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DENOMINATORS IN LUSZTIG'S ASYMPTOTIC HECKE ALGEBRA VIA THE PLANCHEREL FORMULA

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Abstract Let W_{aff} be an extended affine Weyl group, $\mathbf{H}$ be the corresponding affine Hecke algebra over the ring $\mathbb{C}[\mathbf{q}^{\frac{1}{2}},\mathbf{q}^{-\frac{1}{2}}]$, and J be Lusztig's asymptotic Hecke algebra, viewed as a based ring with basis $\{t_w\}$. Viewing J as a subalgebra of the $(\mathbf{q}^{-\frac{1}{2}})$-adic completion of $\mathbf{H}$ via Lusztig's map ϕ, we use Harish-Chandra's Plancherel formula for p-adic groups to show that the coefficient of T_x in t_w is a rational function of $\mathbf{q}$, with denominator depending only on the two-sided cell containing w, and dividing a power of the Poincaré polynomial of the finite Weyl group. As an application, we conjecture that these denominators encode more detailed information about the failure of the Kazhdan-Lusztig classification at roots of the Poincaré polynomial than is currently known.

Along the way, we show that upon specializing $\mathbf{q} = q > 1$, the map from J to the Harish-Chandra Schwartz algebra is injective. As an application of injectivity, we give a novel criterion for an Iwahori-spherical representation to have fixed vectors under a larger parahoric subgroup in terms of its Kazhdan-Lusztig parameter.

Contents

Keywords: Asymptotic Hecke algebra; Iwahori-Hecke algebra; Plancherel formula; parahoric subgroup

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1. Introduction

Let W_{aff} be an affine Weyl group or extended affine Weyl group, and let $\mathbf{H}$ be its associated Hecke algebra over $\mathcal{A} := \mathbb{C}[\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}]$, where $\mathbf{q}$ is a formal variable. The representation theory of $\mathbf{H}$ is very well understood, behaving well and uniformly when $\mathbf{q}$ is specialized to any $q \in \mathbb{C}^\times$ that is not a root of the Poincaré polynomial P_W of the finite Weyl group $W \subset W_{\text{aff}}$.

When $\mathbf{q}$ is specialized to a prime power q, the category of finite-dimensional modules over the specialized algebra H is equivalent to the category of admissible representations with nonzero Iwahori-fixed vector of some p-adic group. A form of local Langlands correspondence, the Deligne-Langlands conjecture, has been established by Kazhdan and Lusztig in [25], where they classified modules over the generic algebra $\mathbf{H}$ using algebraic K-theory. A slightly different approach to this classification due to Ginzburg is explained in [12]. In both treatments, a first step is to fix a central character. In particular, one must choose a complex number $q \in \mathbb{C}^\times$ by which $\mathbf{q}$ will act. Decomposing the K-theory of certain subvarieties of Springer fibres into irreducible representations of a certain finite group yields the *standard modules*. It can happen that the standard modules are themselves simple (for example, simple tempered representations, which play an essential role in the present paper, are of this form), but in general simple modules are obtained as a certain unique nonzero quotient of standard modules. This quotient exists when $q \in \mathbb{C}^\times$ is not a root of unity, but can be zero otherwise. Lusztig conjectured in [30] that this classification would in fact hold whenever q was not a root of P_W, and this result was proven by Xi in [52]. Xi also showed that the classification fails in general at roots of the Poincaré polynomial, and presented this failure by giving an example related to a lack of simple $\mathbf{H}|_{\mathbf{q}=q}$-modules attached to the lowest two-sided cell. Our results in this paper explain that the lowest two-sided cell is, in a precise sense, maximally singular with respect to the parameter $\mathbf{q}$.

One way Lusztig expressed the uniformity in q of the representation theory of the various algebras $\mathbf{H}|_{\mathbf{q}=q}$ is via the asymptotic Hecke algebra J. This is a $\mathbb{C}$-algebra (in fact, a $\mathbb{Z}$-algebra) J with distinguished basis $\{t_w\}_{w \in W_{\text{aff}}}$, equipped with an injection $\phi\colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{C}} \mathcal{A}$. In this way there is a map from J-modules to $\mathbf{H}$-modules, and Lusztig has shown in [29] and [17] that when q is not a root of unity (other than 1), the specialized map ϕ_q induces a bijection between simple $\mathbf{H}|_{\mathbf{q}=q}$-modules and simple J-modules, these last being defined over $\mathbb{C}$. Moreover, he showed that when $P_W(q) \neq 0$, the map ϕ_q induces an isomorphism

$$(\phi_q)_*\colon K_0(J - \mathbf{Mod}) \to K_0(\mathbf{H}|_{\mathbf{q}=q} - \mathbf{Mod})$$

of Grothendieck groups. The map ϕ becomes a bijection after completing $\mathbf{H}$ and $J \otimes_{\mathbb{C}} \mathcal{A}$ by replacing $\mathcal{A}$ with $\mathbb{C}((\mathbf{q}^{-1/2}))$ and allowing infinite sums convergent in the $(\mathbf{q}^{-1/2})$-adic topology. In this way one can write a basis element t_w as an infinite sum

$$^\dagger(-) \circ \phi^{-1}(t_w) = \sum_{x \in W_{\text{aff}}} a_{x,w} T_x, \tag{1.1}$$

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where each $a_{x,w}$ is a formal Laurent series in $\mathbf{q}^{-1/2}$, and ${}^\dagger(-)$ is the involution of $\mathbf{H}$ defined in Definition 1.6. It agrees with the Goldman involution of $\mathbf{H}$ when G is simply connected. In this paper we will almost exclusively work with $\phi \circ {}^\dagger(-)$, for reasons explained in Section 1.1.

In light of the above, it is natural to ask how $a_{x,w}$ behaves when $\mathbf{q}$ is specialized to a root of unity.

1.1. The asymptotic Hecke algebra and p-adic groups

This paper is prompted by the work of Braverman and Kazhdan in [11], who related the asymptotic Hecke algebra to harmonic analysis on p-adic groups. Specifically, in [17], Lusztig relates simple J-modules to certain $\mathbf{H} \otimes_{\mathcal{A}} \mathbb{C}(\mathbf{q}^{-1/2})$-modules termed *tempered* because their definition is made in analogy with Casselman's criterion for temperedness of p-adic groups. In [11], Braverman and Kazhdan showed essentially that the analytic meaning of the word 'tempered' can be substituted into Lusztig's results from [17].

More precisely, let $\mathbf{G}$ be a connected reductive group defined and split over a non-archimedean local field F whose extended affine Weyl group is W_{aff}. Then in [11], Braverman and Kazhdan define a map expressing J as sitting between the Iwahori-Hecke algebra of $G = \mathbf{G}(F)$ and the Harish-Chandra Schwartz algebra $\mathcal{C}$, and propose a spectral characterization of J via the operator Paley-Wiener theorem, obtaining a diagram

$$\begin{array}{ccccc} H(G,I) & & \hookrightarrow & & \mathcal{C}^{I\times I} \\ & \overset{\phi_q \circ {}^\dagger(-)}{\searrow} & & \overset{\tilde{\phi}}{\nearrow} & \\ \Big\downarrow \sim & & J & & \Big\downarrow \sim \\ & & \Big\downarrow \eta & & \\ \mathcal{E}^I & \hookrightarrow & \mathcal{E}^I_J & \hookrightarrow & \mathcal{E}^I_t, \end{array} \tag{1.2}$$

where the outer vertical maps are Fourier transform $f \mapsto \pi(f)$ and the rings $\mathcal{E}^I$ and $\mathcal{E}^I_t$ of endomorphisms of forgetful functors to vector spaces are as described by the operator Paley-Wiener theorem, as we recall in Section 2.3.

The map η is defined in [11] and we will recall its definition below. In particular, it induces the map $\tilde{\phi}$, which then associates a Harish-Chandra Schwartz function to every element of J such that $\eta(j) = \pi(\tilde{\phi}(j))$, giving another way of associating to t_w an expression similar to (1.1).

This prompts several questions: whether η (equivalently $\tilde{\phi}$) is injective, whether it is surjective, and the nature of the relationship between the Schwartz function $\tilde{\phi}(t_w)$ and the expression (1.1).

1.1.1. Denominators in the affine Hecke algebra and injectivity of η. In the first part of this paper we prove that η is injective. Along the way, we prove in Proposition 3.18 that $\tilde{\phi}$ is essentially the map ϕ^{-1}. Our strategy is to determine that the Schwartz functions f_w on the p-adic group G satisfy the statements of Theorem 1.2

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below, and are in addition sufficiently well-behaved so as to lift to elements of a certain completion $\boldsymbol{\mathcal{H}}^-$ of $\mathbf{H}$, thus defining a map

$$\phi_1 \colon J \otimes_{\mathbb{C}} \mathcal{A} \to \boldsymbol{\mathcal{H}}^-.$$

We therefore obtain two maps $J \subset J \otimes_{\mathbb{C}} \mathcal{A} \to \boldsymbol{\mathcal{H}}^-$: the inverse ${}^{\dagger}(-) \circ \phi^{-1}$ of Lusztig's map, and our map ϕ_1 induced by the construction in [11]. We prove in Proposition 3.18 that these maps agree, at which point Theorem 3.21 and the first statement of Theorem 1.2 follow.

In particular, ϕ_1 is injective. Using that the representation theory of J is sufficiently independent of q, we then show in Corollary 3.20 that $\tilde{\phi}$ is injective for any $q > 1$. This is obviously equivalent to

Theorem 1.1 (Corollary 3.20). *The map η is injective for any $q > 1$.*

A weaker form of the following Theorem, which is the main result of this paper, was conjectured by Kazhdan.

Theorem 1.2. *Let W_{aff} be an affine Weyl group, $\mathbf{H}$ its affine Hecke algebra over $\mathcal{A}$, and J its asymptotic Hecke algebra. Let $\phi \colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{C}} \mathcal{A}$ be Lusztig's map.*

1. *For all $x, w \in W_{\mathrm{aff}}$, $a_{x,w}$ is a rational function of $\mathbf{q}$. The denominator of $a_{x,w}$ is independent of x. As a function of w, it is constant on two-sided cells.*
2. *There exists $N_{W_{\mathrm{aff}}} \in \mathbb{N}$ such that upon writing*

$$^{\dagger}(-) \circ \phi^{-1}(t_w) = \sum_{x \in W_{\mathrm{aff}}} a_{x,w} T_x,$$

we have

$$P_W(\mathbf{q})^{N_{W_{\mathrm{aff}}}} a_{x,w} \in \mathcal{A}$$

for all $x, w \in W_{\mathrm{aff}}$.

3. *If d is a distinguished involution in the lowest two-sided cell, then $a_{1,d} = 1/P_W(\mathbf{q})$ exactly.*

In [14], the author proved Theorem 1.2 in type $\tilde{A}_1$, but with different conventions. To translate to the conventions of this paper, the reader should replace j with the involution ${}^{\dagger}(-)$, and the completion of $\mathbf{H}$ with respect to the C_w basis and positive powers of $\mathbf{q}^{1/2}$ with the completion of $\mathbf{H}$ with respect to the basis $\{(-1)^{\ell(w)} C'_w\}_{w \in W_{\mathrm{aff}}}$ and negative powers of $\mathbf{q}^{1/2}$. Note also that we write $a_{x,w}$ instead of $a_{w,x}$ as in [14]. In [36], Neunhöffer described the coefficients $a_{x,w}$ for finite Weyl groups.

In future, it would be desirable to also treat the case of unequal parameters, where a result like that of [7] governing denominators is not yet available. On the other hand, when $G = \mathrm{GL}_n$, we are able to be slightly more precise than Theorem 1.2, while also not appealing to [7]. For this reason we treat the case $G = \mathrm{GL}_n$ separately as Theorem 3.21 in Section 3.4.

Our main tool is Harish-Chandra's Plancherel formula for the p-adic group G associated to $\mathbf{H}$ and the surjection of cocentres induced by Lusztig's map ϕ after inverting $P_W(\mathbf{q})$

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proved in [7][1]. We invoke [7] only at the very end of our argument, which, absent [7], still proves that $a_{x,w}$ are rational functions with denominator depending only on the two-sided cell containing w. We do so with an eye to future work dealing with Hecke algebras with unequal parameters.

In [14], the author related a conjecture of Kazhdan concerning the positivity of some coefficients related to the coefficients $a_{x,w}$. Historically, proofs of such positivity phenomena have also provided interpretations of the positive quantities in question. While we cannot currently prove the conjecture in [14], our results in Section 4 hint at a possible interpretation of $a_{1,d}$ for certain distinguished involutions d.

Remark 1.3. It is tempting to conjecture the following more precise version of Theorem 1.2, based on the factorization $P_{\mathbf{M_P}}(\mathbf{q})P_{\mathbf{G/P}}(\mathbf{q}) = P_{\mathbf{G/B}}(\mathbf{q})$: for every Levi subgroup $\mathbf{M_P}$ of $\mathbf{G}$ and all $\omega \in \mathcal{E}_2(M_P)$, the formal degree $d(\omega)$ is a rational function of q the denominator of which divides a power of $P_{\mathbf{M_P}}(q)$. The integral over all induced twists $\mathrm{Ind}_P^G(\nu\otimes\omega)$ is a rational function of q with denominator dividing a power of the Poincaré polynomial of the partial flag variety $(\mathbf{G/P})(\mathbb{C})$. For example, if $G = \mathrm{GL}_6(F)$ and $M = \mathrm{GL}_3(F)\times\mathrm{GL}_3(F)$. In this case the integral itself (omitting the factor C_M in the notation of Section 2.5) is

$$\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(z_1-z_2)(z_1-z_2)}{(z_1-q^3z_2)(z_1-q^{-3}z_2)}\frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2} = \frac{(1-q^3)^2}{(1-q^6)^2q^3}+1 = \frac{(1-q^3)q^{-3}}{1+q^3}+1,$$

and by [10] Proposition 23.1, (with $\mathbf{q}=t^2$) we have

$$P_{\mathbf{G/P}}(\mathbf{q}) = P_{G(3,6)}(\mathbf{q}) = (1+\mathbf{q}^2)(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3+\mathbf{q}^4)(1+\mathbf{q}^3).$$

In examples such as the above, this does indeed happen, but only after cancellation with some terms in the numerator. In general, we will not track numerators precisely enough to show this version of Theorem 1.2. We shall however see a limited demonstration of this behaviour in Corollary 3.3.

1.1.2. Denominators in the affine Hecke and the Kazhdan-Lusztig classification at roots of unity.

The affine Hecke has a filtration by two-sided ideals

$$\mathbf{H}^{\geq i} = \mathrm{span}\{C_w \,|\, a(w)\geq i\},$$

where a is Lusztig's a-function. As such, for any $q\in\mathbb{C}^\times$ and any simple $H=\mathbf{H}|_{\mathbf{q}=q}$-module M there is an integer $a(M)$ such that $H^{\geq i}M\neq 0$ but $H^{\geq i+1}M=0$. Define $a(M)=i$ to be this integer. One can also define $a(E)=a(\mathbf{c}(E))$ where E is a simple J-module and $J_{\mathbf{c}(E)}$ is the unique two-sided ideal not annihilating E.

The algebra J linearizes the above filtration into an honest direct sum and implements the almost-independence on $q\in\mathbb{C}^\times$ of the representation theory of $H=\mathbf{H}|_{\mathbf{q}=q}$ as follows.

[1] In fact, as proved by Bezrukavnikov-Braverman-Kazhdan in the appendix of *loc. cit.*, ϕ_q induces an isomorphism whenever q is not a root of unity, but we will not need this.

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Theorem 1.4 [52].

1. *Suppose that q is not a root of the Poincaré polynomial of $\mathbf{G}$. Then for each simple J-module E, the H module ${}^{\phi_q}E$ has a unique simple quotient L such that $a(E) = a(L)$. For all other simple subquotients L' of E, we have $a(L') < a(E)$.*
   Equivalently, for all admissible triples (u,s,ρ), the representation $K(u,s,\rho,q)$ of H has a unique nonzero simple quotient $L = L(u,s,\rho,q)$ such that $a(L) = a(\mathbf{c}(u))$. That is, the Deligne-Langlands conjecture is true for $\mathbf{H}|_{\mathbf{q}=q}$.
2. *If q is a root of the Poincaré polynomial of $\mathbf{G}$, then the Deligne-Langlands conjecture is false for the lowest cell. That is, if $u = \{1\}$, then every simple subquotient L' of $K(u,s,\rho,q)$ has $a(L') < a(\mathbf{c}_0)$.*

By Theorem 1.2, the coefficients $a_{1,d}$ have poles at every root of P_W, for all distinguished involutions d in the lowest two-sided cell $\mathbf{c}_0$. On the other hand, as we show in Example 3.10, there do exist cells $\mathbf{c} \neq \mathbf{c}_0$ such that the coefficients $a_{x,w}$ are nonsingular at certain roots of P_W, for all $w \in \mathbf{c}$ and $x \in W_{\mathrm{aff}}$. We encode the hope that this is no accident as

Conjecture 1.5. *Let $\tilde{W}$ be an affine Weyl group, and let $q \in \mathbb{C}^\times$ be a root of P_W. Let $\mathbf{c}$ be a two-sided cell such that if $w \in \mathbf{c}$, then $a_{x,w}$ does not have a pole at $\mathbf{q} = q$ for any $x \in W_{\mathrm{aff}}$. Let $u = u(\mathbf{c})$. Let $K(u,s,\rho)$ be a standard module in the notation of [25]. Then the module ${}^{\dagger *}K(u,s,\rho,q)$ (see Definition 1.6 (a), (b) and the discussion following Theorem 2.7) has a unique simple quotient $L = L(u,s,\rho,q)$ such that $a(L) = a(E)$, where E is the simple J module corresponding via ϕ to (u,s,ρ) under [17, Thm. 4.2]. Two such simple modules are isomorphic if and only if their corresponding triples are conjugate.*

Note that in type $\tilde{A}_n$, the number of two-sided cells grows as $e^{\sqrt{n}}$, whereas the number of subsets of roots of P_W is $2^{n(n+1)/2}$. For example in type $\tilde{A}_1$, there is only one root of $P_W = \mathbf{q}+1$, but there are two two-sided cells (and both are singular at $\mathbf{q} = -1$.) However, already in type $\tilde{A}_3$, one can see from Theorem 3.2 that the two-sided cell corresponding to the partition $4 = 2+2$ is not singular at two of the roots of $P_{A_3}(\mathbf{q}) = (1+\mathbf{q})(1+\mathbf{q}+\mathbf{q}^2)(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3)$; see Example 3.10.

