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Maurer–Cartan Elements in the Lie Models of Finite Simplicial Complexes

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Abstract. In a previous work, we associated a complete differential graded Lie algebra to any finite simplicial complex in a functorial way. Similarly, we also have a realization functor from the category of complete differential graded Lie algebras to the category of simplicial sets. We have already interpreted the homology of a Lie algebra in terms of homotopy groups of its realization. In this paper, we begin a dictionary between models and simplicial complexes by establishing a correspondence between the Deligne groupoid of the model and the connected components of the finite simplicial complex.

Let $\mathrm{MC}(L)$ be the set of Maurer–Cartan elements of a differential graded Lie algebra (L,d) over $\mathbb{Q}$ (henceforth DGL). The group L_0 of elements of degree 0, endowed with the Baker–Campbell–Hausdorff product, acts on $\mathrm{MC}(L)$ by

$$x\mathcal{G}z = e^{\mathrm{ad}_x}(z) - \frac{e^{\mathrm{ad}_x}-1}{\mathrm{ad}_x}(dx),$$

with $x\in L_0$ and $z\in\mathrm{MC}(L)$. We denote by $\widetilde{\mathrm{MC}}(L)$ the orbit space for this action.

In [1], we constructed a functor $\mathcal{L}$ from the category of finite simplicial complexes to the category of complete differential graded Lie algebras (henceforth cDGL), $X\mapsto\mathcal{L}_X$. Rational homotopy has been mainly introduced and used for simply connected spaces [5,10,11]. In [11], there is also an extension to non-simply connected spaces over $\mathbb{R}$ via fiber bundles (see [7] for an adaptation to $\mathbb{Q}$). Recently, the classical approach has been extended to non-simply connected spaces in [6], and the functor $\mathcal{L}$ gives the corresponding extension for DGL's.

In this paper we prove the following relation between $\mathcal{L}_X$ and the topology of X.

Theorem *For any finite simplicial complex X, there is a bijection*

$$\pi_0(X_+)\cong\widetilde{\mathrm{MC}}(\mathcal{L}_X),$$

where $X_+ = X\sqcup\{*\}$.

The case of the interval $X=[0,1]$ was solved in [2]. In Section 1, we make the necessary recalls on Maurer–Cartan elements and the functor $\mathcal{L}$. Section 2 is devoted

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to a decomposition of $\mathcal{L}_X$ when X is connected. Finally, the proof of the theorem is done in Section 3.

1 Functor $\mathcal{L}$ and Maurer–Cartan Elements

Recall that a DGL(L,d) is *complete* if $L = \varprojlim_n L/L^{[n]}$, where $L^{[n]}$ denotes the sequence of ideals defined by

$$L^{[1]} = L \quad \text{and} \quad L^{[n+1]} = [L, L^{[n]}],\ n \geq 2.$$

When V is finite dimensional, $\widehat{\mathbb{L}}(V) = \lim_n \mathbb{L}(V)/\mathbb{L}(V)^{[n]}$ is the completion of the free graded Lie algebra $\mathbb{L}(V)$.

Let (L,d) be a cDGL. An element $u \in L_{-1}$ is a *Maurer–Cartan element* if

$$du = -\frac{1}{2}[u,u].$$

In [8], R. Lawrence and D. Sullivan constructed a cDGL $\mathcal{L}_I$ that is, in a sense that we will make precise later, a model for the interval $I = [0,1]$. More precisely,

$$\mathcal{L}_I = \big(\widehat{\mathbb{L}}(a,b,x), d\big),$$

where a and b are Maurer–Cartan elements and x is an element of degree 0 with

$$dx = \mathrm{ad}_x\, b + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}(b-a) = [x,b] + \sum_{n=0}^{\infty} \frac{B_n}{n!}\, \mathrm{ad}_x^n (b-a).$$

Here, the B_n are the well known Bernoulli numbers. This model has been described in detail in [4, 9].

