INVARIANT SUBSPACES ON RIEMANN SURFACES

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1. Introduction. In this paper we generalize to Riemann surfaces a theorem of Helson and Lowdenslager in **(2)** describing the closed subspaces of $L^2(\{|z| = 1\})$ that are invariant under multiplication by $e^{i\theta}$.

Let R be a region on a Riemann surface with boundary Γ consisting of a finite number of disjoint simple closed analytic curves such that $R \cup \Gamma$ is compact and R lies on one side of Γ. Let $d\mu$ be the harmonic measure on Γ with respect to a fixed point t_0 on R. We shall consider the closed subspaces of $L^2(\Gamma, d\mu)$ that are invariant under multiplication by functions in $A(R) = \{F | F$ continuous on $\bar{R}$, analytic on $R\}$.

For some subspaces it is convenient to consider corresponding spaces on the disk $K = \{z \mid |z| < 1\}$ that arise by a universal covering map $T: K \to R$. The map T can be extended to a (relatively) open subset of $C = \{z \mid |z| = 1\}$ which is of full Lebesgue measure; furthermore, if $F \in L^p(\Gamma)$, then

$$F \circ T \in L^p(C, d\theta);$$

cf. (**8**, Lemma 6.1). The set $IL^p = \{F \circ T \mid F \in L^p(\Gamma)\}$ is a closed subspace of $L^p(C)$, and for V a closed subspace of $L^2(\Gamma)$, $V_T = \{F \circ T \mid F \in V\}$ is a closed subspace of IL^2. For certain invariant subspaces V it is more convenient to describe V_T rather than V.

Let $Q = \{q\}$ be the group of fractional linear transformations of K onto K such that $T \circ q = T$ and let $\{q_1, \ldots, q_n\}$ be generators of Q. For $a = (a_1, \ldots, a_n)$, an n-tuple of unimodular constants, let

$$I_a L^p = \{f \in L^p(C) \mid f \circ q_j = a_j f; j = 1, 2, \ldots, n\}$$

and

$$I_a H^p = \{f \in H^p(C) \mid f \circ q_j = a_j f; j = 1, 2, \ldots, n\}, \qquad 1 \leqslant p \leqslant \infty,$$

where $H^p(C)$ is the class of boundary functions of the Hardy functions H^p on K. Note that $IL^p = I_a L^p$ for $a_j = 1; j = 1, \ldots, n$.

THEOREM 1. *If V is a closed invariant subspace of $L^2(\Gamma)$, then either*

(1) *$V = \chi_S L^2(\Gamma)$, where χ_S is the characteristic function of a measurable set S on Γ; or*

(2) *there is $\Phi \in I_a L^\infty$, for some a, with $|\Phi| = 1$ a.e. on C such that*

$$V_T = \Phi[I_{\bar{a}} H^2(C)]; \qquad \bar{a} = (\bar{a}_1, \ldots, \bar{a}_n).$$

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M. Hasumi (**1**) and F. Forelli independently have proved results equivalent to Theorem 1 by methods different from ours. D. Sarason (**6**; **7**) proved a result equivalent to Theorem 1 for R an annulus. Theorem 1 is an extension of results in (**8**) where the closed invariant subspaces of the Hardy class, $H^2(R)$, on R are described.

2. For the remainder of this paper we assume that V is a closed invariant subspace of $L^2(\Gamma)$ which is *not* of the form $\chi_S L^2(\Gamma)$. In this section we obtain information about V which will be applied to V_T in §3.

For $1 \leqslant p < \infty$, $H^p(R)$ is the class of functions F analytic on R such that $|F|^p$ has a harmonic majorant. $H^\infty(R)$ is the class of bounded analytic functions on R. For $F \in H^p(R)$, $1 \leqslant p \leqslant \infty$, F has non-tangential boundary values $F^*(t)$ a.e. on Γ and $F^* \in L^p(\Gamma)$ and $\log|F^*| \in L^1(\Gamma)$ if $F \not\equiv 0$. These facts are known and follow easily from corresponding results on the disk; cf. (**8**, p. 496). We shall use H^p (Γ) to denote the space of boundary functions of the functions in $H^p(R)$ and $C(\Gamma)$ to denote the space of continuous complex-valued functions on Γ.