1.1.3. Application: representations with parahoric-fixed vectors. In Section 4, we use the existence of the action of the asymptotic Hecke algebra on tempered G-representations to give a simple criterion for the existence of vectors fixed under a parahoric subgroup of G:

Theorem 4.1. *Let $\pi = K(u,s,\rho)$ be a simple tempered I-spherical representation of G. Let $\mathcal{P}$ be a parahoric subgroup of G and let $w_{\mathcal{P}}$ be the longest element in the corresponding subgroup of W_{aff}. Let $\mathcal{B}_u^\vee$ be the Springer fibre for u.*

1. *If $\ell(w_{\mathcal{P}}) > \dim \mathcal{B}_u^\vee$, then $\pi^{\mathcal{P}} = \{0\}$.*
2. *Conversely, let $u_{\mathcal{P}}$ be the unipotent conjugacy class corresponding to the two-sided cell containing $w_{\mathcal{P}}$. Then there exists a semisimple element $s \in Z_{G^\vee}(u_{\mathcal{P}})$, a Levi subgroup*

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$M^\vee$ of $G^\vee$ minimal such that $(u_{\mathcal{P}},s) \in M^\vee$, and a discrete series representation $\omega \in \mathcal{E}_2(M)$ such that

$$\pi^{\mathcal{P}} = i_{P_M}^G(\omega \otimes \nu)^{\mathcal{P}} \neq \{0\}$$

for all ν non-strictly positive and the parameter of π is $(u_{\mathcal{P}},s)$.

Thus starting from the regular unipotent class, $\pi(t_{w_{\mathcal{P}}}) = 0$ until reaching the unipotent attached to $w_{\mathcal{P}}$. At this unipotent, $\mathcal{P}$-fixed vectors are first encountered, and $t_{w_{\mathcal{P}}}$ acts by a nonzero projector with image contained in $\pi^{\mathcal{P}}$. For lower cells, it may still be the case that $\pi^{\mathcal{P}} \neq 0$, but $t_{w_{\mathcal{P}}}$ will act by zero on such representations, too. Therefore the nonzero action of $t_{w_{\mathcal{P}}}$ detects precisely the 'most regular' unipotent attached to $\mathcal{P}$-spherical representations, in the sense that if a representation π such that $\pi^{\mathcal{P}} \neq 0$ has the unipotent part of its parameter equal to u, then $a(u) \geq a(u_{\mathcal{P}})$. In this way the distinguished involutions $t_{w_{\mathcal{P}}}$ are more exact versions of the corresponding indicator functions $1_{\mathcal{P}}$, at the expense of being more complicated to understand.

Remark 1.6. Recall from Section 1.1.2 that for every simple H-module M there is a number $a(M)$ such that $H^{\geq i}M = 0$ for all $i > a(M)$. However, if (u,s,ρ) is the KL-parameter of M, then without knowing that M extends to a simple J-module, it does not follow that $a(M) = \dim \mathcal{B}_u^\vee$.

Remark 1.7. By [17, Theorem 4.8(d)] and the proof of [17, Lemma 5.5], every two-sided cell contains a distinguished involution contained in a finite parabolic subgroup of W_{aff}, but not every distinguished involution of a finite Coxeter group is the longest word of a parabolic subgroup, *i.e.* is of the form $w_{\mathcal{P}}$; approximately half of two-sided cells of the finite Weyl group $W \subset W_{\mathrm{aff}}$ do not contain any distinguished involutions contained in proper parabolic subgroups, because of the cell-preserving bijection $w \mapsto w_0 w$. For example, this happens for the second-lowest cell for E_8.

The existence of parahoric-fixed vectors is a rigid question, in the sense of the rigid cocentre of Ciubotaru-He [13]. We investigate this connection further in forthcoming work.

Some time after completing the present paper, we became aware of [19], which also studies the connection between the asymptotic Hecke algebra and the Plancherel theorem in type $\tilde{G}_2$, for unequal parameters. In *op. cit.* the authors speculate that the 'asymptotic Plancherel measure' of *op. cit.* should be related to the perspective of [11] on J. We defer investigation of this to future work, but note that in light of both the classic work [35] of Morris, and recent work [47] of Solleveld, the unequal parameters case is relevant even to split p-adic groups. In particular, establishing results similar to those of the present paper for unequal parameters may provide an effective way to study the algebra $\mathcal{J}$ of Braverman and Kazhdan given in Definition 1.9 of [11].

1.2. Outline of the argument

This paper is organized according to our strategy for proving Theorems 1.2, Theorem 3.21, and Corollary 3.20.

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These results are each simple corollaries of computations with the Plancherel formula and some of Lusztig's results on J. The remainder of this section will introduce $\mathbf{H}$ and J precisely, and recall their basic representation theory. In Section 2, we introduce Harish-Chandra's Plancherel formula in detail, along with all the numerical constants that appear in it. In Section 2.4, we recall the results of Braverman-Kazhdan from [11]. There is no original material in the first two sections. In Section 2.8, we prepare to apply the Plancherel formula by proving that, if f_w is the Schwartz function associated by Braverman-Kazhdan to t_w, and π is a tempered representation, then $\operatorname{trace}(\pi, f_w)$ is sufficiently regular so as not to complicate the denominators of $a_{x,w}$. This section is also mostly a recollection of standard material, the only original result being Lemma 2.17.

In Section 3, we prove most of our main results. As we are able to be more precise in type $\tilde{A}_n$, we perform each step in parallel for type $\tilde{A}_n$ and for other types: in Sections 3.1 and 3.2 we prove statements like those of Theorem 1.2 for the Schwartz functions f_w. In these sections $\mathbf{q}$ is specialized to a prime power q. In Section 3.3 we relate the functions f_w to the basis elements t_w, turning statements that hold for all prime powers q into statements that hold for the formal variable $\mathbf{q}$. We are then able to prove our main results.

In Section 4, we state our application about the existence of parahoric-fixed vectors.

1.3. The affine Hecke algebra

Let F be a non-archimedean local field, $\mathcal{O}$ its ring of integers and ϖ be a uniformizer. Let q be the cardinality of the residue field. Then $q = p^r$ is a prime power. We write $|\cdot|_F$ for the p-adic absolute value on F; when necessary, $|\cdot|_\infty$ will denote the archimedean absolute value on $\mathbb{C}$.

Let $\mathbf{G}$ be a connected reductive algebraic group defined and split over F, $\mathbf{A}$ a maximal F-split torus of $\mathbf{G}$, and $X_* = X_*(\mathbf{A})$ the cocharacter lattice of $\mathbf{A}$. Let $\pi_1(G) = X_*/\mathbb{Z}\Phi^\vee$ be the fundamental group, the quotient of the cocharacter lattice by the coroot lattice. Let $\mathbf{N}$ be unipotent radical of a chosen Borel subgroup $\mathbf{B}$, so that $\mathbf{B} = \mathbf{AN}$. Let W be the finite Weyl group of $\mathbf{G}$, and $W_{\text{aff}} = W \ltimes X_*(\mathbf{A})$ be the extended affine Weyl group. Write S for the set of simple reflections in W_{aff}. Let $\mathbf{G}^\vee$ be the Langlands dual group of $\mathbf{G}$, taken over $\mathbb{C}$. We write $G = \mathbf{G}(F)$, $A = \mathbf{A}(F)$, etc. Where there is no danger of confusion, we also write $G^\vee$ for $\mathbf{G}^\vee(\mathbb{C})$, $M^\vee$ for $\mathbf{M}^\vee(\mathbb{C})$, etc. Let K be the maximal compact subgroup $\mathbf{G}(\mathcal{O})$. Also let I be the Iwahori subgroup of G that is the preimage of $\mathbf{B}(\mathbb{F}_q)$ in K. We sometimes write $P_{\mathbf{G}/\mathbf{B}}$ for P_W, as this polynomial is also the Poincaré polynomial of the flag variety $(\mathbf{G}/\mathbf{B})(\mathbb{C})$.

We write $\mathbf{H}$ for the affine Hecke algebra of W_{aff}. It is a unital associative algebra over the ring $\mathcal{A} = \mathbb{C}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ (in fact, it is defined over $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ but we will work over $\mathbb{C}$ to avoid having to introduce extra notation later), where $\mathbf{q}^{\frac{1}{2}}$ is a formal variable. We will think of $\mathbb{C}^\times$ as $\operatorname{Spec}\mathcal{A}$. The algebra $\mathbf{H}$ has the Coxeter presentation with standard basis $\{T_w\}_{w\in\tilde{W}}$ with $T_wT_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and quadratic relation $(T_s+1)(T_s - \mathbf{q}) = 0$ for $s \in S$. We write θ_λ for the generators of the Bernstein subalgebra.

Recall from [24] the two Kazhdan-Lusztig bases $\{C_w\}_{w\in W_{\text{aff}}}$ and $\{C'_w\}_{w\in W_{\text{aff}}}$ of $\mathbf{H}$, where

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$$C'_w = \mathbf{q}^{-\frac{\ell(w)}{2}} \sum_{x \le w} P_{x,w}(\mathbf{q}) T_x$$

for the Kazhdan-Lusztig polynomials $P_{x,w}$. Write $C_x C_y = \sum_{z \in W_{\mathrm{aff}}} h_{x,y,z} C_z$. The *inverse Kazhdan-Lusztig polynomials* $Q_{y,x}$ are the unique family of polynomials satisfying

$$T_x = \sum_{y \le x} (-1)^{\ell(x)-\ell(y)} \mathbf{q}^{\frac{\ell(y)}{2}} Q_{y,x}(\mathbf{q}) C'_y,$$

or equivalently, satisfying

$$\sum_{z \le y \le x} (-1)^{\ell(x)-\ell(y)} Q_{y,x}(\mathbf{q}) P_{z,y}(\mathbf{q}) = \delta_{z,x}$$

along with some restrictions on their degrees. For example, we shall use in Section 3.3.2 that $\deg Q_{y,x} \le \frac{1}{2}(\ell(x) - \ell(y) - 1)$. See [9] for further exposition.

If

$$\varphi \colon (W_{\mathrm{aff}}, S) \to (W_{\mathrm{aff}}, S)$$

is a Coxeter group automorphism of W_{aff}, then

$$T_w \mapsto T_{\varphi(w)}$$

is an algebra automorphism of $\mathbf{H}$ commuting with the bar involution, and therefore given equivalently by

$$C'_w \mapsto C'_{\varphi(w)}$$

and

$$C_w \mapsto C_{\varphi(w)}. \tag{1.3}$$

It is well-known that there is an isomorphism of associative $\mathbb{C}$-algebras

$$\mathbf{H}|_{\mathbf{q}=q} := \mathbf{H} \otimes_{\mathcal{A}} \mathbb{C} \to C_c^\infty(G)^{I \times I} =: H,$$

where $\mathbf{q}$ acts on $\mathbb{C}$ by multiplication by q.

1.4. The asymptotic Hecke algebra

Definition 1.1. *Lusztig's* a*-function* $a \colon W_{\mathrm{aff}} \to \mathbb{Z}_{\ge 0}$ is defined such that $a(w)$ is the minimal value such that $\mathbf{q}^{\frac{a(w)}{2}} h_{x,y,w} \in \mathcal{A}^+ = \mathbb{C}[\mathbf{q}^{1/2}]$ for all $x, y \in \tilde{W}$.

The a-function is constant on two-sided cells of W_{aff}. Obviously, $a(1) = 0$, and the a-function obtains its maximum, equal to the number of positive roots, on the two-sided cell containing the longest word $w_0 \in W$. In general, under the bijection between two-sided cells $\mathbf{c}$ of W_{aff} and unipotent conjugacy classes $u = u(\mathbf{c})$ in $G^\vee$ of [17], we have

$$a(\mathbf{c}) = \dim_{\mathbb{C}}(\mathcal{B}_u^\vee),$$

where $\mathcal{B}_u^\vee$ is the Springer fibre of u. We have $a(w) \le \ell(w)$ for all $w \in W_{\mathrm{aff}}$.

In [28], Lusztig defined an associative algebra J over $\mathbb{C}$ equipped with an injection $\phi \colon \mathbf{H} \hookrightarrow J \otimes_{\mathbb{C}} \mathcal{A}$ which becomes an isomorphism after taking a certain completion, to be

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recalled in Section 3.3.1, of both sides. As a $\mathbb{C}$-vector space, J has basis $\{t_w\}_{w\in W_{\mathrm{aff}}}$. The structure constants of J are obtained from those of $\mathbf{H}$ written in the $\{C_w\}_{w\in W_{\mathrm{aff}}}$-basis under the following procedure: first, the integer $\gamma_{x,y,z}$ is defined by the condition

$$\mathbf{q}^{\frac{a(z)}{2}} h_{x,y,z^{-1}} - \gamma_{x,y,z} \in \mathbf{q}\mathcal{A}^+.$$

One then defines

$$t_x t_y = \sum_{z\in W_{\mathrm{aff}}} \gamma_{x,y,z} t_{z^{-1}}.$$

Surprisingly, this defines a unital associative algebra with unit

$$1_J = \sum_{d\in\mathcal{D}} t_d,$$

where $\mathcal{D}$ is the (finite) set of *distinguished involutions* [28]. The elements t_d are orthogonal idempotents in J, which decomposes as a direct sum indexed by the two-sided cells of W_{aff} in the sense of [27]. Each left cell, again in the sense of *op. cit.*, contains a single distinguished involution which is the unit in the ring $t_d J t_d$. If $\mathbf{c}$ is a two-sided cell, then $J_{\mathbf{c}}$ is a is a unital ring with unit

$$1_{J_{\mathbf{c}}} = \sum_{d\in\mathcal{D}\cap\mathbf{c}} t_d.$$

Lusztig further defined a map of algebras

$$\phi\colon \mathbf{H} \to J \otimes_{\mathbb{C}} \mathcal{A}$$

given by

$$\phi(C_w) = \sum_{z\in W,\ d\in\mathcal{D},\ a(z)=a(d)} h_{w,d,z} t_z.$$

Write ϕ_q for the specialization of this map when $\mathbf{q}=q$. It is known [29, Proposition 1.7] that ϕ_q is injective for all $q\in\mathbb{C}^\times$.

Lemma 1.8. *Let*

$$\varphi\colon (W_{\mathrm{aff}}, S) \to (W_{\mathrm{aff}}, S)$$

be a Coxeter group automorphism. Then

1. *The map $t_w \mapsto t_{\varphi(w)}$ defines a based ring automorphism of J, which we also denote φ.*
2. *The map ϕ is φ-linear, in the sense that it commutes with the automorphism from the first statement and the automorphism (1.3).*

Proof. The first statement is [32, 2.2(g)].

For the second statement, note that φ acts on $\mathcal{D}$ and that

$$\phi(C_{\varphi(w)}) = \sum_{d_1\sim_L z_1} h_{\varphi(w),d_1,z_1} t_{z_1}$$

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and

$$\varphi(\phi(C_w)) = \sum_{d_2 \sim_L z_2} h_{w,d_2,z_2} t_{\varphi(z_2)}.$$

Now if $z_1 = \varphi(z_2)$, then $d_1 = \varphi(d_2)$, and

$$h_{\varphi(w),d_1,z_1} = h_{\varphi(w),\varphi(d_2),\varphi(d_2)} = h_{w,d_2,z_2}$$

by (1.3). $\square$

1.4.1. Deformations of the group ring. Upon setting $\mathbf{q} = 1$, $\mathbf{H}|_{\mathbf{q}=1}$ is isomorphic to $\mathbb{C}[W_{\text{aff}}]$, and so $\mathbf{H}$ is a deformation of the group algebra of W_{aff}.

Let, temporarily, W be any finite Coxeter group. Then one can define its Hecke algebra $\mathbf{H}$, an algebra over $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}, \mathbf{q}^{-\frac{1}{2}}]$ which deforms the group ring $\mathbb{Z}[W]$. Let $q \in \mathbb{C}^\times$. For all but finitely many values of q, all roots of unity, the algebras $\mathbf{H}_{\mathbf{q}=q}$ are trivial deformations of $\mathbb{C}[W]$, and hence are all isomorphic. However, this isomorphism requires choosing a square root of q. The affine Hecke algebra provides a canonical isomorphism: away from finitely many q, we have that $\mathbf{H}|_{\mathbf{q}=q}$ is isomorphic to J, and J is defined over $\mathbb{Z}$ (although, as stated above, we will view it as a $\mathbb{C}$- algebra to unburden notation), see [31, Section 20.1 (e)].