In a cDGL(L,d), two Maurer–Cartan elements u_1 and u_2 are *equivalent* if they are in the same orbit for the gauge action. By construction, this is equivalent to the existence of a morphism of DGL's, $f\colon \mathcal{L}_I \to (L,d)$ with $f(a) = u_1$ and $f(b) = u_2$. The map f is called *a path from u_1 to u_2*. The set of equivalence classes of Maurer–Cartan elements is denoted $\widetilde{MC}(L)$.

Our purpose is the determination of $\widetilde{MC}(L)$ for a family of cDGL's directly related to topology. In fact, the cDGL $\mathcal{L}_I$ is the first example of a Lie model for a general simplicial complex. More generally, there is a functor $\mathcal{L}$, unique up to isomorphism, $X \mapsto \mathcal{L}_X$, from the category of finite simplicial complexes to the category of cDGL's. As any finite simplicial complex is a subcomplex of some Δ^n, it is sufficient to construct the models, $\mathcal{L}_{\Delta^n}$, of the Δ^n's.

Proposition 1.1 ([1, Theorem 2.8]) *The* cDGL $\mathcal{L}_{\Delta^n}$ *is defined, up to isomorphism, by the following properties.*

(i) *The* cDGL*'s* $\mathcal{L}_{\Delta^n}$ *are natural with respect to the injections of the subcomplexes* Δ^p*, for all* $p < n$.
(ii) *For* $n = 0$*, we have* $\mathcal{L}_{\Delta^0} = (\widehat{\mathbb{L}}(a), d)$ *where* a *is a Maurer–Cartan element.*
(iii) *The linear part* d_1 *of the differential of* $\mathcal{L}_{\Delta^n}$ *is the desuspension of the differential* δ *of the chain complex* $C_*(\Delta^n)$.

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In the case where $\Delta^1 = [0,1]$, we recover the Lawrence–Sullivan construction. For each finite simplicial complex, X, contained in Δ^n, the Lie subalgebra $\widehat{\mathbb{L}}(s^{-1}C_*(X))$ is preserved by the differential of $\mathcal{L}_{\Delta^n}$ and gives a model $\mathcal{L}_X$ of X.

When a is a Maurer–Cartan element in $\mathcal{L}_X$, we denote by d_a the perturbed differential $d_a = d + \mathrm{ad}_a$. The first properties of $\mathcal{L}_X = (\widehat{\mathbb{L}}(W), d)$ are contained in the following statements extracted from [1, 3].

(a) If d_1 denotes the linear part of the differential d, then (W, d_1) is isomorphic to the desuspension of the simplicial chain complex $C_*(X)$ of X.
(b) If $f\colon X \to Y$ is the inclusion of a subcomplex, then $\mathcal{L}_f\colon \mathcal{L}_X \to \mathcal{L}_Y$ is equal to $\widehat{\mathbb{L}}(s^{-1}C_*(f))$.
(c) $H(\mathcal{L}_X) = 0$ ([3, Theorem 4.1]).
(d) If X is simply connected, and a is the Maurer–Cartan element associated with a 0-simplex, then $(\widehat{\mathbb{L}}(W), d_a)$ is quasi-isomorphic to the usual rational Quillen model of X [1, Theorem 7.4(ii)].
(e) If X is connected and a is the Maurer–Cartan element associated with a 0-simplex, then $H_0(\widehat{\mathbb{L}}(W), d_a)$ is isomorphic to the Malcev Completion of $\pi_1(X)$ ([1, Theorem 9.1]).