Let ω be a non-vanishing analytic differential on $\bar{R}$ and ω^* be the restriction of ω to Γ. The following theorem is due to Royden (**5**; **8**) and is a generalization of the well-known theorem by F. and M. Riesz about measures on C.

(2.1) THEOREM. *If ν is a measure on Γ such that $\int_\Gamma F\,d\nu = 0$ for all $F \in A(R)$, then $d\nu = H\omega^*$ for some $H \in H^1(\Gamma)$.*

Note that the measures $d\mu$ and ω^* are mutually absolutely continuous; indeed, $P = \omega^*/d\mu$ is bounded away from 0 and ∞.

(2.2) LEMMA. *If $F \in L^1(\Gamma)$, $F \not\equiv 0$, and $\int_\Gamma FW\,d\mu = 0$ for all $W \in A(R)$, then* $\log|F| \in L^1(\Gamma)$.

Proof. By Theorem (2.1) $F\,d\mu = H\,d\omega^*$, for some $H \in H^1(\Gamma)$. Thus $F = HP$, which implies that $\log|F| = \log|H| + \log|P| \in L^1(\Gamma)$.

(2.3) LEMMA. *For $F \in V$, let $S(F) = \{t \in \Gamma \mid F(t) = 0\}$. Then $\mu(S(F)) = 0$, if $F \not\equiv 0$.*

Proof. Let $S = S(F)$. Suppose $\mu(S) > 0$. Let $V(F)$ be the smallest closed invariant subspace of $L^2(\Gamma)$ that contains F. Suppose $G \perp V(F)$. Then $\int_\Gamma WF\bar{G}\,d\mu = 0$ for all $W \in A(R)$. It follows from Lemma (2.2) that $F\bar{G} = 0$ a.e. since $\mu(S) > 0$ implies that $\log|F\bar{G}| \notin L^1(\Gamma)$. Hence $G \in \chi_S L^2(\Gamma)$. It follows that $V(F)^\perp = \chi_S L^2(\Gamma)$ and thus $V(F) = \chi_{\tilde{S}}L^2(\Gamma)$ where $\tilde{S} = \Gamma - S$. Now for $H \in V$, $H = F_1 + G_1$, where $F_1 \in V(F)$ and $G_1 \in V(F)^\perp \cap V$. Now $G_1 \in V$ and $G_1 = 0$ a.e. on $\tilde{S}$. Then for $S_1 = S(G_1)$, it follows that $V(G_1) = \chi_{\tilde{S}_1}L^2(\Gamma)$. Hence for $W \in C(\Gamma)$, $WH = WF_1 + WG_1 \in V$. That is, V is invariant under multiplication by $C(\Gamma)$, which implies that $V = \chi_{S_2}L^2(\Gamma)$ for some set S_2, contradicting our assumption.

(2.4) LEMMA. *If* $F \in V$ *and* $F \not\equiv 0$, *then* $\log |F| \in L^1(\Gamma)$.

Proof. Choose $G \perp F$, $G \not\equiv 0$. Then $\int_\Gamma WF\bar{G}\, d\mu = 0$ for all $W \in A(R)$ and by Lemmas (2.2) and (2.3)

$$\log |F| + \log |G| = \log |FG| \in L^1(\Gamma),$$

which implies that $\log |F| \in L^1(\Gamma)$.

(2.5) LEMMA. *If* F *and* $G \in V$, *then* F/G *is the quotient of two functions in* $H^1(\Gamma)$.

Proof. Consider $Q \perp V$, $Q \not\equiv 0$. Then

$$\int WF\bar{Q}\, d\mu = \int WG\bar{Q}\, d\mu = 0 \qquad \text{for all } W \in A(R).$$

Thus there are functions $H_1, H_2 \in H^1(\Gamma)$ such that $F\bar{Q}\, d\mu = H_1\, \omega^*$ and $G\bar{Q}\, d\mu = H_2\, \omega^*$. Then $F/G = H_1/H_2$.