Example 1.9. Let $W = \langle 1, s \,|\, s^2 = 1 \rangle$ be the Weyl group of type A_1. The Kazhdan-Lusztig C_w-basis elements are $C_1 = T_1$ and $C_s = \mathbf{q}^{-\frac{1}{2}} T_s - \mathbf{q}^{\frac{1}{2}} T_1$, and $\mathcal{D} = W$ in this case. There are two two-sided cells in W, and one can easily check that

$$\phi \colon C_1 \mapsto t_1 + t_s$$

and

$$\phi \colon C_s \mapsto -\left(\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{-\frac{1}{2}}\right) t_s.$$

Specializing $\mathbf{q} = q$, we see that ϕ becomes an isomorphism whenever $\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) \neq 0$, that is, whenever $q \neq -1$.

1.5. Representation theory of H and J

Recall the classification of finite-dimensional $\mathbf{H}$-modules given in [25]. For an extended exposition with slightly different conventions, we refer the reader to [12]. The primary difference between the setup we require and that of [12] is that we must be able to defer specializing $\mathbf{q}$ until the last possible moment, whereas specializing $\mathbf{q}$ is the first step of the construction as given in [12]. In particular, let $u \in \mathbf{G}^\vee(\mathbb{C})$ be a unipotent element, and $s \in \mathbf{G}^\vee(\mathbb{C})$ be a semisimple element such that $us = su$. Let ρ be an irreducible representation of the simultaneous centralizer $Z_{G^\vee}(s,u)$. The *standard H-module* $K(s,u,\rho)$ is a certain, and in general reducible, H-module defined using the geometry of the flag variety of the Langlands dual group. It may be the zero module; we say that (u,s,ρ) is *admissible* when this does not happen. Having fixed s and u, we say that ρ is admissible if (u,s,ρ) is.

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We now recall an algebraic version of the Langlands classification. As we do not have access to the notion of absolute value of $\mathbf{q}$, the classical definitions of tempered and discrete-series representations of the corresponding p-adic group G are not available to us. However, Kazhdan-Lusztig provide the following algebraic generalization. Let $\mathcal{K} = \mathbb{C}(\mathbf{q}^{-\frac{1}{2}})$ and $\bar{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$. We write $\mathbf{H}_{\bar{\mathcal{K}}}$ for $\mathbf{H} \otimes_{\mathcal{A}} \bar{\mathcal{K}}$ and recall another definition of Lusztig's from [17].

Definition 1.2. Let M be a $H_{\bar{\mathcal{K}}}$-module finite-dimensional over $\bar{\mathcal{K}}$. Say that $m \in M$ is an *eigenvector* if $\theta_x \cdot m = \chi_m(x)m$ for all dominant x in X_*. As χ_m is a character of the coweight lattice, it corresponds to an element $\sigma_m \in \mathbf{A}^\vee(\bar{\mathcal{K}})$ in the sense that, for all cocharacters x of A, we have

$$\chi_m(x) = x(\sigma_m)$$

where x is viewed as a character of $\mathbf{A}^\vee$. Then M is of *constant type* if there is a semisimple element $s' \in \mathbf{G}^\vee(\mathbb{C})$ and a morphism of algebraic groups

$$\phi' \colon \mathrm{SL}_2(\mathbb{C}) \to Z^0_{G^\vee}(s')$$

such that for all eigenvectors m of M, the element σ_m is $\mathbf{G}^\vee(\bar{\mathcal{K}})$-conjugate to

$$\phi'(\mathrm{diag}(\mathbf{q}^{1/2}, \mathbf{q}^{-1/2}))s',$$

where by abuse of notation we have written ϕ' again for the base-change to $\bar{\mathcal{K}}$.

The idea of the name of the definition is that $s' \in \mathbf{G}^\vee(\mathbb{C})$ is a 'constant element' not depending on $\mathbf{q}$.

Next is a generalization of Casselman's criterion, which as such, requires a choice of dominant weights. Following [25] (see the proof of Prop. 1.6 of *loc. cit.*) and [17, Section 1.6], we choose the positive roots to be those occurring in $\mathfrak{g}/\mathfrak{b}$.

Following [17], we choose a morphism of groups $V \colon \bar{\mathcal{K}}^\times \to \mathbb{R}$ such that $V(\mathbf{q}^{\frac{1}{2}}) = 1$ and $V(a\mathbf{q}^{\frac{1}{2}} + b) = 0$ for all $a \in \mathbb{C}$, $b \in \mathbb{C}^\times$.

Definition 1.3 [17], c.f. [25]. Let M be any finite-dimensional $\mathbf{H}_{\bar{\mathcal{K}}}$-module. We say that M is V-*tempered* if all eigenvalues ν of θ_λ for all dominant $\lambda \in X_*(\mathbf{A})$ satisfy $V(\nu) \leq 0$.

The representation theory of J is very well understood. We shall recall some notation and then state some major classification results of Lusztig, which relate the representation theory of J to certain $\mathbf{H}$-modules defined by Kazhdan-Lusztig.

Definition 1.4. Let E be a J-module. Then $E \otimes_{\mathbb{C}} \mathcal{K}$ is a $J \otimes_{\mathbb{C}} \mathcal{K}$-module. Hence $\mathbf{H}_{\mathcal{K}}$ acts on E via ϕ. Denote the resulting $\mathbf{H}_{\mathcal{K}}$ module by ${}^\phi E$.

1.5.1. Involutions on H. For $x \in W_{\mathrm{aff}}$, let $\omega(x) \in \pi_1(G)$ label the $W \ltimes \mathbb{Z}\Phi^\vee$-coset of W_{aff} containing x, and write $\omega(x) = \omega(x)_f \omega(x)_t \in W \ltimes X_*$. Then every $y \leq x$ is also in the coset of $\omega(x)$.

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Definition 1.5. Let $j\colon \mathbf{H}\to\mathbf{H}$ be the ring (and not $\mathcal{A}$-algebra) involution of $\mathbf{H}$ defined by $j(\sum_w a_w T_w)=\sum_w \bar{a}_w(-1)^{\ell(w)}\mathbf{q}^{-\ell(w)}T_w$.

The j-involution exchanges the $\{C_w\}$ and $\{C'_w\}$-bases [24].

The reason for our choice of conventions, which differ slightly from those of [29] and [17], is the presence of the involution † and its exchange of temperedness and anti-temperedness in the relationship between H-modules and J-modules; see Theorem 1.13 and Lemma 1.10 below.

Definition 1.6.

(a) Define the $\mathcal{A}$-algebra involution $^{\dagger}(-)$ of $\mathbf{H}$ by setting

$$^{\dagger}T_w=(-1)^{\ell(w_f)}\mathbf{q}^{\ell(w)}T_{w^{-1}}^{-1},\quad w=w_f\lambda\in W_{\mathrm{aff}}.$$

Note that the sign factor depends only on w_f.

(b) Let $h\mapsto {}^{*}h$ be the $\mathcal{A}$-involution defined in terms of the Bernstein presentation of $\mathbf{H}$: by

$$^{*}T_w=(-1)^{\ell(w)}\mathbf{q}^{\ell(w)}T_{w^{-1}}^{-1},\quad w\in W$$

and

$$^{*}\theta_\lambda=\theta_\lambda^{-1},\quad \lambda\in X_*.$$

(c) ([40, Section 5.a].) Let $\bar{\kappa}$ be the $\mathcal{A}$-linear involution defined by

$$\bar{\kappa}(T_w)=T_{\bar{\kappa}(w)}$$

induced by the Coxeter group automorphism given by $\bar{\kappa}(s)=w_0sw_0$ for $s\in S\setminus\{s_0\}$ and $\bar{\kappa}(s_0)=s_0$, so that $\bar{\kappa}(\lambda)=-w_0(\lambda)$ for $\lambda\in X_*$; equivalently

$$\bar{\kappa}(T_w)=T_{w_0ww_0^{-1}},\quad w\in W,\quad \bar{\kappa}(\theta_\lambda)=\theta_{-w_0(\lambda)},\quad \lambda\in X_*.$$

(d) ([5, Section 5].) Let $h\mapsto {}^{\bullet}h$ be the $\mathcal{A}$-linear anti-involution defined by

$$^{\bullet}T_w=T_{w^{-1}},\quad w\in W,\quad {}^{\bullet}\theta_\lambda=\theta_\lambda,\quad \lambda\in X_*.$$

(e) ([5, Section 5].) Let $h\mapsto {}^{\star}h$ be the $\mathcal{A}$-linear anti-involution defined by

$$^{\star}T_w=T_{w^{-1}},\quad w\in W,\quad {}^{\star}\theta_\lambda=T_{w_0}\theta_{\bar{\kappa}(\lambda)}T_{w_0}^{-1},\quad \lambda\in X_*.$$

By [38, Prop. 2.9], $^{\star}T_w=T_{w^{-1}}$ for $w\in W_{\mathrm{aff}}$. When G is simply connected, the involution † agrees with the *Goldman involution* of $\mathbf{H}$. Now we have

Lemma 1.10. *We have*

(a) *We have* $^{\dagger}C_x=(-1)^{\ell(\omega(x)_f)}j(C_w)=(-1)^{\ell(x)+\ell(\omega(x)_f)}C'_x$ *for all* $x\in W_{\mathrm{aff}}$.

(b) *We have* $^{\dagger}\theta_\lambda=T_{w_0}\theta_{-\bar{\kappa}(\lambda)}T_{w_0}^{-1}=T_{w_0}\theta_{\bar{\kappa}(\lambda)}^{-1}T_{w_0}^{-1}$.

(c) *We have* $^{\bullet\star\dagger}(-)={}^{*}(-)$ *as involutions on* $\mathbf{H}$.

(d) *We have* $^{\bullet}(-)\circ{}^{\star}(-)=T_{w_0}^{-1}\bar{\kappa}(-)T_{w_0}$ *as automorphisms of* $\mathbf{H}$.

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(e) *We have the commutative diagram*

$$\begin{array}{ccccc} \mathbf{H} & \xrightarrow{{}^{\dagger}(-)} & \mathbf{H} & \xrightarrow{\quad\phi\quad} & J\otimes\mathcal{A} \\ {\scriptstyle {}^{*}(-)}\big\downarrow & & \big\downarrow{\scriptstyle T_{w_0}^{-1}\bar{\kappa}(-)T_{w_0}} & & \big\downarrow{\scriptstyle \phi(T_{w_0})^{-1}\bar{\kappa}(-)\phi(T_{w_0})} \\ \mathbf{H} & \xrightarrow{\ \mathrm{id}\ } & \mathbf{H} & \xrightarrow{\quad\phi\quad} & J\otimes\mathcal{A}. \end{array}$$

Proof. By definition, if $x=\omega x'$ for $x'\in W\ltimes\mathbb{Z}\Phi^\vee$ and $\omega\in\pi_1(G)$, then $C'_x=T_\omega C'_{x'}$ and $C_x=T_\omega C_{x'}$. Further, if $\omega=\omega_f\omega_t\in W_{\mathrm{aff}}$ has $\ell(\omega)=0$, then ${}^{\dagger}T_\omega=(-1)^{\ell(\omega_f)}T_\omega$. Therefore it suffices to show that

$$\overline{{}^{\dagger}h}={}^{\dagger}\left(\bar{h}\right)=j(h)$$

as $\mathcal{A}$-antilinear automorphisms of $\mathbf{H}$ for G simply connected. In this case, $\ell(\lambda)\in 2\mathbb{Z}$ for any dominant $\lambda\in X_*$, whence ${}^{\dagger}(-)$ agrees with the involution $T_w\mapsto(-1)^{\ell(w)}\mathbf{q}^{\ell(w)}T_{w^{-1}}^{-1}$, $w\in W\ltimes\mathbb{Z}\Phi^\vee$. Therefore we compute

$$\overline{\sum_x b_x{}^{\dagger}T_x}=\sum_x\bar{b}_x(-1)^{\ell(x)}\overline{\mathbf{q}^{\ell(x)}\bar{T}_x}=\sum_x\bar{b}_x(-1)^{\ell(x)}\mathbf{q}^{-\ell(x)}T_x=j\left(\sum_x b_xT_x\right)$$

whereas

$${}^{\dagger}\left(\overline{\sum_x b_xT_x}\right)={}^{\dagger}\left(\sum_x\bar{b}_x(-1)^{\ell(x)}\mathbf{q}^{-\ell(x)}{}^{\dagger}T_x\right).$$

Thus we have $\overline{{}^{\dagger}C_w}=(-1)^{\ell(w)}C'_w$ for G simply connected, whence (a).

It suffices to prove (b) for λ dominant, in which case

$$\begin{aligned}{}^{\dagger}\theta_\lambda=\mathbf{q}^{-\frac{\ell(\lambda)}{2}}{}^{\dagger}T_\lambda=\mathbf{q}^{\frac{\ell(\lambda)}{2}}T_{-\lambda}^{-1}=\mathbf{q}^{\frac{\ell(\lambda)}{2}}\left({}^{\star}T_\lambda\right)^{-1}&\\ =\left({}^{\star}\theta_\lambda\right)^{-1}=\left(T_{w_0}\theta_{\bar{\kappa}(\lambda)}T_{w_0}^{-1}\right)^{-1}=T_{w_0}\theta_{\bar{\kappa}(\lambda)}^{-1}T_{w_0}^{-1},&\end{aligned}$$

where we used the equivalence of the definitions in Definition 1.6 (e). This shows (b).

For (c), we again compute, for λ dominant, that

$${}^{\bullet\star\dagger}\theta_\lambda={}^{\bullet\star}\left(T_{w_0}\theta_{-\bar{\kappa}(\lambda)}T_{w_0}^{-1}\right)={}^{\bullet}\theta_{-\lambda}=\theta_{-\lambda}.$$

Agreement on T_w for $w\in W$ follows from the fact that ${}^{\bullet\star}(-)$ is the identity on the finite Hecke algebra. This shows (c).

On the Bernstein subalgebra, part (d) follows from the definitions. On the finite Hecke algebra, we must show that the right hand side is the identity automorphism. For s a finite reflection, write $w_0sw_0=\bar{\kappa}(s)$, again a finite reflection, so that $\bar{\kappa}(s)w_0s=w_0$, and

$$T_{w_0}T_sT_{w_0}^{-1}=T_{w_0}T_sT_s^{-1}T_{w_0s}^{-1}=T_{w_0}T_{w_0s}^{-1}=T_{\bar{\kappa}(s)}T_{w_0s}T_{w_0s}^{-1}=T_{\bar{\kappa}(s)}.$$

Therefore on the finite Hecke algebra, $\bar{\kappa}$ is given by conjugation by T_{w_0}.

Commutativity of the right square in (e) follows from Lemma 1.8, and commutativity of the left square follows from (c) and (d). □

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Given an $\mathbf{H}$ (or $\mathbf{H}_{\mathcal{K}}$)-module M, define ${}^{\dagger}M$ to be the same vector space with the $\mathbf{H}$-action twisted by this involution, and likewise for other automorphisms.

Corollary 1.11. *Twisting by* ${}^{\dagger}(-)$ *exchanges tempered and anti-tempered* H*-modules.*

Proof. Immediate from Lemma 1.10 (b). □

In fact, more is true: by [23, Thm. 2], ${}^{\dagger}M$ is the Aubert-Zelevinski dual of M.

Remark 1.12. There is another natural auto-equivalence of the category of admissible representations of G which exchanges tempered and anti-tempered representations, namely Bernstein's cohomological duality. This functor differs from Aubert-Zelevinski duality by the contragredient [43] (c.f. [37]). On semisimple H-modules π, we have ${}^{\bar{\kappa}}\pi = \tilde{\pi}$, by [40, Prop. 6.3]. Therefore twist by the involution ${}^{*}(-)$, which manifestly exchanges tempered and anti-tempered representations, induces on semisimple modules the cohomological duality. However, it does not do so in general, because $\bar{\kappa}$ does not induce the contragredient in general. Indeed, for $G = \mathrm{SL}_2(F)$, $\bar{\kappa} = \mathrm{id}$ but not all non-unitary principal series of G are self-dual. Therefore the operation ${}^{*}(-)$ is less natural than ${}^{\dagger}(-)$; this is perhaps reflected by the fact that the formulas in Theorem 1.2 are nicer than those for $(\phi \circ {}^{*}(-))^{-1}(t_w)$, which are related to those of the Theorem by Lemma 1.10 (e).

We now summarize the relationship between representations of $\mathbf{H}$ and of J.

Theorem 1.13 ([17], Prop. 2.11, Thm. 4.2, Prop. 4.4). *There are bijections of sets*

$$\begin{array}{ccc}
(u,s,\rho) & & \{(u,s,\rho) \,|\, \rho \text{ admissible, } us = su\}/\mathbf{G}^{\vee}(\mathbb{C}) \\
\downarrow & & \downarrow \\
{}^{*}K(s,u,\rho) \otimes_{\mathcal{A}} \mathcal{K} & & \{M \in \mathbf{H}_{\mathcal{K}} - \mathbf{Mod} \,|\, {}^{*}M \otimes_{\mathcal{K}} \bar{\mathcal{K}} \text{ simple, } V-\text{tempered} \\
\| & & \\
{}^{\phi}E = E \otimes_{\mathbb{C}} \mathcal{K} \in \mathbf{H}_{\mathcal{K}} - \mathbf{Mod} & & \mathbf{H}_{\bar{\mathcal{K}}} - \text{module of constant type}\} \\
\uparrow & & \uparrow \\
E & & \{E \in J - \mathbf{Mod} \,|\, E \text{ is simple}\},
\end{array}$$

where $K(u,s,\rho)$ *is a standard module as in [25]. Moreover, for a simple* J*-module* E*,*

1. E *is finite-dimensional over* $\mathbb{C}$*;*
2. *There is a unique two-sided cell* $\mathbf{c} = \mathbf{c}(E)$ *of* W_{aff} *such that* $\mathrm{trace}(E,t_w) \neq 0$ *implies* $w \in \mathbf{c}$.
3. $\mathrm{trace}(E,t_w)$ *is the constant term of the polynomial*

$$(-\mathbf{q}^{1/2})^{a(\mathbf{c}(E))}\mathrm{trace}(M,C_w) \in \mathbb{C}[\mathbf{q}^{\frac{1}{2}}]$$

where $M \simeq {}^{\phi}E$.