Recall that the Lawrence–Sullivan interval $\mathcal{L}_I$ is isomorphic to the cylinder construction ([12]) on a Maurer–Cartan element ([3, Theorem 6.3]). More precisely, consider the cDGL$(\widehat{\mathbb{L}}(a,c,y), d)$ with $|y| = 0$, $|c| = -1$, $da = -\frac{1}{2}[a,a]$, $dy = c$ and $dc = 0$ that we equip with a derivation s of degree +1, defined by $s(a) = y$, $s(c) = s(y) = 0$. Then the morphism

$$\psi\colon \big(\widehat{\mathbb{L}}(a,b,x), d\big) \longrightarrow \big(\widehat{\mathbb{L}}(a,c,y), d\big) \tag{1.1}$$

defined by $\psi(a) = a$, $\psi(b) = e^{sd+ds}(a)$, $\psi(x) = y$ is an isomorphism of DGL's. In particular,

$$\psi(b) = a + c + \sum_{n\geq 1} \frac{(sd)^n}{n!}(a) = e^{\mathrm{ad}_{-y}}(a) + \frac{e^{\mathrm{ad}_{-y}} - 1}{\mathrm{ad}_{-y}}(c).$$

Definition 1.2 Two Maurer–Cartan elements u, v in a cDGL$(\widehat{\mathbb{L}}(V), d)$ are called *equivalent of order r* if there is a morphism

$$\varphi\colon \big(\widehat{\mathbb{L}}(a,b,x), d\big) \longrightarrow \big(\widehat{\mathbb{L}}(V), d\big)$$

with $\varphi(x) \in \mathbb{L}^{\geq r}(V)$, $\varphi(a) = u$ and $\varphi(b) = v$. We denote this relation by $u \sim_{O(r)} v$.

This relation is a key point in the proof of Proposition 2.1. We end this section with two properties of $\sim_{O(r)}$.

Lemma 1.3 *Let u be a Maurer–Cartan element in $(\widehat{\mathbb{L}}(V), d)$. We suppose $u = v + w$ with $w \in \mathbb{L}^{\geq r}(V)$, and the existence of an element $z \in \mathbb{L}^{\geq r}(V)$ with $dz = w + t$ and $t \in \mathbb{L}^{\geq r+1}(V)$. Then, we have $u \sim_{O(r)} v + w'$ with $w' \in \mathbb{L}^{\geq r+1}(V)$.*

Proof Let $f\colon (\widehat{\mathbb{L}}(a,c,y), d) \to (\widehat{\mathbb{L}}(V), d)$ be the morphism defined by $f(a) = u$, $f(y) = -z$, and $f(c) = -dz$. Then $f \circ \psi$ is a path in $(\widehat{\mathbb{L}}(V), d)$ with $f\psi(a) = u$,

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$f\psi(x) = -z$. To determine $f\psi(b)$, we first observe that

$$\psi(b) = a + c + \sum_{n\geq 1} \frac{(sd)^n}{n!}(a).$$

Remark also that $f(sd)^n(a) \in \mathbb{L}^{\geq r+1}(V)$, for $n \geq 1$. Therefore,

$$f \circ \psi(b) \in f(a) + f(c) + \mathbb{L}^{\geq r+1}(V) = u - dz + \mathbb{L}^{\geq r+1}(V) = v - t + \mathbb{L}^{\geq r+1}(V),$$

with $t \in \mathbb{L}^{\geq r+1}(V)$. ∎

Lemma 1.4 *Let $(u_r)_{r\geq n_0}$ be a sequence of Maurer–Cartan elements in $(\widehat{\mathbb{L}}(V), d)$ such that $u_r = z + v_r$ with $v_r \in \mathbb{L}^{\geq r}(V)$. If $u_r \sim_{O(r)} u_{r+1}$ for each $r \geq n_0$, then we have $u_{n_0} \sim_{O(n_0)} z$.*