In §3 we shall need the fact that for each $a = (a_1, \ldots, a_n)$ there is a function $h_a \in I_a H^\infty$ such that $1/h_a \in I_{\bar{a}} H^\infty$. This is a consequence of the known result that for $\gamma_1, \ldots, \gamma_n$, a homology basis for R, there is an analytic differential α on $\bar{R}$ with periods $\operatorname{Log} a_j$ on $\gamma_j; j = 1, \ldots, n$; cf. (**4**, p. 198). Then for

$$H(t) = \exp\left(\int_{t_0}^{t} \alpha\right),$$

$h_a = H \circ T$ is the desired function.

3. We now consider $V_T = \{F \circ T \mid F \in V\}$. For $f = F \circ T \in V_T$, $F \not\equiv 0$, we have $\log |f| \in L^1(C)$ since $\log |F| \in L^1(\Gamma)$. Let f_1 be the outer function such that $|f_1| = |f|$ a.e. on C (**3**, p. 62) and let $f_0 = f/f_1$. Since $f \in L^2(C)$ and $\log |f| \in IL^1$, it follows that $f_1 \in I_a H^2$ for some a and thus $f_0 \in I_{\bar{a}} L^\infty$. We now fix $g \in V_T$. Then $g_0 \in I_{\bar{b}} L^\infty$ for some b. By Lemma (2.5), $f_0 f_1/g_0\, g_1$ is the quotient of two functions in IH^1 and it follows easily that $f_0 f_1/g_0\, h_b$ is of the form $(\phi/\psi)h$ where ϕ and ψ are inner functions and h is an outer function. Let $\tilde{V}_T = \{f/g_0\, h_b \mid f \in V_T\}$. Note that $\tilde{V} = \{f \circ T^{-1} \mid f \in \tilde{V}_T\}$ is a closed invariant subspace of $L^2(\Gamma)$. For $f \in \tilde{V}_T$, $f = (\phi_f/\psi_f)f_1$ where ϕ_f and ψ_f are inner functions and f_1 is the outer function with $|f_1| = |f|$ a.e. on C.

For α and β inner functions, we say that α *divides* β if β/α is an inner function. It is well known that any collection of inner functions has a greatest common divisor (**3**, p. 85). In particular, for each $f \in \tilde{V}_T$, we can take ϕ_f and ψ_f to be relatively prime. Then ϕ_f and ψ_f are *modulus invariant* (i.e., $|\phi_f \circ q| = |\phi_f|$ and $|\psi_f \circ q| = |\psi_f|$ on K for all $q \in Q$) by the following argument. For $q \in Q$,

$$(\phi_f/\psi_f)f_1 = f = f \circ q = [\phi_f \circ q/\psi_f \circ q]f_1 \circ q = \lambda[\phi_f \circ q/\psi_f \circ q]f_1 \quad \text{a.e. on } C$$

where λ is a unimodular constant. Thus $\phi_f/\psi_f = \lambda(\phi_f \circ q/\psi_f \circ q)$ on K. Note that $\phi_f \circ q$ and $\psi_f \circ q$ have no common factors since ϕ_f and ψ_f have no common divisors. Since $\phi_f(\psi_f \circ q)/\psi_f$ and $(\phi_f \circ q)\psi_f/\psi_f \circ q$ are inner functions, it follows that ψ_f and $\psi_f \circ q$ divide each other and thus $|\psi_f| = |\psi_f \circ q|$ on K.

This implies that the same relation holds between ϕ_f and $\phi_f \circ q$. Thus ϕ_f and ψ_f are modulus invariant.

We have already observed that a collection of inner functions has a greatest common divisor. It follows that if a collection of inner functions has a common multiple, then it has a least common multiple.

(3.1) LEMMA. *$\{\psi_f | f \in \tilde{V}_T\}$ has a least common multiple.*

Proof. Consider $Q \perp \tilde{V}$, $Q \not\equiv 0$. Then for $F \in \tilde{V}$,

$$\int_\Gamma WF\bar{Q}\,d\mu = 0 \qquad \text{for all } W \in A(R),$$

and thus $F\bar{Q}\,d\mu = H_F\,\omega^*$ for some $H_F \in H^1(\Gamma)$. Fix $M \in \tilde{V}$, $M \not\equiv 0$. Then $F/M = H_F/H_M$ a.e. on Γ and it follows that $f/m = h_f/h_m$ a.e. on C where $f = F \circ T$, $m = M \circ T$, $h_f = H_F \circ T$, and $h_m = H_M \circ T$. Then