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In particular, $\mathrm{trace}(E, t_w)$ is independent of $\mathbf{q}$, and upon specializing $\mathbf{q} = q$ a prime power, will be a regular function in the twisting character in the setting of the Paley-Wiener theorem for the Iwahori-Hecke algebra of the p-adic group G, as we will explain in greater detail below.

We will comment even further in Section 2.4 on the necessity of twisting by some $\mathbf{H}$-involution exchanging tempered and anti-tempered modules, and how it is sufficient to twist ϕ by either ${}^*(-)$ or ${}^\dagger(-)$, respectively but that the latter twist leads to nicer formulas.

2. Harish-Chandra's Plancherel formula

We recall the notation and classical results we will need about the Plancherel formula. For Iwahori-biinvariant Schwartz functions, the Plancherel formula is known explicitly for all connected reductive groups, and is due to Opdam in [38]. In the case of $G = \mathrm{GL}_n(F)$, we shall refer instead to [4] (where in fact the Plancherel formula is computed explicitly in its entirety for GL_n). In the case $\mathbf{G} = \mathrm{Sp}_4$, we shall refer to the unpublished work [3] of Aubert and Kim. For $\mathbf{G} = G_2$, we will refer to Parkinson [39].

In this section q is a prime power (or at least a real number of absolute value strictly greater than 1). The formal variable $\mathbf{q}$ will not appear in this section.

2.1. Tempered and discrete series representations

We take our definitions of discrete series and tempered representations from [48, III.1] and [48, III.2] respectively. By parabolic induction we always understand normalized induction.

Definition 2.1. A smooth admissible representation ω of G belongs to the *discrete series* if ω admits a unitary central character and all matrix coefficients of ω are square-integrable modulo $Z(G)$. We write $\mathcal{E}_2(G)$ for the space of irreducible discrete series, and $\mathcal{E}_2(G)^I$ for the space of irreducible discrete series with nontrivial Iwahori-fixed vectors.

Let v be the K-fixed vector in the self-contragredient representation $\mathrm{Ind}_B^G(\mathrm{triv})^K$ such that $v(1) = 1$. Define $\Xi(g) = \langle \pi(g)v, v\rangle$ to be the corresponding matrix coefficient.

Definition 2.2. A smooth function f on G is *tempered* if there is $C > 0$ and $r \in \mathbb{R}$ such that

$$|f(g)| \leq C\Xi(g)\left(1 + \log \|g\|\right)^r,$$

where $\|g\| \geq 1$ is defined as in [48, p.242].

Definition 2.3. A smooth admissible representation π of G is *tempered* if all its matrix coefficients are tempered functions in the sense above.

We write $\mathcal{M}_t(G)$ for the category of tempered representations of G. If a tempered representation admits a central character, the central character takes values in the circle group $\mathbb{T} \subset \mathbb{C}^\times$.

The tempered representations are built from the discrete series according to the following theorem of Harish-Chandra, as related in [48, Prop. III.4.1].

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Theorem 2.1 (Harish-Chandra). *Let P be a parabolic subgroup of G with Levi subgroup M, and let $\omega \in \mathcal{E}_2(M)$. Let ν be a unitary character of M. Then $\pi = \mathrm{Ind}_P^G(\omega \otimes \nu)$ is a tempered representation. Every simple tempered representation is a direct summand of a representation of this form.*

2.1.1. Formal degrees of discrete series representations. We will soon study the Plancherel decomposition $f = \sum_M f_M$ of the Schwartz function f determined by an element of J as explained in Section 2.4. As will be explained below, each function f_M is given by an integral formula that involves several constants that depend on the Levi subgroup M, or are functions on the discrete series of M. These constants are rational functions of q, the most sensitive of which is the formal degree $d(\omega)$ of $\omega \in \mathcal{E}_2(M)^I$. Much is known about formal degrees for I-spherical ω; the most general current result seems to be

Theorem 2.2 ([46], [16], Theorem 5.1 (b), [18] Proposition 4.1). *Let $\mathbf{G}$ be connected reductive over F. Let ω be any unipotent – in particular, any Iwahori-spherical – discrete series representation of $G = \mathbf{G}(F)$. Then $d(\omega)$ is a rational function of q, the numerator and denominator of which are products of factors of the form $q^{m/2}$ with $m \in \mathbb{Z}$ and $(q^n - 1)$ with $n \in \mathbb{N}$. Moreover, there is a polynomial Δ_G depending only on G and F such that $\Delta_G d(\omega)$ is a polynomial in q.*

This result is proven by first proving that the Hiraga-Ichino-Ikeda conjecture [22] holds for unipotent discrete series representations. Note that [46], [16] and [22] all use the normalization of the Haar measure on G defined in [22]. This normalization gives in our setting $\mu_{\mathrm{HII}}(K) = q^{\dim \mathbf{G}} \# \mathbf{G}(\mathbb{F}_q)$. Hence, noting that $\#\mathbf{G}(\mathbb{F}_q) = P_{\mathbf{G}/\mathbf{B}}(q) \cdot \#\mathbf{B}(\mathbb{F}_q)$ and that, as $\mathbb{F}_q$ is perfect, $\#B(\mathbb{F}_q)$ is a polynomial in q, we have

$$\mu_I = \frac{1}{q^{\dim \mathbf{G}} \# \mathbf{B}(\mathbb{F}_q)} \mu_{\mathrm{HII}},$$

and so this question of normalization cannot affect the denominators of $d(\omega)$, for any Levi subgroup.

In the Iwahori-spherical case, Opdam showed the above result in [38, Proposition 3.27 (v)], although with less control over the possible factors appearing in the numerator and denominator of $d(\omega)$. We emphasize that *op. cit.* does not make the splitness assumption we allow ourselves.

Remark 2.3. Proposition 4.1 of [18] studies not the γ-factor we are interested in, but rather its quotient by the γ-factor for the Steinberg representation. However, accounting for the use of the Euler-Poincaré measure μ_{HII} on G, and known formula for the formal degree of the Steinberg representation, one may recover our desired statement about formal degrees from the main theorem and equation (61) of [18].

2.2. Harish-Chandra's canonical measure

In this section, we recall the standard coordinates used in [4], and [48]. We follow both references closely. These conventions differ slightly from the original [21]. Everything in

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this section is standard, but we include details because we do require explicit measures with which to compute. We state the below for general $\mathbf{G}$, for application to each standard Levi subgroup of $\mathbf{G}$.

2.2.1. Unramified characters. Let $X^*(\mathbf{G}) = \mathrm{Hom}(\mathbf{G},\mathbb{G}_\mathrm{m})$ denote the rational characters of $\mathbf{G}$ defined over F. Let $\mathfrak{a}_G := (X^*(\mathbf{A}) \otimes_\mathbb{Z} \mathbb{R})^* = (X^*(\mathbf{G}) \otimes_\mathbb{Z} \mathbb{R})^*$ be the real Lie algebra of the maximal split central torus $\mathbf{A} = \mathbf{A}_G$ of $\mathbf{G}$ and let $\mathfrak{a}_{G\mathbb{C}}$ be its complexification. We have a map

$$X^*(\mathbf{G}) \to \mathrm{Hom}(G/G^1,\mathbb{C}^\times) =: \mathcal{X}(G)$$

given by $\chi \mapsto |\chi|_F$, where $|\chi|_F(g) = |\chi(g)|_F$ and $G^1 = \bigcap_{\chi \in X^*(\mathbf{G})} \ker|\chi|_F$. This gives the unramified characters $\mathcal{X}(G)$ a complex manifold structure under which $\mathcal{X}(G) \simeq (\mathbb{C}^\times)^{\dim_\mathbb{R} \mathfrak{a}_G}$. For indeed, we have

$$\mathfrak{a}_{G}{}^*_\mathbb{C} \longrightarrow \mathcal{X}(G) \longrightarrow 1$$

given by

$$\chi \otimes s \mapsto (g \mapsto |\chi(g)|_F^s = q^{-s\mathrm{val}(\chi(g))}).$$

The kernel is spanned by all $\chi \otimes s$ such that $s\mathrm{val}(\chi(g)) \in \frac{2\pi i}{\log q}\mathbb{Z}$ for all $g \in G$. Hence the kernel is $\frac{2\pi i}{\log q}R$, where $R \subset X^*(\mathbf{A}) \otimes_\mathbb{Z} \mathbb{Q}$ is a lattice. In this way the quotient $\mathcal{X}(G)$ is a complex manifold.

2.2.2. Unitary unramified characters. Denote by $\mathrm{Im}\,\mathcal{X}(G)$ the set of unitary unramified characters taking values in the unit circle $\mathbb{T} \subset \mathbb{C}^\times$. This notation is justified, as if $|\chi|_F = |\chi'|_F$, then $\mathrm{Re}(\chi) = \mathrm{Re}(\chi') \in \mathfrak{a}_G^*$. Hence we can define $\mathrm{Im}\,\mathcal{X}(G)$ to be the unramified characters coming from pure imaginary elements of $\mathfrak{a}_{G}{}^*_\mathbb{C}$.

The surjection $\mathrm{Im}\,\mathcal{X}(G) \to \mathrm{Im}\,\mathcal{X}(A)$ has finite kernel, and $\mathrm{Im}\,\mathcal{X}(A)$ is compact. We choose the Haar measure on it with volume one.

2.2.3. Action by twisting and the canonical measure. The group $\mathcal{X}(G)$ acts on admissible representations of G by twisting: $\omega \mapsto \omega \otimes \nu$ for $\nu \in \mathcal{X}(G)$. This restricts to an action of $\mathrm{Im}\,\mathcal{X}(G)$ on $\mathcal{E}_2(G)$.

Pulling this action back to $i\mathfrak{a}_G^*$, and given a representation ω, let L^* be its stabilizer in $i\mathfrak{a}_G^*$, so that the orbit $\mathfrak{o}$ of ω is identified as $i\mathfrak{a}_G{}^*/L^*$. This gives $\mathfrak{o} = \mathrm{Im}\,\mathcal{X}(G)\cdot\omega$ the structure of a real submanifold of the larger orbit $\mathfrak{o}_\mathbb{C} = \mathfrak{a}_{G}{}^*_\mathbb{C}/L^* = \mathcal{X}(G)\cdot\omega$.

Definition 2.4. Given an orbit $\mathfrak{o} \subset \mathcal{E}_2(G)$, the *Harish-Chandra canonical measure* on $\mathfrak{o}$ is the Euclidean measure on $\mathfrak{o}$ whose pullback to $\mathrm{Im}\,\mathcal{X}(G)$ agrees with the pullback of the Haar measure on $\mathrm{Im}\,\mathcal{X}(A)$.

Hence even though the construction of the canonical measure is slightly involved, in practice it will be easy to recognize as being essentially the Haar measure on the compact torus $\mathrm{Im}\,\mathcal{X}(A)$.

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Example 2.4. If $G = \mathrm{SL}_2(F)$ and $M = A$ is the diagonal torus, we have $\mathfrak{a}_G^* \simeq \mathbb{R}$ and $R = \mathbb{Z}$ so that a fundamental domain for $\mathfrak{a}_G^*/\frac{2\pi}{\log q}R$ is $\left[-\frac{\pi}{\log q}, \frac{\pi}{\log q}\right)$ and the canonical measure $d\nu = \frac{\log q}{2\pi}\mathrm{d}x$, where $\mathrm{d}x$ is the Lebesgue measure. To obtain quasicharacters of G, we associate to $\nu \in \mathfrak{a}_{G\mathbb{C}}^*$ the quasicharacter $\chi_\nu(g) = q^{\langle \nu, H_G(g)\rangle} = |\nu(g)|_F$, where the second equality defines $H_G\colon G \to \mathfrak{a}_G$.

Therefore to compute the integral of a function f on $\mathcal{E}_2(G)$ supported on the unramified unitary characters of A, we compute

$$\int_{\mathcal{E}_2(A)} f(\omega)\,\mathrm{d}\omega = \int f(\chi_\nu)\,\mathrm{d}\nu = \frac{\log q}{2\pi}\int_{\frac{-\pi}{\log q}}^{\frac{\pi}{\log q}} f(e^{it\log q})\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z}\,\mathrm{d}z$$

if $t \mapsto e^{it\log q} = q^s =: z$ (here $s = it$) parameterizes the unit circle $\mathbb{T}$.

In general, $\mathcal{E}_2(M_P)$ is a disjoint union of compact tori, and the Plancherel density descends to the quotients of these compact tori by certain finite groups, namely, the Weyl groups of (P, A_P). The set $\mathcal{E}_2(M_P)^I$ is finite up to twists by unramified characters, by a result of Harish-Chandra [48].

2.3. The Harish-Chandra Schwartz algebra

Let $\mathcal{C} = \mathcal{C}(G)$ be the Harish-Chandra Schwartz algebra of G; see [20], [48], or [11] for the definition and associated notation. In particular, we will record for future comparison with the argument in the proof of Proposition 3.13 that

$$q^{\frac{\ell(w)}{2}} q^{-\#W} \leq \Delta(IwI) \leq q^{\frac{\ell(w)}{2}}.$$

Let $\mathcal{C}^{I\times I}$ be the subalgebra of Iwahori-biinvariant functions. As explained in [48], the Fourier transform $f \mapsto \pi(f)$ defines an endomorphism of π for every $f \in \mathcal{C}$ and every tempered representation π.

The Plancherel theorem is the statement (see [48, Thm. VIII.1.1]) that this assignment defines an isomorphism of rings

$$\mathcal{C} \to \mathcal{E}_t(G),$$

where $\mathcal{E}_t(G)$ is the subring of the endomorphism ring of the forgetful functor $\mathcal{M}_t(G) \to \mathbf{Vect}_{\mathbb{C}}$ defined by the following conditions:

1. For all $\pi = \mathrm{Ind}_P^G(\nu \otimes \omega)$, the endomorphism $\eta_\pi = \eta_{\nu,\omega}$ is a smooth function of the unramified unitary character ν and $\omega \in \mathcal{E}_2(M_P)$;
2. The endomorphism η_π is biinvariant with respect to some open compact subgroup of G.

We have the obvious inclusion $\iota\colon H \hookrightarrow \mathcal{C}^{I\times I}$.

Define the subring $\mathcal{E}(G)$ of the endomorphism ring of the forgetful functor $\mathcal{M}(G) \to \mathbf{Vect}_{\mathbb{C}}$ from the category of all smooth representations, by replacing, in (1) above, unitary characters with all unramified characters, and 'smooth' with 'algebraic', and adding

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3. The endomorphisms η_π are compatible with supercuspidal support, *i.e.* if (M_1,σ_1) is the supercuspidal support of π, so that

$$\pi = i_P^G(\omega \otimes \nu) \hookrightarrow \pi_1 = i_{P_1}^G(\sigma_1 \otimes \nu_1),$$

then η_{π_1} preserves π and $\eta_{\pi_1}|_\pi = \eta_\pi$.

The matrix Paley-Wiener theorem of Bernstein [6] says that $f \mapsto \pi(f)$ is an isomorphism from the full Hecke algebra of G onto $\mathcal{E}(G)$. Denote by $\mathcal{E}^I$ and $\mathcal{E}_t$ the subrings of I-invariant endomorphisms.

Remark 2.5. Property 3 means that unlike Schwartz functions, the Fourier transform of a compactly supported function may be freely specified only on inductions of supercuspidal representations. In particular, an algebraic family of endomorphisms defined for all unramified characters may fail to come from a compactly supported function. An example of a non-compactly supported function with regular Fourier transform is the element of J given in Corollary 3.19, because $\pi(t_1) = 0$ except if π is the Steinberg representation of G.

For computational purposes such as ours, we require that these isomorphisms be explicit. In the Iwahori-spherical case, harmonic analysis on $\mathcal{C}^{I\times I}$ can be phrased internally to H and various completions of H. In this setting Opdam gave an explicit Plancherel formula in [38]. In more general settings there are explicit formulas for $\mathrm{GL}_n(F)$, $\mathrm{Sp}_4(F)$, and $G_2(F)$, which we will also make use of.

2.4. The algebra J as a subalgebra of the Schwartz algebra

In [11], Braverman and Kazhdan constructed a map of $\mathbb{C}$-algebras $J \to \mathcal{C}^{I\times I}$. We shall review this construction now.

Definition 2.5 ([11], Section 1.7). Let $P = M_P N_P$. A character $\chi\colon M_P \to \mathbb{C}^\times$ of M_P is *non-strictly positive* if for all root subgroups $U_\alpha \subset N_P$, we have $|\chi(\alpha^\vee(x))|_\infty \geq 1$ for $|x|_F \geq 1$.

We say a non-strictly positive character χ is *strictly positive* if for all root subgroups $U_\alpha \subset N_P$, we have $|\chi(\alpha^\vee(x))|_\infty > 1$ for $|x|_F > 1$.

Of course, it suffices to test this for $x = \varpi^{-1}$.

Example 2.6. For $\mathbf{G} = \mathrm{SL}_2$, in the conventions fixed in Example 2.4, an unramified character χ of A is non-strictly positive if it corresponds to z such that $|z| \geq 1$.

If $\mathbf{G} = \mathrm{GL}_n$ and ν corresponds to the vector $(z_1,\dots,z_n) \in \mathbb{C}^n$, then the condition that ν is non-strictly positive translates to $|z_1| \geq |z_2| \geq \cdots \geq |z_n|$. Such conditions divide $\mathbb{C}^n$ into chambers, on which the Weyl group $\mathfrak{S}_n$ clearly acts simply transitively. Interior points correspond to strictly positive ν.