Proof By hypothesis, for $r \geq n_0$, there is a morphism

$$\varphi_r \colon (\widehat{\mathbb{L}}(a, b, x), d) \longrightarrow (\widehat{\mathbb{L}}(V), d)$$

with $\varphi_r(a) = u_r$, $\varphi_r(b) = u_{r+1}$ and $\varphi_r(x) \in \mathbb{L}^{\geq r}(V)$. For $r > n_0$, we define w_r to be the Baker–Campbell–Hausdorff product

$$w_r = \varphi_{n_0}(x) * \varphi_{n_0+1}(x) * \cdots * \varphi_{r-1}(x).$$

From the associativity established in [8], the element w_r is a path from u_{n_0} to u_r. We form the infinite product

$$w = \varphi_{n_0}(x) * \varphi_{n_0+1}(x) * \cdots,$$

which is well defined in $\widehat{\mathbb{L}}(V)$ as the limit of the w_r. Now we claim that the element w is a path of order n_0 from u_{n_0} to z; *i.e.*, we have $u_{n_0} \sim_{O(n_0)} z$. Consider the element

$$y = dw - [w, z] - \sum_{n\geq 0} \frac{B_n}{n!} \operatorname{ad}_w^n(z - u_{n_0}),$$

where the B_n are the Bernoulli numbers. The element y has the same image in $\mathbb{L}(V)/\mathbb{L}^{\geq r}(V)$ as

$$dw_r - [w_r, u_r] - \sum_{n\geq 0} \frac{B_n}{n!} \operatorname{ad}_{w_r}^n(u_r - u_{n_0}).$$

This last expression is equal to 0, because w_r is a path from u_{n_0} to u_r. This implies $y = 0$ and proves the result. ∎

2 Model of a Finite Connected Simplicial Complex

Proposition 2.1 *Let X be a connected finite simplicial complex of dimension n; then we have an isomorphism of* cDGL's

$$\mathcal{L}_X \cong \big(\widehat{\mathbb{L}}(V), d\big) \,\widehat{\amalg}_i \big(\widehat{\mathbb{L}}(u_i, v_i), d\big),$$

where $dv_i = u_i$, $du_i = 0$, $V = V_{\leq n-1}$, $V = \mathbb{Q}a \oplus V_{\geq 0}$, a is a Maurer–Cartan element and $\widehat{\amalg}$ denotes the completion of the coproduct. Moreover, the differential of any $x \in V_{\geq 0}$ verifies $dx + [a, x] \in \widehat{\mathbb{L}}^{\geq 2}(V_{\geq 0})$.

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Proof By Lemma 2.2, this is true if $\dim X = 1$. Proceed by induction on n. We can therefore suppose that

$$X = Y \cup \cup_{j=1}^{k} \Delta_j^n \quad \text{and} \quad (\mathcal{L}_Y, d) \cong \big(\widehat{\mathbb{L}}(V), d\big) \widehat{\amalg}_i \big(\widehat{\mathbb{L}}(u_i, v_i), d\big)$$

with $n \geq 2$, $\dim Y \leq n-1$, $V = V_{\leq n-2} = \mathbb{Q}a \oplus W$, $W = W_{\geq 0}$, $|v_i| \leq n-2$, $dv_i = u_i$. We set $u_i' = u_i + [a, v_i]$ and we get an isomorphism of DGL's

$$\big(\widehat{\mathbb{L}}(V), d_a\big) \widehat{\amalg\amalg}_i \big(\widehat{\mathbb{L}}(u_i', v_i), d_a\big) \longrightarrow (\mathcal{L}_Y, d_a),$$

with $d_a v_i = u_i'$, $d_a u_i' = 0$. Now, by construction of the model $\mathcal{L}_X$, there are cycles $\Omega_j \in (\mathcal{L}_Y)_{n-2}$ such that

$$(\mathcal{L}_X, d_a) = \big(\mathcal{L}_Y \widehat{\amalg\amalg}_{j=1}^{k} \mathbb{L}(x_j), d_a\big), \quad |x_j| = n-1, \quad d_a x_j = \Omega_j.$$

Since the inclusion $(\widehat{\mathbb{L}}(V), d_a) \hookrightarrow (\widehat{\mathbb{L}}(V), d_a) \widehat{\amalg\amalg}_i (\widehat{\mathbb{L}}(u_i', v_i), d_a)$ is a quasi-isomorphism, we can choose $\Omega_j \in \widehat{\mathbb{L}}(W)$.

