every $a \in A_0$. In particular, if $F: X \longrightarrow 2^{X^*}$ is a (maximal) monotone operator (for example, the subdifferential map of a continuous convex function on X) and if f is the norm on the dual X^* of a Banach space X, we reobtain a well known result of Kenderov [6]. (Recall that in this situation F is automatically norm-to-weak* usco and the norm on X^* is weak* lower semicontinuous.) The extensions of Kenderov's result proved in [9] and [10] are also particular cases of the previous theorem.

(ii) The second assertion of the theorem can be reformulated as follows: if $a \in A_0$, $z \in F(a)$, (a_α) is a net in A converging to a, and (z_α) is a net in Z such that $z_\alpha \in F(a_\alpha)$, then the net $\big(f(z_\alpha)\big)$ converges to $f(z)$. In the particular case when F is a (maximal) monotone operator on the Banach space X and f is the dual norm on X^*, we reobtain a result of Fitzpatrick [5].

(iii) It is obvious that a convex set contained in a level set of a strictly convex function must be a singleton. As a consequence of the above theorem (as reformulated in (i)) we obtain:

COROLLARY 4. *Let A be a Baire space and Z be a topological vector space (resp. a topological vector space satisfying $(*)$) such that there exists a strictly convex, lower semicontinuous function $f: Z \longrightarrow R$. Let also $F: A \longrightarrow 2^Z$ be a minimal convex (resp. minimal) usco map. Then there exists a dense G_δ subset of A on which F is single valued.*

It is well known (see [1]) that a Banach space X, which can be equivalently renormed such that the dual norm is strictly convex, is a weak Asplund space (i.e., every continuous, convex function defined on an open, convex subset of X is Gâteaux differentiable at each point of some G_δ subset of its domain); more generally, such a space is in Stegall's class C (i.e., any minimal weak* usco map defined on a Baire space with values in 2^{X^*} is single valued on a dense G_δ subset of its domain; see [8] for properties of this class and [2] or [9] for a proof of this assertion). As an immediate consequence of Corollary 4 we have the following generalization of these results.

THEOREM 5. *Let X be a Banach space and assume that there exists a weak* lower semicontinuous, strictly convex function $f: X^* \longrightarrow R$. Then X is is in Stegall's class C. In particular, it is a weak Asplund space.*

As a second application of Theorem 2, we shall give next a short proof for another result of Asplund [1]. Recall first that a Banach space X is called an *Asplund space* if every continuous, convex function defined on an open, convex subset of X is Fréchet differentiable on a dense G_δ subset of its domain.

THEOREM 6. *Let X be a Banach space such that its dual X^* is (norm) separable. Then X is an Asplund space.*

PROOF. Let C be an open, convex subset of X and $\varphi: C \longrightarrow R$ be a continuous, convex function. Then the subdifferential map $\partial\varphi: C \longrightarrow 2^{X^*}$ is a minimal convex weak* usco map (see [7], Theorem 3.25 and Theorem 7.9). Our assertion is now a direct consequence of the following lemma and Proposition 2.8 in [7].

LEMMA 7. *Let A be a Baire space, X be a Banach space with separable dual and $F: A \longrightarrow 2^{X^*}$ be a minimal (convex) weak* usco map. Then there exists a dense G_δ subset D of A such that F is single valued and norm usco at each point of D.*

PROOF. Let (x_k^*) be a dense sequence in X^*. For every k define $f_k: X^* \longrightarrow R$ by $f_k(x^*) = \|x^* - x_k^*\|$. Then f_k is weak* lower semicontinuous and by Theorem 2 there exists a dense G_δ subset A_k of A such that $f_k \circ F: A \longrightarrow 2^R$ is single valued and usco at each point of A_k. Let $a \in D = \bigcap A_k$ and assume that x^* and y^* are different elements in $F(a)$. Then there exists x_k^* such that $\|x^* - x_k^*\| \neq \|y^* - x_k^*\|$. Since this contradicts the fact that $f_k \circ F$ is single valued at a, it follows that F must be single valued at a.

It remains to prove that F is norm usco at each $a \in D$. To this end let (a_α) be a net in A converging to $a \in D$. Let $y_\alpha^* \in F(a_\alpha)$ and let $F(a) = \{y^*\}$. Take any $\epsilon > 0$ and choose x_k^* such that $f_k(y^*) = \|y^* - x_k^*\| < \epsilon/3$. Since $f_k \circ F$ is usco at a, there exists α_ϵ such that

$$f_k(y_\alpha^*) < f_k(y^*) + \epsilon/3 < 2\epsilon/3 \text{ if } \alpha \geq \alpha_\epsilon.$$

Thus

$$|y_\alpha^* - y^*\| \leq \|y^* - x_k^*\| + \|y_\alpha^* - x_k^*\| = f_k(y^*) + f_k(y_\alpha^*) < \epsilon, \text{ if } \alpha \geq \alpha_\epsilon,$$

which proves that (y_α^*) norm converges to y^*. Lemma 1 implies that F is norm usco at a.

REFERENCES

1. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. **121**(1968), 31–47.
2. J. M. Borwein, S. P. Fitzpatrick and P. S. Kenderov, *Minimal convex uscos and monotone operators on small sets*. Preprint, 1989.
3. L. Drewnowski and I. Labuda, *On minimal upper semicontinuous, compact-valued maps*, Rocky Mountain J. of Math. **20**(1990), 1–16.
4. ———, *On minimal convex usco and maximal monotone maps*, Real Analysis Exchange **15**(1989/1990), 729–742.

5. S. P. Fitzpatrick, *Continuity of non-linear monotone operators*, Proc. Amer. Math. Soc. **62**(1977), 111–116.
6. P. S. Kenderov, *The set-valued monotone mappings are almost everywhere single-valued*, C. R. Acad. Bulgare Sci. **27**(1974), 1173–1175.
7. R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics **1364**, Springer-Verlag, Berlin, 1989.
8. C. Stegall, *Gâteaux differentiation of functions on a certain class of Banach spaces*, Functional Analysis: Surveys and Recent Results III, North-Holland, 1984, 35–46.
9. M. E. Verona, *Generic differentiability of convex functions and monotone operators*. Ph.D. Thesis, California Institute of Technology, Pasadena, 1989.
10. A. Verona and M. E. Verona, *Locally efficient monotone operators*, Proc. Amer. Math. Soc. **109**(1990), 195–204.

Department of Mathematics
California State University
Los Angeles, CA
USA 90032

Department of Mathematics
University of California
Riverside, CA
USA 92521