Following *op. cit.*, let $\mathcal{E}_J^I(G)$ denote the subring of $\mathcal{E}_t(G)$ defined by the following conditions on the endomorphisms η_π:

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1. For all $\pi = \mathrm{Ind}_P^G(\nu \otimes \omega)$, the endomorphism $\eta_\pi = \eta_{\nu,\omega}$ is a rational function of ν, regular on the set of non-strictly positive ν.
2. The endomorphism η_π is $I \times I$-biinvariant.

Theorem 2.7 ([11], Theorem 1.8). *Let* $\mathbf{G}$ *be a connected reductive group defined and split over* F*. Then the following statements hold:*

1. *Let* π *be a tempered representation of* G*. Then the action of* H *on* π^I *extends uniquely to* J *via* $\phi \circ {}^\dagger(-)$*.*
2. *Let* $P = MN$ *be a parabolic subgroup of* G *with Levi subgroup* M *and let* ω *be an irreducible tempered representation of* M*. Let* ν *be a non-strictly positive character of* M *and let* $\pi = \mathrm{Ind}_P^G(\omega \otimes \nu)$*. Then the action of* $H(G,I)$ *on* π^I *extends uniquely to an action of* J*.*
3. *The action of* J *on the representations* π^I *in (2) extends rationally in* ν *to define a homomorphism*

$$\eta \colon J \to \mathcal{E}_J^I(G).$$

denoted

$$t_w \mapsto (\eta_\pi(w))_{\pi \in \mathcal{M}_t(G)}.$$

The proof of Theorem 2.7 (1) and (2) in [11] uses Theorem 1.13 and [52], to show that, in the notation of Section 1.5.1,

$$E(u,s,\rho)|_H \simeq {}^*K(u,s,\rho,q) = {}^*i_P^G(\sigma \otimes \nu)$$

where the restriction is via ϕ, and hence that

$$E(u,s,\rho)_{H,\phi\circ^*(-)} \simeq K(u,s,\rho,q) = i_P^G(\sigma \otimes \nu),$$

where the restriction is now via $\phi \circ {}^*(-)$.

By Lemma 1.8, the map ϕ is $\bar{\kappa}$-linear, where $\bar{\kappa}$ is as in Definition 1.6. Of course, ϕ also intertwines conjugation by T_{w_0} and $\phi(T_{w_0})$.

As noted in Section 1.5.1, if $\pi = K(u,s,\rho)$ is simple tempered (or more generally, is any semisimple module), then by [40, Prop. 6.3],

$$E(u,s,\rho)|_{\phi\circ^\dagger(-)} = {}^{\dagger *}K(u,s,\rho) \simeq \widetilde{K(u,s,\rho)} \tag{2.1}$$

is simple tempered. Moreover, by Lemma 1.10 (e) and (2.1), we have, if $\Psi_{w_0}(h) = T_{w_0} h T_{w_0}^{-1}$, that

$$\begin{aligned}
{}^{\bar{\kappa}\circ\phi(\Psi_{w_0}^{-1})}E(u,s,\rho)|_{\phi\circ^\dagger(-)} &= {}^{\phi(\Psi_{w_0}^{-1})}E(u,s,\rho)|_{\bar{\kappa}\circ\phi\circ^\dagger(-)} \\
&= E(u,s,\rho)|_{\phi(\Psi_{w_0}^{-1})\circ\bar{\kappa}\circ\phi\circ^\dagger(-)} \\
&= E(u,s,\rho)|_{\phi\circ\Psi_{w_0}^{-1}\circ\bar{\kappa}\circ^\dagger(-)} \\
&= E(u,s,\rho)|_{\phi\circ^*(-)} \\
&= K(u,s,\rho)
\end{aligned}$$

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for any standard H-module $K(u,s,\rho)$. Therefore any standard H-module extends to a simple J-module via $\phi \circ {}^\dagger(-)$.

Composing the morphism η with the inverse Fourier transform, Braverman and Kazhdan define an algebra map

$$\tilde{\phi}\colon J \to \mathcal{C}^{I\times I}$$

sending

$$t_w \mapsto (\eta_\pi(w))_{\pi\in\mathcal{M}_t(G)} \mapsto f_w = \sum_{x\in W_{\mathrm{aff}}} A_{x,w}T_x \in \mathcal{C}^{I\times I},$$

where $A_{x,w} = f_w(IxI)$. By definition, $\eta_\pi(w) = \pi(f_w)$ as endomorphisms of π. We will show later that $\tilde{\phi}$ is essentially the map ϕ^{-1}.

Remark 2.8. There are gaps in the proofs of injectivity and, as pointed out to us by R. Bezrukavnikov and I. Karpov after an early version of the present paper was completed, of surjectivity of the map η in [11]. In the present paper we prove injectivity by proving injectivity of $\tilde{\phi}$ in Corollary 3.20. We prove surjectivity in [15] for all but a small number of cells for exceptional groups; [8] proves that η is an isomorphism for all two-sided cells.

Implicit in [11] is

Lemma 2.9. *We have the commutative diagram (1.2).*

Proof. Let π be a tempered representation of G. By Theorem 2.7 and (2.1), the H-action on it extends to a J-action via $\phi\circ^\dagger(-)$, such that

$$\eta_\pi(\phi_q({}^\dagger f)) = \pi(f)$$

in $\mathcal{E}_t^I$ for any $f \in H$. Therefore $\tilde{\phi}\circ\phi_q({}^\dagger f) = f$ by the Plancherel theorem. □

2.5. The Plancherel formula for GL_n

For $G = \mathrm{GL}_n(F)$, we have access to an explicit Plancherel measure and its Bernstein decomposition, thanks to [4].

Recall that for $G = \mathrm{GL}_n(F)$, we have bijections

$$\{\text{partitions of } n\} \leftrightarrow \{\text{Standard Levi subgroups } M \text{ of } \mathrm{GL}_n(F)\} \quad (2.2)$$
$$\leftrightarrow \{\text{unipotent conjugacy classes in } \mathrm{GL}_n(\mathbb{C})\} \quad (2.3)$$
$$\leftrightarrow \mathcal{N}/\mathrm{GL}_n(\mathbb{C})$$
$$\leftrightarrow \{2-\text{sided cells } \mathbf{c} \text{ in } \tilde{W}\} \quad (2.4)$$
$$\leftrightarrow \{\text{direct summands } J_{\mathbf{c}} \text{ of } J\}$$

where (2.2) $\leftrightarrow$ (2.3) sends a unipotent conjugacy class u to the standard Levi M such that a member of u is distinguished in $M^\vee$, and (2.3) $\leftrightarrow$ (2.4) is Lusztig's bijection from [17].

Definition 2.6. Let u be a unipotent element of a semisimple group S over the complex numbers. Then u is *distinguished in S* if $Z_S(u)$ contains no nontrivial torus.

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Let $P = M_P N_P$ be a parabolic subgroup of G and let $\mathfrak{o}$ be an orbit in $\mathcal{E}_2(M_P)$ under the action of the unitary unramified characters of M_P as explained in Section 2.2. Write $W_{M_P} \subset W$ for the finite Weyl group of (M_P, A_P). Let $\mathrm{Stab}_{W_M}(\mathfrak{o})$ be the stabilizer of $\mathfrak{o}$. Recall that a parabolic subgroup of G is said to be *semistandard* if it contains A. Then the Plancherel decomposition reads

$$f = \sum_{(P=M_PN_P,\mathfrak{o})/\text{association}} f_{M_P,\mathfrak{o}}$$

where $f \in \mathcal{C}(G)$, the sum is taken over semistandard parabolic subgroups P up to association, and

$$f_{M_P,\mathfrak{o}}(g) = c(G/M)^{-2}\gamma(G/M)^{-1}\#\mathrm{Stab}_{W_M}(\mathfrak{o})^{-1}\int_{\mathfrak{o}} \mu_{G/M}(\omega)d(\omega)\mathrm{trace}(\pi, R_g(f))\,\mathrm{d}\omega,$$

where $R_g f(x) = f(xg)$ is the right translation of f and $\pi = \mathrm{Ind}_P^G(\nu \otimes \omega)$ is the normalized parabolic induction of the twist of ω by a unitary unramified character ν of M. In [4], each term above is explicitly calculated as a rational function of q.

Lemma 2.10. *There is a finite set $S = \left\{(M,\omega)\,\middle|\,\omega \in \mathcal{E}_2(M)^I\right\}$ such that $\mathrm{trace}\left(\mathrm{Ind}_P^G(\omega \otimes \nu), f\right)$ is nonzero only for $\omega \in S$, for all I-biinvaraiant Schwartz functions f.*

Proof. This is entirely standard. As $\mathcal{E}_2(M)^{I_M}$ is finite for every M, and there are finitely many standard parabolics of G, we need only show that $\mathrm{Ind}_P^G(\omega \otimes \nu)^I \neq 0$ only if $\omega^{I_M} \neq 0$, where I_M is the Iwahori subgroup of the reductive group M relative to the Borel subgroup $M(\mathbb{F}_q) \cap B(\mathbb{F}_q)$ of $M(\mathbb{F}_q)$. Note that I_M is naturally a subgroup of I. For any representation σ of M, if $f \in \mathrm{Ind}_P^G(\sigma)$ is I-fixed, then for $i_M \in I_M$, we have

$$f(i_M) = \sigma(i_M)\delta_P^{\frac{1}{2}}(i_M)f(1) = f(1).$$

As $\delta_P = 1$ on every compact subgroup of P, we have $\delta_P^{\frac{1}{2}}(i_M) = 1$, and $f(1) \in \sigma^{I_M}$. □

Let $\pi = \mathrm{Ind}_P^G(\omega \otimes \nu)$ be a tempered representation and let (u,s) be the KL-parameter of its discrete support. Then by [25, Theorem 8.3], $\mathbf{M}_\mathbf{P}^\vee$ is minimal such that $(u,s) \in M_P^\vee$. By *op. cit.*, this condition is equivalent to $Z_M(s)$ being semisimple and u being distinguished in $Z_M(s)$.

Proposition 2.11 (c.f. [38] Proposition 8.3). *The Plancherel decomposition is compatible with the decomposition $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ in the sense that if $w \in \mathbf{c}$, $f = f_w$ and $u = u(\mathbf{c})$ under Lusztig's bijection, then $f_M \neq 0$ only for those M such that there exists $s \in M^\vee(\mathbb{C})$ such that $Z_{M^\vee}(s)$ is semisimple and $Z_{M^\vee}(s) \cap Z_{M^\vee}(u)$ contains no nontrivial torus.*

Proof. Let $\pi := \mathrm{Ind}_P^G(\nu \otimes \omega)$ be a tempered irreducible representation of G induced as usual from a standard parabolic subgroup P with Levi subgroup M. Then π^I is a tempered irreducible H-module, and is of the form $K(u,s,\rho,q)$ for $u,s \in \mathbf{G}^\vee(\mathbb{C})$ and ρ a representation of $\pi_0(Z_{G^\vee}(u,s))$ with s compact. By Theorem 2.7, $K(u,s,\rho,q)$ extends to a J-module. By definition of Lusztig's bijection, $\pi(f_w) \neq 0$ only if w is in the two-sided

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cell $\mathbf{c} = \mathbf{c}(u)$ of W_{aff} corresponding to u. On the other hand, $K(u,s,\rho,q)$ is induced from a square-integrable standard module $K_M(u,s,\tilde{\rho},q)$ of $H(M,I_M)$. But now [25, Theorem 8.3] says that (u,s) must be precisely as in the statement of the proposition. Thus only such summands $(f_w)_M$ in the Plancherel decomposition of f_w are nonzero. □

Of course, when $\mathbf{G} = \mathrm{GL}_n$, the bijections (2.2), (2.3), and (2.4), imply that there is a unique nonzero summand $(f_w)_M$ for each w.

2.6. Plancherel measure for GL_n

We refer to [4, Section 5], for a summary of the Bernstein decomposition of the tempered irreducible representations of GL_n, in particular we use the description in *loc. cit.* of the Bernstein component parameterizing the I-spherical representations of G.

As we shall be applying the Plancherel formula only to Iwahori-biinvariant functions, it suffices to consider only irreducible tempered representations with Iwahori-fixed vectors. For GL_n, the only such representations are of the form

$$\pi = \mathrm{Ind}_P^G(\nu_1\mathrm{St}_1 \boxtimes \cdots \boxtimes \nu_k\mathrm{St}_k)$$

where St_i is the Steinberg representation of GL_i, and $P \supset M = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_k}$.

These representations are parameterized as follows. Let M be a Levi subgroup corresponding to the partition $l_1 + \cdots + l_k = n$, and recall that we write $\mathbb{T}$ for the circle group. Define $\gamma \in \mathfrak{S}_n$ by $\gamma = (1 \ldots l_1)(1 \ldots l_2)\cdots(1 \ldots l_k)$, so that the fixed-point set $(\mathbb{T}^n)^\gamma = \{(z_1,\ldots,z_1,\ldots,z_k,\ldots,z_k)\} \simeq \mathbb{T}^k$. Then the irreducible tempered representations with Iwahori-fixed vectors induced from M are parameterized by the compact orbifold $(\mathbb{T}^n)^\gamma / Z_{\mathfrak{S}_n}(\gamma)$.

Theorem 2.12 ([4], Remark 5.6). *Let $G = \mathrm{GL}_n$ and $M = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_k}$ be a Levi subgroup. Then the Plancherel measure of H on $(\mathbb{T}^n)^\gamma / Z_{\mathfrak{S}_n}(\gamma)$ is*

$$\mathrm{d}\nu_H(\omega) = \prod_{i=1}^k \frac{q^{l_i^2 - l_i}(q-1)^{l_i}}{l_i(q^{l_i}-1)} \cdot q^{\frac{n-n^2}{2}} \cdot \prod_{(i,j,g)} \frac{q^{2g+1}(z_i - z_j q^g)(z_i - q^{-g} z_j)}{(z_i - z_j q^{-g-1})(z_i - q^{g+1} z_j)},$$

where the tuples $(i,j,g) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ are tuples such that $1 \le i < j \le k$ and $|g_i - g_j| \le g \le g_i + g_j$, where $g_i = \frac{l_i - 1}{2}$.

This is the measure that we will integrate against, by successively applying the residue theorem. When carrying out explicit calculations, we will usually elide the constant

$$\prod_{i=1}^k \frac{q^{l_i^2 - l_i}(q-1)^{l_i}}{l_i(q^{l_i}-1)} \cdot q^{\frac{n-n^2}{2}}$$

as it depends only on M. We shall abbreviate

$$\Gamma_{i,j,g} := q^{-2g-1} \left| \Gamma_F\left(q^{-g}\frac{z_i}{z_j}\right)\right|^2 = \frac{(z_i - q^g z_j)(z_i - q^{-g} z_j)}{(z_i - z_j q^{-g-1})(z_i - q^{g+1} z_j)},$$

and recall that, as noted in the proof of Theorem 5.1 in [4], the function $(z_1,\ldots,z_k) \mapsto \prod_{(i,j,g)} \Gamma_{i,j,g}$ is $Z_{\mathfrak{S}_n}(\gamma)$-invariant. Hence for the purposes of integration, we may allow

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ourselves to integrate simply over $\mathbb{T}^k$. Moreover, there are many cancellations between the $\Gamma_{i,j,g}$ for a fixed pair $i<j$ as g varies. Indeed, putting $q_{ij}=q^{|g_i-g_j|}$, $q^{ij}=q^{g_i+g_j+1}$, and

$$\Gamma^{ij}:=\frac{(z_i-q_{ij}z_j)(z_i-(q_{ij})^{-1}z_j)}{(z_i-q^{ij}z_j)(z_i-(q^{ij})^{-1}z_j)}$$

we have

$$\prod \Gamma_{i,j,g}=\Gamma^{ij},$$

where the product is taken over all integers g appearing in triples (i,j,g) for $i<j$ fixed. We set

$$c_M:=\prod_{(i,j,g)} q^{2g+1}\cdot\prod_{i=1}^{k}\frac{q^{l_i^2-l_i}(q-1)^{l_i}}{l_i(q^{l_i}-1)}\cdot q^{\frac{n-n^2}{2}},$$

where the first product is taken over (i,j,g) such that $1\le i<j\le k$ and $|g_i-g_j|\le g\le g_i+g_j$.

2.7. Beyond type A: the Plancherel formula following Opdam

Beyond type A, we still have available Opdam's explicit Plancherel formula for the Iwahori-Hecke algebra [38]. Let $\mathbf{G}$ be a connected reductive algebraic group defined and split over F, of Dynkin type other than type A (to avoid redundancy). Let f be an Iwahori-biinvariant Schwartz function on G and let M be a Levi subgroup of G. Given a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, let $R_{1,+}$ and $R_{P,1,+}$ be defined as in [38], Section 2.3. Recall that the group of unramified characters of a Levi subgroup M has the structure of a complex torus, and is in fact a maximal torus of $\mathbf{M}^\vee(\mathbb{C})$. In particular, if P is a parabolic subgroup and α is a root of (M_P,A_P), then it makes sense to write $\alpha(\nu)$ for any unramified character ν of M. Then, altering Opdam's notation to match our own from Section 2.5, the Plancherel formula reads

Theorem 2.13. ([38] Thm. 4.43).

$$f_{M,\mathfrak{o}}(1)=\frac{q^{-\ell(w^P)}}{\#\mathrm{Stab}_{W_M}(\mathfrak{o})}\int_{\mathfrak{o}} d(\omega)\cdot\prod_{\alpha^\vee\in R_{1,+}\setminus R_{P,1,+}}\frac{|1-\alpha^\vee(\nu)|^2}{|1+q_\alpha^{\frac{1}{2}}\alpha^\vee(\nu)^{1/2}|^2|1-q_\alpha^{\frac{1}{2}}q_{2\alpha}\alpha^\vee(\nu)^{1/2}|^2}\mathrm{trace}(\pi,f)\,\mathrm{d}\omega, \tag{2.5}$$

where $\pi=\mathrm{Ind}_P^G(\omega\otimes\nu)$, $P\supset M$, and where q_α and $q_{2\alpha}$ are powers of q, and w^P is the longest element in the complement W^P to the parabolic subgroup W_P of W.