Let $(x_j)_{j \in \mathcal{A}}$ be the family of the x_j's such that the differential $dx_j = \Omega_j$ has a non-zero linear part Ω_j^1. We set $\mathcal{B} = \{1, \dots, k\} \backslash \mathcal{A}$ and denote by $\mathcal{K}$ the ideal generated by $\{x_j, \Omega_j^1 \mid j \in \mathcal{A}\}$. If V' is a direct summand of $\oplus_{j \in \mathcal{A}} \mathbb{Q}\Omega_j^1$ in V, we have an isomorphism $(\widehat{\mathbb{L}}(V'), d) \cong (\widehat{\mathbb{L}}(V), d)/\mathcal{K}$. From [1, Proposition 2.4], we deduce that the canonical surjection $\rho\colon (\widehat{\mathbb{L}}(V), d) \to (\widehat{\mathbb{L}}(V), d)/\mathcal{K}$ is a quasi-isomorphism. Since the DGL$(\widehat{\mathbb{L}}(V'), d)$ is cofibrant ([3, Proposition 5.5]), we can lift ρ in a quasi-isomorphism

$$\varphi\colon \big(\widehat{\mathbb{L}}(V'), d\big) \widehat{\amalg\amalg}_{j \in \mathcal{A}} \widehat{\mathbb{L}}(x_j, \Omega_j) \longrightarrow \big(\widehat{\mathbb{L}}(V), d\big)$$

and get an isomorphism

$$\mathcal{L}_X \cong \widehat{\mathbb{L}}\big(V' \oplus \oplus_{j \in \mathcal{B}} \mathbb{Q}x_j\big) \widehat{\amalg}\big(\widehat{\amalg}_{j \in \mathcal{A}} \widehat{\mathbb{L}}(x_j, \Omega_j) \widehat{\amalg}_i \widehat{\mathbb{L}}(u_i, v_i)\big). \qquad \blacksquare$$

Lemma 2.2 *Let X be a* 1*-dimensional connected finite simplicial complex; then we have an isomorphism of* cDGL*'s*

$$\mathcal{L}_X \cong \big(\widehat{\mathbb{L}}(V), d\big) \widehat{\amalg} \big(\widehat{\mathbb{L}}(u_i, v_i), dv_i = u_i\big),$$

with $V = \mathbb{Q}a \oplus V_0$, $da = -\frac{1}{2}[a, a]$ and $dx = -[a, x]$ for any $x \in V_0$.

Proof Let x_0 be a vertex of X and let a denote the corresponding Maurer–Cartan element in $\mathcal{L}_X$. By hypothesis, X is a connected finite graph, and we denote by $\mathcal{T}$ a maximal tree in X. For each vertex v_i different from x_0, there is a unique path $\mathcal{P}_{v_i} \in \mathcal{T}$ of minimal length from x_0 to v_i. We remark that each edge in $\mathcal{T}$ is the terminal edge of some path $\mathcal{P}_{v_i}$ for some vertex v_i different from x_0. The vertices v_i correspond to Maurer–Cartan elements a_i in $\mathcal{L}_X$. With each path $\mathcal{P}_{v_i}$ we associate the Baker–Campbell–Hausdorff product p_i of the edges composing this path.

If b_k is an edge that does not belong to $\mathcal{T}$, we denote by v_{k_0} and v_{k_1} its endpoints. If each of them is different from x_0, we form the loop consisting of the path $\mathcal{P}_{v_{k_0}}$ followed by b_k and $(\mathcal{P}_{v_{k_1}})^{-1}$. If $v_{k_0} = x_0$, we form the loop consisting of b_k and $(\mathcal{P}_{v_{k_1}})^{-1}$ and do similarly if $v_{k_1} = x_0$. Then we denote by c_k the Baker–Campbell–Hausdorff product of the edges composing this loop.