$$\left(\frac{\phi_f}{\psi_f}\right)\left(\frac{\psi_m}{\phi_m}\right) = \frac{(h_f)_0}{(h_m)_0} \qquad \text{a.e. on } C,$$

where $(h_f)_0$ and $(h_m)_0$ are the inner factors of h_f and h_m respectively. Hence $(h_m)_0\,\psi_m\,\phi_f/\psi_f = (h_f)_0\,\phi_m$. Thus ψ_f divides $(h_m)_0\,\psi_m\,\phi_f$, and since ψ_f and ϕ_f have no common factors, ψ_f divides $(h_m)_0\,\psi_m$. That is, $(h_m)_0\,\psi_m$ is a common multiple of $\{\psi_f | f \in \tilde{V}_T\}$. It follows that $\{\psi_f | f \in \tilde{V}_T\}$ has a least common multiple.

Let ϕ be a greatest common divisor of $\{\phi_f \mid f \in \tilde{V}_T\}$ and let ψ denote a least common multiple of $\{\psi_f \mid f \in \tilde{V}_T\}$. By **(8**, Lemma (4.7)**)**, ϕ is modulus invariant and by a similar argument ψ is modulus invariant. Then $\phi/\psi \in I_c L^\infty$ for some c and $(\phi/\psi)h_{\bar{c}} \in IL^\infty$.

LEMMA. *Let $V'_T = (\phi/\psi)h_{\bar{c}}(IH^2)$. Then $V'_T = \tilde{V}_T$.*

Proof. Clearly $\tilde{V}_T \subset V'_T$. We shall prove that $\tilde{V}_T \supset V'_T$ by showing that $\tilde{V} \supset V'$ where $V' = \{f \circ T^{-1} \mid f \in V'_T\}$. Let $Q \perp \tilde{V}$, $Q \not\equiv 0$. Then as in the proof of the previous lemma $F\bar{Q}\,d\mu = H_F\,\omega^*$, $H_F \in H^1(\Gamma)$. That is, $F\bar{Q} = H_F P$ a.e. on Γ. Let $p = P \circ T$ and $q = Q \circ T$. Then $\log|p|$ and $\log|q| \in IL^1$. Let p_1 and q_1 be the outer functions such that $|p_1| = |p|$ and $|q_1| = |q|$ a.e. on C. Then for $p_0 = p/p_1$ and $q = \bar{q}/q_1$,

$$(\psi_f/\phi_f)f_1\,q_0\,q_1 = f\bar{q} = h_f\,p = (h_f)_0(h_f)_1\,p_0\,p_1$$

and it follows that $(\varphi_f/\psi_f)q_0 = (h_f)_0\,p_0$. Then if h_0 is a greatest common divisor of $\{(h_f)_0 \mid f \in \tilde{V}_T\}$, $(\phi/\psi)q_0 = h_0\,p_0$ and

$$(\phi/\psi)h_{\bar{c}}\,\bar{q} = h_0\,h_{\bar{c}}\,q_1\,p_0 = (h_0\,h_{\bar{c}}\,q_1/p_1)p = hp$$

where $h \in IH^2$. Let $A = [(\phi/\psi)h_{\bar{c}}] \circ T^{-1}$. Then $A\bar{Q} = HP$ where

$$H = h \circ T^{-1} \in H^2(\Gamma).$$

For $W \in H^2(\Gamma)$, $WH \in H^1(\Gamma)$ and thus

$$\int_\Gamma AW\bar{Q}\,d\mu = \int_\Gamma WH\,\omega^* = 0.$$

Therefore, $Q \perp AH^2(\Gamma)$. It is easy to see that $V' = AH^2(\Gamma)$. Thus $Q \perp V'$. We have shown that $\tilde{V}^{\perp} \subset V'^{\perp}$; thus $\tilde{V} \supset V'$.

Proof of Theorem 1. We have

$$V_T = g_0\, h_b\, \tilde{V}_{T'} = g_0\, h_b(\phi/\psi)h_{\bar{c}}(IH^2) = \Phi I_{\bar{a}}\, H^2$$

for $\Phi = g_0(\phi/\psi)$ and $\bar{a} = b\bar{c}$.

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