Note that whenever $q_{2\alpha}=1$, which holds whenever $\alpha^\vee\notin 2X_*$, the factor for α reduces to

$$\frac{|1-\alpha^\vee(\nu)|^2}{|1-q_\alpha\alpha^\vee(\nu)|^2}. \tag{2.6}$$

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In types A (as we have used above) and D, this simplification always occurs. In types B and C, it happens for all roots except $\alpha = 2\varepsilon_i \in R_{1,+}(B_n)$ and $\alpha = 4\varepsilon_i \in R_{1,+}(C_n)$, where ε_i is the character $\operatorname{diag}(a,\dots,a_n) \mapsto a_i$.

For explicit evaluation we rewrite (2.5) in coordinates as follows. Recalling the setup of Section 2.2.1, a chosen basis $\{\beta_i^\vee\}$ of the coweight lattice of $\mathbf{G}$. Then we obtain coordinates z_i, such that if $\alpha^\vee = \sum_i e_i\beta_i^\vee$, then the factor in (2.5) labelled by $\alpha^\vee$ is

$$\frac{|1-z_1^{e_1}\cdots z_n^{e_n}|^2}{|1+q_\alpha^{\frac{1}{2}}(z_1^{e_1}\cdots z_n^{e_n})^{1/2}|^2|1-q_\alpha^{\frac{1}{2}}q_{2\alpha}(z_1^{e_1}\cdots z_n^{e_n})^{1/2}|^2}. \tag{2.7}$$

When integrating, the coordinates z_i are restricted to the residual coset corresponding to $\mathfrak{o}$, in the sense of [38]. For $\mathbf{G} = \mathrm{GL}_n$, we used the basis afforded by the characters ε_i.

2.8. Regularity of the trace

In order to extract information about the expansion of the elements t_w in terms of the T_x-basis via the Plancherel formula, we must establish a regularity property of $\operatorname{trace}(\pi, f_w)$ where π is an irreducible tempered representation of G. The needed property follows trivially from Theorem 2.7.

2.8.1. Intertwining operators. The goal is to use the property that elements of $\mathcal{E}_J^I(G)$ commute with all intertwining operators in $\mathcal{M}_t(G)$, and regularity of the trace for unitary and non-strictly positive characters of M to deduce regularity of the trace at all characters of M.

Let ω be a discrete series representation of a Levi subgroup M of G, and let ν be any unramified character of M, not necessarily unitary. Then we may form the representation $\pi = \operatorname{Ind}_P^B(\nu\otimes\omega)$ of G, where P is a parabolic with Levi factor M. We will now recall some well-known facts about the action of the Weyl group of M on such representations π. Let θ and θ' be two subsets of Δ corresponding to Levi subgroups M and M'. Let $w \in W$ be such that $w\theta = \theta'$. Then there is an intertwining operator

$$J_{P|P'}(\omega,\nu)\colon \operatorname{Ind}_P^B(\nu\otimes\omega) \to \operatorname{Ind}_{P'}^B(\nu\otimes\omega)$$

for each $w \in W(\theta,\theta') = \{w \in W \mid w(\theta) = \theta'\}$. For $\mathbf{G} = \mathrm{GL}_n$, this set is nonempty only if

$$M_\theta = \mathrm{GL}_{l_1} \times \cdots \times \mathrm{GL}_{l_N} \quad M_{\theta'} = \mathrm{GL}_{l'_1} \times \cdots \times \mathrm{GL}_{l'_N}$$

and $\{l_1,\dots,l_N\} = \{l'_1\dots,l'_N\}$ are equal multisets. In this case, $W(\theta,\theta') \simeq \mathfrak{S}_N$ can be viewed as acting by permuting the blocks of M. It is well-known that $J_{P|P'}(\omega,\nu)$ is a meromorphic function of ν with simple poles. The poles of these operators have been studied by Shahidi in [44], and in the language of modules over the full Hecke algebra by Arthur in [1]. The results will be stated for certain renormalizations $A(\nu,\omega,w)$ of the operators $J_{P|P'}(\omega,\nu)$ as explained in equation (2.2.1) of [44].

2.8.2. Conventions on parabolic subgroups. We now recall the notation of Shahidi [44]. Given a subset θ, we set $\Sigma_\theta = \operatorname{span}_{\mathbb{R}}\theta$, and $\Sigma_\theta^+ = \Psi^+ \cap \Sigma_\theta$ and likewise

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define Σ_θ^-. Let $\Sigma(\theta)$ be the roots of (P,A_P). Define the positive roots $\Sigma^+(\theta)$ to be the roots obtained by restriction of an element of $\Psi^+ \setminus \Sigma_\theta^-$.

Given two subsets $\theta,\theta' \subset \Delta$, following [44] we set

$$W(\theta,\theta') = \{w \in W \,|\, w(\theta) = \theta'\},$$

and then for $w \in W(\theta,\theta')$, we define

$$\Sigma(\theta,\theta',w) = \left\{[\beta] \in \Sigma^+(\theta) \,\middle|\, \beta \in \Psi^+ - \Sigma_\theta^+ \text{ and } w(\beta) \in \Psi^-\right\},$$

and then

$$\Sigma^\circ(\theta,\theta',w) := \left\{[\beta] \in \Sigma(\theta,\theta',w) \,\middle|\, w_{[\beta]} \in W(A_P)\right\}.$$

Remark 2.14. In [44] additional care about the relative case is taken in the notation. In our simple case this is of course unnecessary, and we omit it.

In the case $\mathbf{G} = \mathrm{GL}_n$, this specializes as follows. Let $\alpha_{ij} : \mathrm{diag}(t_i) \mapsto t_i t_j^{-1}$ be characters of T. The α_{ij} for $j > i$ are the positive roots of $(\mathbf{B},\mathbf{A})$ and $\beta_i := \alpha_{ii+1}$ are our chosen simple roots. If P corresponds to the partition $n_1 + \cdots + n_p$ of n and subset $\theta \subset \Delta$, then

$$\Sigma(\theta) = \mathrm{span}\{\beta_{n_1}, \beta_{n_1+n_2}, \dots\},$$

where we view β_i as restricted to $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_G$. Note that all the positive roots $\Sigma^+(\theta)$ of (P_θ,N_θ) are in N_P. Denoting by $[\alpha]$ the coset representing a root α of G restricted to P, the positive roots in N_P are the α_{ij} such that $[\alpha_{ij}] = [\beta_{n_1+\cdots+n_k}]$.

Example 2.15. If $\mathbf{G} = \mathrm{GL}_6$ and P be the parabolic subgroup of block upper-triangular matrices corresponding to the partition $6 = 2+2+1+1$ and $\theta = \{\alpha_{12},\alpha_{34}\}$, then we have

$$A_P = \{\mathrm{diag}(t_1,t_1,t_2,t_2,t_3,t_4)\}.$$

The positive simple roots are $\Sigma^+(\theta) = \{[\beta_2],[\beta_4],[\beta_5]\}$. The Weyl group $W(A_P) \simeq \mathfrak{S}_2 \times \mathfrak{S}_2$ acts by permuting the blocks. Note that the simple reflection sending $\beta_4 \mapsto -\beta_4$ does not arise by permuting the blocks (*i.e.* $\begin{pmatrix}4 & 5\end{pmatrix}$ does not send blocks to blocks), hence $w_{[\beta_4]} \notin W(A_P)$. Hence for any $w \in W(\theta,\theta')$ we have $\Sigma^\circ(\theta,\theta',w) \subseteq \{[\beta_2],[\beta_5]\}$.

2.8.3. Regularity of the trace. We have the following information about the poles of intertwining operators, due, according to [44], to Harish-Chandra:

Theorem 2.16 ([44], Theorem 2.2.1). *Let ω be an irreducible unitary representation of M. Say that ω is a subrepresentation of $\mathrm{Ind}_{P_*}^M(\omega_*)$ for a parabolic subgroup $P_* = M_*N_*$ and ω_* is an irreducible supercuspidal representation of M_*. Let $\theta_* \subset \Delta$ be such that $P_* = P_{\theta_*}$ as parabolic subgroups of M_*.*

Then the operator

$$\prod_{\alpha \in \Sigma_r^0(\theta_*, w\theta_*, w)} (1 - \chi_{\omega,\nu}^2(h_\alpha)) A(\nu,\omega,w)$$

is holomorphic on $\mathfrak{a}_{\theta\mathbb{C}^*}$. Here $\chi_{\omega,\nu}$ is the central character of the twisted representation $\omega \otimes q^{\langle \nu, H_\theta(-)\rangle}$.

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In particular for the purposes of the Plancherel formula, the only relevant ω are unitary, hence have unitary central characters. Therefore $A(\nu,\omega,w)$ is holomorphic at ν if $|q^{\langle \nu, H_\theta(-)\rangle}| \neq 1$, or equivalently if

$$\Re(\langle \nu, H_\theta(-)\rangle) \neq 0.$$

In particular, there is a finite union of hyperplanes away from which each operator $A(\nu,\pi,w)$ is holomorphic, for any $w \in W$.

Lemma 2.17. *Let M be a Levi subgroup of $G = \mathbf{G}(F)$ and let ω be a discrete series representation of M. Let $P = M_P A_P N_P$ be the standard parabolic subgroup containing $M = M_P$ and let $k = \operatorname{rk} A_P$. Let $z_1,\dots,z_k \in (\mathbb{C}^\times)^k = X^*(A_P)\otimes_{\mathbb{Z}} \mathbb{C}$ define an unramified quasicharacter $\nu = \nu(z_1,\dots,z_k)$ of A_P as in Section 2.2. Let $\pi = \operatorname{Ind}_P^G(\omega\otimes\nu)$. Let $f \in J$. Then*

$$\operatorname{trace}(\pi,f) \in \mathbb{C}[z_1,\dots,z_k,z_1^{-1},\dots,z_k^{-1}].$$

That is, the trace is a regular function on $(\mathbb{C}^\times)^k$.

Proof. We know *a priori* that

$$\operatorname{trace}(\pi,f) \in \mathbb{C}(z_1,\dots,z_k)$$

is a rational function of ν, as the operator $\pi(f)$ itself depends rationally on the variables z_i by Theorem 2.7. Therefore

$$\operatorname{trace}(\pi,f) = \frac{p(\nu)}{h(\nu)} \in \mathbb{C}(z_1,\dots,z_k).$$

By Theorem 2.16 and the discussion following it, there is an open subset U of the unitary characters, such that for all $\nu \in U$, and all $w \in W$ we have

$$\frac{p(\nu)}{h(\nu)} = \frac{p(w(\nu))}{h(w(\nu))}. \tag{2.8}$$

Holding $z_2,\dots,z_n$ constant and in U, (2.8) becomes an equality of meromorphic functions of z_1 that holds on a set with an accumulation point, and hence (2.8) holds for all $z_1 \in \mathbb{C}^\times$. Now holding $z_1 \in \mathbb{C}^\times$ constant and arbitrary, and $z_3,\dots,z_k$ constant and in U, we see that (2.8) holds also for all $z_2 \in \mathbb{C}^\times$. Therefore (2.8) actually holds for all ν, *i.e.* $\operatorname{trace}(\pi,f)$ is a W-invariant rational function of ν.

When ν is non-strictly positive with respect to M, by Theorem 2.7, $\operatorname{trace}(\pi,f)$ has poles only of the form $z_i^{n_i} = 0$. The claim now follows from the W-invariance, and thus $\operatorname{trace}(\pi,f)$ is a regular function on $(\mathbb{C}^\times)^k$. □

We will therefore allow ourselves to write $\operatorname{trace}(\pi,f)$ for functions $f \in J$ even for ν such that the operator $\pi(f)$ itself is not defined.

Lemma 2.18. *Let $d \in W_{\mathrm{aff}}$ be a distinguished involution and π be a tempered representation induced from one of the Levi subgroups attached to the two-sided cell containing d in the sense of Proposition 2.11. Then $\operatorname{trace}(\pi,f_d)$ is constant and a natural number. In fact, the same is true for any idempotent in J.*

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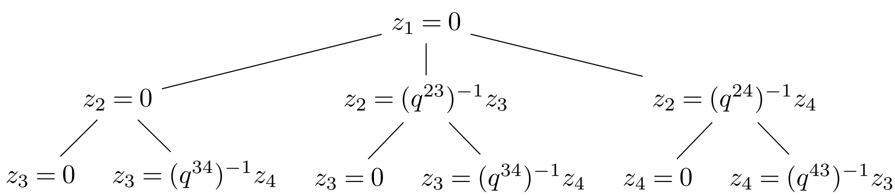
Proof. Let j be an idempotent in J. We have $\mathrm{trace}(\pi, f_j) = \mathrm{rank}(\pi(f_j))$. The trace is continuous in ν by Lemma 2.17, and the present lemma follows as $\mathbb{T}$ is connected. □

3. Proof of Theorem 1.2 for general G and the case of GL_n

3.1. The functions f_w for GL_n

In this section, $q > 1$.

To compute with the Plancherel formula, we will need to apply the residue theorem successively in each variable z_i, and in doing so will we will need to sum over a certain tree that will track, for each variable, at which residues we evaluated. Upon integrating with respect to each variable z_i, we will have poles of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$. For example, if we have 4 variables z_1, z_2, z_3, z_4 corresponding to a Levi subgroup $\mathrm{GL}_{l_1} \times \mathrm{GL}_{l_2} \times \mathrm{GL}_{l_3} \times \mathrm{GL}_{l_4}$, then some of the summands obtained by successively applying the residue theorem are labelled by paths on the tree



Of course, to evaluate the entire integral for M, we would also need to consider trees whose roots are decorated with $z_1 = (q^{12})^{-1} z_2$, and so on, for a total of four trees.

Definition 3.1. Given a Levi subgroup M with $N+1$ blocks, a *bookkeeping tree* T for M is a rooted tree with N levels such that the vertices on the i-th level below the root each have $N-i$ child vertices, and each vertex is decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$, where the index j does not appear along the path from the vertex to the root, and the parent root is decorated with an equation $z_k = (q^{ki})^{-1} z_i$ for some k. Moreover, we require that the root be decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1} z_j$ for i minimal. A *branch* of T is a simple path in T from the root to one of the leaves.

Definition 3.2. Given a branch B of a bookkeeping tree, a *clump in* B is an ordered subset of indices i appearing in the decorations of successive parent-child vertices, such that no decoration of the form $z_i = 0$ occurs along the path from the closest index to the root to the farthest index from the root. We write $C \prec B$ if C is a clump of B.

Example 3.1. The sets of indices {3,4}, {2,3,4}, {2,3}, {2,4}, and {2,4,3} (note the ordering) are all the clumps of the above tree. The sets {1,2,3} and {1,2,4} are not clumps.

Theorem 3.2. *Let $G = \mathrm{GL}_n(F)$ and let d be a distinguished involution such that the two-sided cell containing d corresponds to the Levi subgroup M. Let $N+1$ be the number*

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of blocks in M such that the i-th block has size l_i. Let m_j be the number l_i that are equal to j. Define for $r \leq k$

$$Q_{rk} = q^{i_k i_{k+1}} q^{i_{k-1} i_k} \cdots q^{i_r i_{r+1}}$$

in the notation of 2.6. Then if f_d is as in Section 2.4,

$$f_d(1) = \frac{\operatorname{rank}(\pi(f_d))}{m_1! \cdots m_n!} c_M \sum_{\text{trees } T} \sum_{\text{branches } B \text{ of } T} \prod_{\substack{C \prec B \\ C = \{i_0, \dots, i_t\}}} \frac{(1-q^{l_{i_0}})(1-q^{l_{i_1}})}{1-q^{l_{i_0}+l_{i_1}}} \cdot \prod_{k=1}^{t-1} \frac{(1-q^{l_{i_{k+1}}})}{(1-q^{l_{i_k}+l_{i_{k+1}}})} \prod_{r=0}^{k-1} \frac{R_{rk}}{1-Q_{rk} q^{i_r i_{k+1}}}, \tag{3.1}$$

where

$$R_{rk} = \begin{cases} 1 - Q_{rk} q^{g_{i_r} - g_{i_k}} & \text{if } k < t-1 \\ (1 - Q_{r,t-1} q^{g_{i_r} - g_{i_{t-1}}})(1 - Q_{r,t-1} q^{g_{i_k} - g_{i_r}}) & \text{if } k = t-1 \end{cases}.$$

Corollary 3.3. *The denominator of $f_d(1)$ divides a power of the Poincaré polynomial $P_{\mathbf{G}/\mathbf{B}}(q) = P_{\mathfrak{S}_n}(q)$ of $\mathbf{G}$. Moreover, when d is in the lowest two-sided cell, corresponding to $M = T$, $f_d(1) = \operatorname{rank}(\pi(f_d))/P_{\mathbf{G}/\mathbf{B}}(q)$ exactly.*

Proof of Corollary 3.3. We will show that each of the three forms of denominators that appear in the conclusion of Theorem 3.2 divide $P_{\mathbf{G}/\mathbf{B}}(q)$, and thus that their product divides a power of $P_{\mathbf{G}/\mathbf{B}}(q)$. The denominators of c_M are all of the form $1 + q + \cdots + q^{l_i - 1}$, and so divide $P_{\mathbf{G}/\mathbf{B}}(q)$ as $l_i \leq n$ for all i. Note that as $l_{i_k} + l_{i_{k+1}} \leq n$, the leftmost denominators in (3.1) satisfy the conclusion of the corollary also. Finally, $Q_{rk} q^{i_r i_{k+1}} = q^{l_{i_{k+1}} + l_{i_k} + \cdots + l_{i_r}}$, and R_{rk} is likewise always a polynomial in q (as opposed to q^{-1}) divisible by $1 - q$. Again using that $l_{i_{k+1}} + l_{i_k} + \cdots + l_{i_r} \leq n$, we are done with the first statement.