From these two constructions, we get a morphism of DGL's

$$f\colon(\mathcal{L}',d) := \big(\widehat{\mathbb{L}}(a,a_i,p_i,c_k),d\big) \longrightarrow \mathcal{L}_X.$$

The map f induces an isomorphism on the indecomposable elements, and thus it is an isomorphism. In $(\mathcal{L}',d)$, for each i, $(\widehat{\mathbb{L}}(a,a_i,p_i),d)$ is a Lawrence-Sullivan interval connecting a to a_i. On the other hand (see [1, Proposition 2.7]), for each k we have $dc_k = -[a,c_k]$.

Recall now from (1.1) that for each i, there is an isomorphism

$$\psi_i\colon\big(\widehat{\mathbb{L}}(a,a_i,p_i),d\big) \longrightarrow \big(\widehat{\mathbb{L}}(a,u_i,v_i),d\big)$$

with $\psi_i(a) = a$, $\psi_i(p_i) = v_i$, $du_i = 0$ and $dv_i = u_i$. The morphisms ψ_i can be pasted together and give an isomorphism

$$\psi\colon(\mathcal{L}',d) \longrightarrow \big(\widehat{\mathbb{L}}(a,u_i,v_i,c_k),d\big)$$

with $dc_k = -[a,c_k]$ and $dv_i = u_i$. Therefore,

$$\mathcal{L}_X \cong \big(\widehat{\mathbb{L}}(V),d\big)\,\widehat{\amalg}\,\big(\widehat{\mathbb{L}}(u_i,v_i),d\big)$$

with $V = \mathbb{Q}a \oplus V_0$ and $dx = -[a,x]$ for any $x \in V_0$. ∎

Corollary 2.3 *Using the notation of Proposition 2.1, we have*

$$\widetilde{MC}(\mathcal{L}_X) = \widetilde{MC}\big(\widehat{\mathbb{L}}(V),d\big).$$

Proof This follows directly from [3, Proposition 2.4]. ∎

3 Maurer–Cartan Elements and Connected Components

Proof of the Theorem Let X be a finite simplicial complex and denote by X_i its connected components for $i = 1,\dots,k$. Then $\mathcal{L}_X = \widehat{\amalg}_{i=1}^k \mathcal{L}_{X_i}$. For each $i = 1,\dots,k$, we have

$$\mathcal{L}_{X_i} \cong \big(\widehat{\mathbb{L}}(V(i)),d\big)\,\widehat{\amalg}\,\big(\widehat{\mathbb{L}}(u_{ij},v_{ij}),d\big),$$

with $d(u_{ij}) = v_{ij}$, and $V(i) = \mathbb{Q}a_i \oplus V(i)_{\geq 0}$ verifies the properties established in Proposition 2.1. Moreover, we deduce from Corollary 2.3 that

$$\widetilde{MC}(\mathcal{L}_X) = \widetilde{MC}\big(\widehat{\amalg}_{i=1}^k(\widehat{\mathbb{L}}(V(i)),d)\big).$$

A Maurer–Cartan element $u \in \mathcal{L}_X$ can be written in the form

$$u = \sum_{i=1}^k \lambda_i a_i + \mu,$$

where μ is a decomposable element and $\lambda_i \in \mathbb{Q}$. From a short computation, we observe that all the numbers λ_i, except at most one, are equal to zero.