We now take up the second statement, the proof[2] of which does not require any computations at all and which holds for any $w \in \mathbf{c}_0$. Recalling that the only K-spherical tempered representations of G are principal series representations [33], let μ_K be the Haar measure on G such that $\mu_K(K) = 1$, μ_I be the Haar measure such that $\mu_I(I) = 1$, and denote temporarily $\mathrm{d}\pi_\mu$ the Plancherel measure normalized according to a chosen Haar measure μ on G. Likewise write $\pi_K(f)$ for the Fourier transform with respect to μ_K and $\pi_I(f)$ for the Fourier transform with respect to μ_I.

As we have

$$\mu_I = P_{\mathbf{G}/\mathbf{B}}(q) \mu_K,$$

and formal degrees scale inversely to the Haar measure, we have

$$d\pi_{\mu_I} = \frac{1}{P_{\mathbf{G}/\mathbf{B}}(q)} d\pi_{\mu_K} \tag{3.2}$$

[2]We thank A. Braverman for explaining to us the observation whose obvious generalization we present below.

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for tempered principal series. Now let $w \in \mathbf{c}_0$ and consider $f_w = \tilde{\phi}(t_w)$. By Lemma 2.17 and the Satake isomorphism, there is a function h_w in the spherical Hecke algebra such that for all principal series π,

$$\operatorname{trace}(\pi_I(f_w)) = \operatorname{trace}(\pi_K(h_w))$$

as regular Weyl-invariant functions of the Satake parameter. By the Plancherel formula and (3.2),

$$\begin{aligned} f_w(1) = \int \operatorname{trace}(\pi_I(f_w))\, \mathrm{d}\pi_{\mu_I} &= \int \operatorname{trace}(\pi_K(h_w))\, \mathrm{d}\pi_{\mu_I} \\ &= \frac{1}{P_{\mathbf{G}/\mathbf{B}}(q)} \int \operatorname{trace}(\pi_K(h_w))\, \mathrm{d}\pi_{\mu_K} = \frac{h_w(1)}{P_{\mathbf{G}/\mathbf{B}}(q)}, \end{aligned} \tag{3.3}$$

In particular, if $w = d$ is a distinguished involution in the lowest two-sided cell, then

$$\operatorname{trace}(\pi_I(f_d)) = \operatorname{rank}(\pi_I(f_d)) = \operatorname{rank}(\pi_I(f_d)) \cdot \operatorname{trace}(\pi_K(1_K)),$$

and (3.3) becomes

$$f_d(1) = \frac{\operatorname{rank}(\pi(f_d))}{P_{\mathbf{G}/\mathbf{B}}(q)} = \frac{1}{P_{\mathbf{G}/\mathbf{B}}(q)}. \tag{3.4}$$

Here the rank is given by [49], Proposition 5.5. (In *op. cit.* there is the assumption of simple-connectedness, but it is easy to see that the distinguished involutions for the extended affine Weyl group $W_{\mathrm{aff}}(\tilde{\mathbf{G}}(F))$ of the universal cover $\tilde{\mathbf{G}}$ are distinguished involutions for W_{aff} using the definition in [28] and uniqueness of the $\{C_w\}$-basis, and that the lowest cell is just the lowest cell of $W_{\mathrm{aff}}(\tilde{\mathbf{G}}(F))$ intersected with W_{aff}.) □

Once we have established injectivity of $\tilde{\phi}$, we will show by a counting argument that $\operatorname{rank}(\pi(t_d)) = 1$ for any distinguished involution d, in the case $G = \mathrm{GL}_n$, see Theorem 3.21. Note that (3.4) is an example of the behaviour conjectured in Remark 1.3.

Remark 3.4. It would be interesting to find I-biinvariant Schwartz functions h playing the role of h_w for the other two-sided cells, namely such that $\operatorname{trace}(\pi, h)$ was regular, nonzero only for a single pair (M, ω), and the value $h(1)$ was known as a function of $q^{1/2}$.

Remark 3.5. If $\mathcal{P}$ is a maximal parahoric subgroup of G, then the longest word $w_{\mathcal{P}}$ is a distinguished involution in the lowest two-sided cell. As above, and as we will explain again in Section 4, $\pi(f_{w_{\mathcal{P}}})$ is nonzero only for the principal series representations, and its image is contained in $\pi^{\mathcal{P}}$. Thus after we will have shown injectivity, [26] gives another proof that $\operatorname{rank}(\pi(f_{w_{\mathcal{P}}})) = 1$, and more generally that $\pi(t_d)$ has rank 1 on the principal series for any distinguished involution $d \in \mathbf{c}_0$.

Lemma 3.6. *Let $e_0, \ldots, e_n \in \mathbb{Z}$. Then*

$$\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} z_0^{e_0} \cdots z_N^{e_N} \prod_{i<j} \Gamma^{ij} \mathrm{d}z_0 \cdots \mathrm{d}z_N = 0 \tag{3.5}$$

unless $e_0 + \cdots + e_N = -N$.

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Although we do not use the Lemma in the sequel, we include it because it illustrates an efficient way to compute the functions f_w in practice, as we explain after its proof.

Corollary 3.7. *Let $w \in W_{\mathrm{aff}}$. Then $f_w(1)$ is a rational function of q with denominator dividing a power of the Poincaré polynomial of $\mathbf{G}$. The numerator is a Laurent polynomial $p_1(q^{1/2}) + p_2(q^{-1/2})$ in $q^{1/2}$, where the degree of p_1 is bounded uniformly in terms of W_{aff}. The denominator of $f_w(1)$ depends only on the two-sided cell containing w.*

Remark 3.8. In light of Lemma 2.17, Lemma 3.6 and Corollary 3.7 have the following interpretation. Let Γ be a left cell in a two-sided cell $\mathbf{c}$. Then in [51], Xi shows that all the rings $J_{\Gamma\cap\Gamma^{-1}}$ are isomorphic to the representation ring of the associated Levi subgroup $M_{\mathbf{c}}$, and $J_{\mathbf{c}}$ is a matrix algebra over $J_{\Gamma\cap\Gamma^{-1}}$. Therefore $w \in \Gamma\cap\Gamma^{-1}$ are labelled by dominant weights of $M_{\mathbf{c}}$, and if t_λ is such an element, Xi's results show that if $\mathbf{q} = q$ and $\pi = \mathrm{Ind}_B^G(\nu)$ is an irreducible representation of $J_{\mathbf{c}}$, then

$$\mathrm{trace}(\pi, t_\lambda) = \mathrm{trace}(V(\lambda), \nu),$$

where we view ν as a semisimple conjugacy class in $M_{\mathbf{c}}$, and $V(\lambda)$ is the irreducible representation of $M_{\mathbf{c}}$ of highest weight λ. Then we have that $f_\lambda(1) \neq 0$ only if λ is of height 0 with respect to the basis $\varepsilon_i \colon \mathrm{diag}(a_1, \ldots, a_n) \mapsto a_i$.

The proofs of Lemma 3.6 and Corollary 3.7 will use the notation of the proof of Theorem 3.2, and we defer them until after the proof of the theorem.

3.1.1. Example computations and a less singular cell.

In this section we provide two example computations to elucidate the coming proof of Theorem 3.2.

Example 3.9. Let $G = \mathrm{GL}_2$ and $M = A$. Then $l_1 = l_2 = 1$ and $g_1 = g_2 = 0$. Let $d = s_0$ or s_1. Then

$$\begin{aligned}
f_d(1) &= \frac{1}{2}\frac{1}{2\pi i}\frac{1}{2\pi i}q^{-1}\int_{\mathbb{T}}\int_{\mathbb{T}} q\frac{(z_1-z_2)(z_1-z_2)}{(z_1-q^{-1}z_2)(z_1-qz_2)}\frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}\\
&= \frac{1}{2}\frac{1}{2\pi i}\int_{\mathbb{T}}\mathrm{Res}_{z_1=q^{-1}z_2}\frac{(z_1-z_2)(z_1-z_2)}{(z_1-q^{-1}z_2)(z_1-qz_2)}\frac{1}{z_1}\\
&\quad + \mathrm{Res}_{z_1=0}\frac{(z_1-z_2)(z_1-z_2)}{(z_1-q^{-1}z_2)(z_1-qz_2)}\frac{1}{z_1}\frac{\mathrm{d}z_2}{z_2}\\
&= \frac{1}{2}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(q^{-1}z_2-z_2)(q^{-1}z_2-z_2)}{(q^{-1}z_2-qz_2)q^{-1}z_2} + \frac{z_2^2}{z_2^2}\frac{\mathrm{d}z_2}{z_2}\\
&= \frac{1}{2}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(q^{-1}-1)^2}{q^{-2}-1} + 1\frac{\mathrm{d}z_2}{z_2}\\
&= \frac{1}{q+1}.
\end{aligned}$$

The factor $\frac{1}{2}$ reflects the fact that we integrate with the respect to the pushforward of the above $\mathfrak{S}_2$-invariant measure to the quotient $\mathbb{T}\times\mathbb{T}/\mathfrak{S}_2$.

This agrees with the theorem, which instructs us to calculate $f_d(1)$ as follows: There are two trees, each of which has one vertex and no edges. The trees are $z_1 = 0$ and $z_1 = q^{-1}z_2$.

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Each has one branch. The first has no clumps, so the entire product is empty. The second tree has one clump $C=\{1,2\}$ for which $t=1$, and we obtain

$$f_d(1)=\frac{1}{2}\left(1+\frac{1-q}{1+q}\right)=\frac{1}{2}\frac{2}{1+q}=\frac{1}{1+q}.$$

Now we give an example of a less singular cell.

Example 3.10. Let $G=\mathrm{GL}_4$, and let $\mathbf{c}$ be the two-sided cell corresponding to the partition $4=2+2$. Then $P_W(\mathbf{q})=(1+\mathbf{q})(1+\mathbf{q}+\mathbf{q}^2)(1+\mathbf{q}+\mathbf{q}^2+\mathbf{q}^3)$ has five distinct roots $\mathbf{q}=-1,\pm i,\zeta_1,\zeta_2$, where ζ_i are primitive third roots of unity. We compute $P_{\mathbf{c}}$ using the Plancherel formula. We have

We have $l_1=l_2=2$, $q_{12}=1$, $q^{12}=q^2$,

$$c_M=\frac{q^4(q-1)^2}{2(q+1)^2},$$

and

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^2\iint_{\mathbb{T}^2}\Gamma^{12}\frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}&=\left(\frac{1}{2\pi i}\right)^2\iint_{\mathbb{T}^2}\frac{(z_1-z_2)(z_1-z_2)}{(z_1-q^2z_2)(z_1-q^{-2}z_2)}\frac{\mathrm{d}z_1}{z_1}\frac{\mathrm{d}z_2}{z_2}\\&=\frac{1}{2\pi i}\int_{\mathbb{T}}\mathrm{Res}_{z_1=0}\frac{\Gamma^{12}}{z_1}+\mathrm{Res}_{z_1=q^{-2}z_2}\frac{\Gamma^{12}}{z_1}\frac{\mathrm{d}z_2}{z_2}\\&=1+\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(q^{-2}-1)^2q^2}{(q^{-2}-q^2)}\frac{\mathrm{d}z_2}{z_2}\\&=1+\frac{(1+q)(1-q^2)}{1+q+q^2+q^3}\\&=\frac{2}{1+q^2}.\end{aligned}$$

Accounting for c_M, we see that $J_{\mathbf{c}}$ is regular at $\mathbf{q}=\zeta_1,\zeta_2$.

3.1.2. Proofs of Theorem 3.2, Lemma 3.6, and Corollary 3.7.

Proof of Theorem 3.2. Let $n=l_0+\cdots+l_N$. We may assume that $l_0\le l_2\le\cdots\le l_N$. It suffices to evaluate the integral

$$\left(\frac{1}{2\pi i}\right)^{N+1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod\Gamma_{i,j,g}\frac{\mathrm{d}z_0}{z_0}\cdots\frac{\mathrm{d}z_N}{z_N}=\left(\frac{1}{2\pi i}\right)^{N+1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_0}{z_0}\cdots\frac{\mathrm{d}z_N}{z_N}.$$

We claim that the value of this integral is

$$\begin{aligned}c_M\sum_{\text{trees }T}\sum_{\text{branches }B\text{ of }T}\prod_{\substack{C\prec B\\C=\{i_0,\ldots,i_t\}}}\prod_{k=0}^{t-1}&\frac{(1-q^{i_ki_{k+1}}q_{i_ki_{k+1}})(1-q^{i_ki_{k+1}}(q_{i_ki_{k+1}})^{-1})}{1-(q^{i_ki_{k+1}})^2}\\&\cdot\prod_{r=0}^{k-1}\frac{(1-Q_{rk}q_{i_ri_{k+1}})(1-Q_{rk}(q_{i_ri_{k+1}})^{-1})}{(1-Q_{rk}q^{i_ri_{k+1}})(1-Q_{rk}(q^{i_ri_{k+1}})^{-1})},\end{aligned}\tag{3.6}$$

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where the sum over trees is taken over all bookkeeping trees for the integral. When $k=0$, we interpret the product over r as being empty.

First we explain how (3.6) simplifies to (3.1). All cancellations will take place within the same clump C of some branch B, which we now fix. We have

$$1-Q_{rk}(q^{i_r i_{k+1}})^{-1}=1-q^{g_{i_k}+g_{i_{k+1}}+1}q^{i_k i_{k+1}}\cdots q^{i_r i_{r+1}}q^{-g_{i_r}-g_{i_{k+1}}-1}=1-Q_{r,k-1}q^{g_{i_k}-g_{i_r}},$$

which is one of the factors in the product $(1-Q_{r,k-1}q_{i_r i_k})(Q_{r,k-1}(q_{i_r i_k})^{-1})$. The surviving factor in the numerator at index $(k-1,r)$ is then equal to $(1-Q_{r,k-1}q^{g_{i_r}-g_{i_k}})$. In short, the above factors in the denominator cancel with a numerator occurring with the same r-index but k-index one lower. Such a factor occurs whenever $r<k-1$ (note that this inequality does not hold when $k=1$ and $r=0$). When $r=k-1$, we have

$$\begin{aligned}1-Q_{k-1,k}(q^{i_k i_{i+1}})^{-1}&=1-q^{i_k i_{k+1}}q^{i_{k-1}i_k}(q^{i_{k-1}i_{k+1}})^{-1}\\&=q^{g_{i_k}+g_{i_{k+1}}+1}q^{g_{i_{k-1}}-g_{i_{k+1}}}=1-q^{2g_{i_k}+1},\end{aligned}$$

which is one of the factors in $(1-q^{i_{k+1}i_k}q_{i_k i_{k+1}})(1-q^{i_k i_{k+1}}(q_{i_k}i_{k+1})^{-1})$. The cancellation leaves behind the factor $1-q^{l_{i_{k+1}}}$ in the numerator, except for $k=0$; this term keeps both its denominators. At this point we have shown that the factor corresponding to C in (3.6) simplifies to

$$\frac{(1-q^{l_{i_0}})(1-q^{l_{i_1}})}{1-q^{l_{i_0}+l_{i_1}}}\prod_{k=1}^{t-1}\frac{(1-q^{l_{i_{k+1}}})}{(1-q^{l_{i_k}+l_{i_{k+1}}})}\prod_{r=0}^{k-1}\frac{R_{rk}}{1-Q_{rk}q^{i_r i_{k+1}}},$$

where

$$R_{rk}=\begin{cases}1-Q_{rk}q^{g_{i_r}-g_{i_k}} & \text{if } k<t-1\\(1-Q_{r,t-1}q^{g_{i_r}-g_{i_{t-1}}})(1-Q_{r,t-1}q^{g_{i_k}-g_{i_r}}) & \text{if } k=t-1\end{cases}.$$

This means that (3.6) simplifies to (3.1).

To prove (3.6), we will use the residue theorem for each variable consecutively, keeping track of the constant expressions in q that we extract after integrating with respect to each variable z_i. More precisely, we will track what happens in a single summand corresponding to some set of successive choices of poles to take residues at. Note that all the rational functions that will appear, namely the Γ^{ij} or the rational functions that result from substitutions into the Γ^{ij}, become equal to 1 once z_i or z_j is set to zero. Therefore poles at $z_i=0$ serve simply to remove all factors involving z_i from inside the integrand (we will see what these rational functions are below). It follows that the summand whose branch we are computing is a product over clumps in the corresponding branch, so it suffices to compute the value of a given clump for some ordered subsets $\{i_0,i_1,\ldots,i_l\}$ of the indices $\{0,\ldots,N\}$. As we are inside a clump, we will consider only poles occurring at nonzero complex numbers. Thus we are left only to determine what happens within a single clump.

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We first integrate with respect to the variable z_{i_0}. The residue theorem gives

$$\left(\frac{1}{2\pi i}\right)^{l+1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_0}}{z_{i_0}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}$$
$$=\left(\frac{1}{2\pi i}\right)^{l}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\sum_{l\neq i_0}\mathrm{Res}_{z_{i_0}=(q^{i_0 l})^{-1}z_l}\frac{1}{z_{i_0}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}$$
$$+\left(\frac{1}{2\pi i}\right)^{l}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{\substack{i<j\\ i,j\neq i_0}}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}.$$

As noted above, the second integral belongs to a different branch (in fact, in the case of i_0, to a different tree); our procedure will deal with it separately, and we will now consider what happens with the first integral.