• If $\lambda_1 \neq 0$, then $\lambda_1 = 1$ and we set $a = a_1$, $V = V(1)$ and $W = \oplus_{i\geq 2} V(i)$. We denote by E_r the subvector space of $\mathcal{L}_X$ generated by the Lie words containing exactly r elements of $V_{\geq 0}$. The differential d can be written as a series $d = \sum_{i\geq 1} d_i$, with $d_i(V) \subset E_i$. By hypothesis, we have $d_1(v) = -[a,v]$ if $v \in V_{\geq 0}$ and $d_1(w) = 0$ if $w \in W$. Remark now that since a is in degree -1 and $V \oplus W$ is finite dimensional, the ideal $E_{\geq 1}$ generated by $V_{\geq 0}$ is the free complete DGL on the elements $a^r \boxtimes v_k := \mathrm{ad}_a^r(v_k)$

and $a^r \boxtimes w_k := \mathrm{ad}_a^r(w_k)$, where $r \geq 0$, the v_k's run over a graded basis of $V_{\geq 0}$ and the w_k over a graded basis of W. Recall that $v \in V_{\geq 0}$ and $w \in W$. A simple computation gives

$$d_1(a^r \boxtimes v) = \begin{cases} -a^{r+1} \boxtimes v, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd,} \end{cases}$$

$$d_1(a^r \boxtimes w) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ -a^{r+1} \boxtimes w, & \text{if } r \text{ is odd.} \end{cases}$$

The derivation defined by $\theta = -\mathrm{ad}_a - d_1$ verifies that

$$\theta(a^r \boxtimes v) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ -a^{r+1} \boxtimes v, & \text{if } r \text{ is odd,} \end{cases}$$

$$\theta(a^r \boxtimes w) = \begin{cases} -a^{r+1} \boxtimes w, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$$

Clearly, we have $\theta^2 = 0$ and $H(E_{\geq 1}, \theta) = \widehat{\mathbb{L}}(V)$. In particular, $H_{-1}(E_{\geq 1}, \theta) = 0$. We construct a sequence of Maurer–Cartan elements (u_n) such that $u_1 = u$, $u_n - a \in E_{\geq n}$ and $u_n \sim_{O(n)} u_{n+1}$. Suppose u_n has been constructed; then we can write it as

$$u_n = a + \omega_n + \gamma, \quad \text{with} \quad \omega_n \in E_n,\ \gamma \in E_{>n}.$$

Since u_n is a Maurer–Cartan element, we have $d_1(\omega_n) = -[a, \omega_n]$ and $\theta(\omega_n) = 0$. From $H_{-1}(E_{\geq 1}, \theta) = 0$, we deduce the existence of $t \in E_n$ such that $\omega_n = \theta(t)$. This implies that $\omega_n = -[a, t] - d_1(t)$. Recall from (1.1) the morphism

$$\psi\colon \big(\widehat{\mathbb{L}}(a, b, x), d\big) \longrightarrow \big(\widehat{\mathbb{L}}(a, e, c), d\big)$$

and construct a morphism $\mu\colon (\widehat{\mathbb{L}}(a, e, c), d) \to (\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), d)$, by $\mu(a) = u_n$, $\mu(e) = t$ and $\mu(c) = dt$. A short computation gives

$$\mu \circ \psi(b) = a + \gamma', \quad \gamma' \in E_{>n}.$$

The path $\mu \circ \psi$ defines u_{n+1} such that $u_n \sim_{O(n)} u_{n+1}$, and the result follows from Lemma 1.4.

• Suppose now $\lambda_i = 0$ for $i = 1, \dots, k$. We write $u = \sum_{i \geq 1} \omega_i$ with $\omega_i \in E_i$. Since u is a Maurer–Cartan element, we have $d\omega_1 = 0$. From $H(\mathcal{L}_X, d) = 0$, we deduce the existence of ω_1' such that $\omega_1 = d\omega_1'$ and Lemma 1.3 implies $u \sim_{O(1)} u_2$ with $u_2 \in E_{\geq 2}$. With the same process, we get a sequence of Maurer–Cartan elements $u_n \in E_{\geq n}$ such that $u_n \sim_{O(n)} u_{n+1}$. Finally, Lemma 1.4 gives $u \sim 0$. ∎

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