For the first integral, consider one of the summands corresponding to $z_{i_0}=(q^{i_0i_1})^{-1}z_{i_1}$ for some i_1. We have

$$\left(\frac{1}{2\pi i}\right)^{l}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\mathrm{Res}_{z_{i_0}=(q^{i_0i_1})^{-1}z_{i_1}}\frac{1}{z_{i_0}}\prod_{i<j}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}$$
$$=\frac{(1-q^{i_0i_1}q_{i_0i_1})(1-q^{i_0i_1}(q_{i_0i_1})^{-1})}{1-(q^{i_0i_1})^2}\left(\frac{1}{2\pi i}\right)^{l}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\cdot$$
$$\prod_{j\neq i_1,i_0}\frac{(z_{i_1}-q^{i_0i_1}q_{i_0j}z_j)(z_{i_1}-q^{i_0i_1}(q_{i_0j})^{-1}z_j)}{(z_{i_1}-q^{i_0i_1}q^{i_0j}z_j)(z_{i_1}-q^{i_0i_1}(q^{i_0j})^{-1}z_j)}\prod_{\substack{i<j\\ i,j\neq i_0}}\Gamma^{ij}\frac{\mathrm{d}z_{i_1}}{z_{i_1}}\frac{\mathrm{d}z_{i_2}}{z_{i_2}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}.$$

Recall that, if $i_1>2$, even though formally we have defined the symbols q_{ij} and q^{ij} only for $i<j$, the symmetry of the factors allows us to write q_{ij} even if $i>j$, thanks to the factor with $(q_{ij})^{-1}$ also present in the numerator.

Now we integrate with respect to z_{i_1}. Observe that the leftmost product over $j\neq i_1,i_0$ does not contribute poles. Indeed, the first factor in each denominator does not have its zero contained in $\mathbb{T}$, and the second factor in each denominator has its zero at $z_{i_1}=q^{i_0i_1}(q^{i_0i_2})^{-1}z_{i_2}$. The power of q appearing is

$$g_{i_0}+g_{i_1}+1-(g_{i_0}+g_{i_2}+1)=g_{i_1}-g_{i_2},$$

and so $q^{i_0i_1}(q^{i_0i_2})^{-1}$ is equal to $q_{i_1i_2}$ or $(q_{i_2i_1})^{-1}$, whichever is defined. Thus this zero cancels with a zero in the numerator of $\Gamma^{i_1i_2}$ or $\Gamma^{i_2i_1}$, whichever is defined. Therefore we need only consider the simple poles at $z_{i_1}=(q^{i_1i_2})^{-1}z_{i_2}$ for $i_1<i_2$ and $z_{i_1}=(q^{i_2i_1})^{-1}z_{i_2}$ for $i_2<i_1$. Observe that the residues will be the same for either inequality. In the case $i_2<i_1$, for example, $\Gamma^{i_2i_1}$ needs to be rewritten so that its simple pole inside $\mathbb{T}$ is in the correct format to calculate the residue by substitution:

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$$\begin{aligned}
&\operatorname{Res}_{z_{i_1}=(q^{i_2i_1})^{-1}z_{i_2}} \frac{(z_{i_2}-q_{i_2i_1}z_{i_1})(z_{i_2}-(q_{i_2i_1})^{-1}z_{i_1})}{(z_{i_2}-q^{i_2i_1}z_{i_1})(z_{i_2}-(q^{i_2i_1})^{-1}z_{i_1})}\frac{1}{z_{i_1}}\\
&\quad=\operatorname{Res}_{z_{i_1}=(q^{i_2i_1})^{-1}z_{i_2}} \frac{-(q^{i_2i_1})^{-1}(z_{i_2}-q_{i_2i_1}z_{i_1})(z_{i_2}-(q_{i_2i_1})^{-1}z_{i_1})}{(z_{i_2}-q^{i_2i_1}z_{i_1})(z_{i_2}-(q^{i_2i_1})^{-1}z_{i_1})}\frac{1}{z_{i_1}}\\
&\quad=\frac{(1-q^{i_2i_1}q_{i_2i_1})(1-q^{i_2i_1}(q_{i_2i_1})^{-1})}{1-(q^{i_2i_1})^2}.
\end{aligned}$$

Therefore after integrating within the clump at hand with respect to z_{i_0} and then z_{i_1}, we have the expression

$$\begin{aligned}
&\frac{(1-q^{i_0i_1}q_{i_0i_1})(1-q^{i_0i_1}(q_{i_0i_1})^{-1})}{1-(q^{i_0i_1})^2}\frac{(1-q^{i_2i_1}q^{i_0i_1}q_{i_0i_2})(1-q^{i_1i_2}q^{i_0i_1}(q_{i_0i_2})^{-1})}{(1-q^{i_2i_1}q^{i_0i_1}q^{i_0i_2})(1-q^{i_2i_1}q^{i_0i_1}(q^{i_0i_2})^{-1})} \\
&\cdot\frac{(1-q^{i_2i_1}q_{i_2i_1})(1-q^{i_2i_1}(q_{i_2i_1})^{-1})}{(1-(q^{i_2i_1})^2)}\\
&\cdot\left(\frac{1}{2\pi i}\right)^{l-1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{j\neq i_0,i_1,i_2}\frac{(z_{i_2}-q^{i_1i_2}q^{i_0i_1}q_{i_0j}z_j)(z_{i_2}-q^{i_1i_2}q^{i_0i_1}(q_{i_0j})^{-1}z_j)}{(z_{i_2}-q^{i_1i_2}q^{i_0i_1}q^{i_0j}z_j)(z_{i_2}-q^{i_1i_2}q^{i_0i_1}(q^{i_0j})^{-1}z_j)}\\
&\cdot\prod_{j\neq i_0,i_1,i_2}\frac{(z_{i_2}-q^{i_1i_2}q_{i_1j}z_j)(z_{i_2}-q^{i_1i_2}(q_{i_1j})^{-1}z_j)}{(z_{i_2}-q^{i_1i_2}q^{i_1j}z_j)(z_{i_2}-q^{i_1i_2}(q^{i_1j})^{-1}z_j)}\prod_{\substack{i<j\\ i,j\neq i_0,i_1}}\Gamma^{ij}\frac{\mathrm{d}z_{i_2}}{z_{i_2}}\cdots\frac{\mathrm{d}z_{i_l}}{z_{i_l}}.
\end{aligned}\tag{3.7}$$

Now we integrate with respect to z_{i_2}. Again, only poles from the product of Γ^{ij}'s occur with nonzero residues: in total we have simple poles contained in $\mathbb{T}$ possibly at $z_{i_2}=q^{i_1i_2}q^{i_0i_1}(q^{i_0j})^{-1}z_j$, at $z_{i_2}=q^{i_1i_2}(q^{i_1j})^{-1}z_j$ and at $z_{i_2}=(q^{i_2,j})^{-1}z_j$ for $j\neq i_0,i_1,i_2$. It may happen that these poles are not all distinct, but all zeros in the denominator of the former two types are in fact cancelled by zeros of denominator anyway. The necessary factors occur in the product immediately adjacent on the right. Indeed, we have

$$g_{i_1}+g_{i_2}+1+g_{i_0}+g_{i_1}+1-g_{i_0}-g_j-1=g_{i_1}+g_{i_2}+1+g_{i_1}-g_j$$

and so either $q^{i_1i_2}q^{i_0i_1}(q_{i_0j})^{-1}=q^{i_1i_2}q_{i_1j}$ or $q^{i_1i_2}q^{i_0i_1}(q_{i_0j})^{-1}=q^{i_1i_2}(q_{i_1j})^{-1}$ (or q_{ji_1} or $(q_{ji_1})^{-1}$).

Likewise we have

$$g_{i_1}+g_{i_2}+1-g_{i_1}-g_j-1=g_{i_2}-g_j$$

as happened when we integrated with respect to z_{i_0}.

Now we see the following pattern: At each stage, we integrate with respect to the variable we took a residue at in the previous step. When integrating with respect to z_{i_r}, there will be $r+1$ products of rational functions, and each rational function in the leftmost r products will contribute a pole, in addition to the pole contributed by $\Gamma^{i_ri_{r+1}}$ or $\Gamma^{i_{r+1}i_r}$, whichever is defined. However, each is nullified by having zero residue thanks to a denominator in the product immediately to the right. Integrating with respect to z_{i_r} will result in extracting $r+1$ new rational factors, each of the form claimed in the theorem. When it comes to integrating with respect to z_{i_l}, all variables z_{i_l} will cancel

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from the remains of the Gamma functions by homogeneity, and the factor $\frac{1}{z_{i_l}}$ will result in the final integral contributing just the remaining $l+1$ rational factors. □

Lemma 3.6 is a porism of the preceding proof.

Proof of Lemma 3.6. We will evaluate the integral (3.5) by applying the residue theorem successively for each variable as in the proof of Theorem 3.2. Note that as functions of any variable z_k, the functions $\prod_{i<j}\Gamma^{ij}$ and all the other products, for example those appearing in (3.7), have numerator and denominator with equal degrees. Thus the overall sum of powers of all z_i in the integrand of (3.5) is $e_0+\cdots+e_N$, and in general, the sum of powers of all z_i in the integrand of an expression like (3.7) is the sum of the degrees of the monomial z_{i_j} terms.

When evaluating (3.5) along a single branch, we find again that the only poles that appear are of the form $z_i=(q^{ij})^{-1}z_j$, or $z_i^{e_i}=0$ for $e_i<0$. We will track the effect that evaluating each successive residue has on the total degree of the integrand, and then conclude using the fact that $\int_{\mathbb{T}} z^r\,\mathrm{d}z=2\pi i\delta_{r,-1}$. First, observe that evaluating a residue of the form $z_i=(q^{ij})^{-1}z_j$ increases the sum of all powers by 1; a factor z_j is contributed to the resulting integrand. The sum of all powers is likewise increased by 1 when evaluating the residue at a simple pole of z_i^{-1} at 0. To compute the residue at a pole of $z_i^{-e_i}$ at 0 for $e_i>1$, consider the Taylor expansion at 0 of Γ^{ij}. We have

$$\frac{1}{z_i-q^{ij}z_j}=-\frac{1}{q^{ij}z_j}-\frac{z_i}{(q^{ij}z_j)^2}-\frac{z_i^2}{(q^{ij}z_j)^3}-\cdots$$

and

$$\frac{1}{z_i-(q^{ij})^{-1}z_j}=-\frac{q^{ij}}{z_j}-\left(\frac{q^{ij}}{z_j}\right)^2 z_i-\left(\frac{q^{ij}}{z_j}\right)^3 z_i^2-\cdots.$$

Multiplying these series and further multiplying by the denominator $z_i^2-(q_{ij}+q_{ij}^{-1})z_iz_j+z_j$, it follows that the Taylor expansion of Γ^{ij} is

$$\begin{aligned}1&+\frac{q^{ij}+(q^{ij})^{-1}-q_{ij}-q_{ij}^{-1}}{z_j}z_i+\frac{(q^{ij})^2+2+(q^{ij})^{-2}-(q_{ij}+q_{ij}^{-1})(q^{ij}+q^{-ij})}{z_j^2}z_i^2+\cdots\\ &+\frac{\begin{array}{c}-(q_{ij}+q_{ij}^{-1})((q^{ij})^{n-1}+\cdots+(q^{ij})^{-n+1})+\\ (q^{ij})^{n-2}+\cdots+(q^{ij})^{-n+2}+(q^{ij})^{n}+\cdots+(q^{ij})^{-n}\end{array}}{z_j^n}z_i^n+\cdots.\end{aligned}\tag{3.8}$$

It is clear that the salient point of (3.8), that z_i^n appears with a coefficient proportional to z_j^{-n}, holds also for the all the products like those in (3.7). Thus computing a residue at the pole of $z_i^{-e_i}$ at 0 will result in a new integrand, the total degree of which has increased by $e_i-e_i+1=1$. It now follows that after integrating with respect to $N-1$ variables, the final integral to be computed will be a constant times $(2\pi i)^{-1}\int_{\mathbb{T}} z_N^{e_0+\cdots+e_N+N-1}\mathrm{d}z_N$. This is nonzero if and only if $e_0+\cdots+e_N=-N$. Summing over all branches of a bookkeeping tree, we see that (3.5) is nonzero only if $e_0+\cdots+e_N=-N$. □

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The lemma is helpful for computing explicit examples, as it points out that one can always avoid dealing with higher-order poles. Indeed, given an integral of the form (3.5), we may assume by Lemma 3.6 that $e_0+\cdots+e_N=-N$. If $e_i=-1$ for all i then the integral (3.5) is just the integral from Theorem 3.2. Otherwise there is some $e_{i_0}\geq 0$. We may assume that $i_0=0$. Then the only poles in z_0 are of the form $z_0=(q^{0i_1})^{-1}z_{i_1}$ for indices $i_1>0$. Therefore we compute that (3.5) is equal to a sum of terms of the form

$$\frac{(1-q^{0i_1}q_{0i_1})(1-q^{0i_1}(q_{0i_1})^{-1})}{1-(q^{0i_1})^2}(q^{0i_1})^{-e_0-1}\left(\frac{1}{2\pi i}\right)^{N-1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}} z_{i_1}^{e_{i_1}+1+e_0}z_{i_2}^{e_{i_2}}\cdots z_{i_N}^{e_{i_N}} \cdot\prod_{j\neq i_1,0}\frac{(z_{i_1}-q^{0i_1}q_{0j}z_j)(z_{i_1}-q^{0i_1}(q_{0j})^{-1}z_j)}{(z_{i_1}-q^{0i_1}q^{0j}z_j)(z_{i_1}-q^{0i_1}(q^{0j})^{-1}z_j)}\prod_{\substack{i<j\\ i,j\neq 0}}\Gamma^{ij}\mathrm{d}z_{i_1}\cdots\mathrm{d}z_{i_N}. \tag{3.9}$$

The total degree of the integrand is now $e_0+\cdots+e_N+1=-(N-1)$. Therefore either $e_{i_0}+e_{i_1}+1=e_{i_2}=\cdots=e_{-N}=-1$, or we may again assume without loss of generality that the exponent of some z_{i_j} is nonnegative, and proceed with evaluating (3.9) by integrating with respect to z_{i_j}. We may continue in this way, never having to deal with more than a simple pole at 0. (Of course, the resulting order of integration need not be the same as the order used in the proofs of Theorem 3.2 and Corollary 3.7.)

We can now prove Corollary 3.7.

Proof of Corollary 3.7. In light of Lemma 2.17, it is enough to prove that the conclusions of the present corollary hold for integrals of the form (3.5). We again follow the algorithm from the proof of Theorem 3.2, so it is sufficient to consider a single branch, and for this it suffices to observe that, given, for example, a variant

$$\left(\frac{1}{2\pi i}\right)^{l-1}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{j\neq i_0,i_1,i_2}\frac{(z_{i_2}-q^{i_1i_2}q^{i_0i_1}q_{i_0j}z_j)(z_{i_2}-q^{i_1i_2}q^{i_0i_1}(q_{i_0j})^{-1}z_j)}{(z_{i_2}-q^{i_1i_2}q^{i_0i_1}q^{i_0j}z_j)(z_{i_2}-q^{i_1i_2}q^{i_0i_1}(q^{i_0j})^{-1}z_j)} \cdot\prod_{j\neq i_0,i_1,i_2}\frac{(z_{i_2}-q^{i_1i_2}q_{i_1j}z_j)(z_{i_2}-q^{i_1i_2}(q_{i_1j})^{-1}z_j)}{(z_{i_2}-q^{i_1i_2}q^{i_1j}z_j)(z_{i_2}-q^{i_1i_2}(q^{i_1j})^{-1}z_j)}\prod_{\substack{i<j\\ i,j\neq i_0,i_1}}\Gamma^{ij}z_{i_2}^{e_{i_2}}\cdot\cdots\cdot z_{i_l}^{e_{i_l}}\mathrm{d}z_{i_2}\cdots\mathrm{d}z_{i_l}$$

of (3.7) in which $e_{i_2}<-1$, the quotient rule gives that the residue at $z_{i_2}=0$ is equal to a linear combination over $\mathbb{Q}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$ of integrals of the form

$$\left(\frac{1}{2\pi i}\right)^{l-2}\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\prod_{\substack{i<j\\ i,j\neq i_0,i_1,i_2}}\Gamma^{ij}z_{i_3}^{e'_{i_3}}\cdot\cdots\cdot z_{i_l}^{e'_{i_l}}\mathrm{d}z_{i_3}\cdots\mathrm{d}z_{i_l}$$

for new exponents e'_{i_k}. This and the calculations in the proof of Theorem 3.2 make clear that the only positive powers of $q^{1/2}$ that appear in the any denominator are those that appeared as in Theorem 3.2. These are controlled by the possible block sizes of M, and hence are bounded in terms of W_{aff}. Throughout this procedure, the denominator has been contributed to only by the Plancherel density itself, which depends only on M. The last claim of the corollary now follows from Proposition 2.11. □

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3.2. The functions f_w for general $\mathbf{G}$

For general $\mathbf{G}$, we will follow the same plan as for $\mathbf{G}=\mathrm{GL}_n$. The only difference is that we have less control over which denominators can appear. Indeed, this is true even for formal degrees, but complications are also introduced by residual coset we integrate over, or equivalently, by lack of explicit control of the Satake parameter of $\pi = i_P^G(\omega\otimes\nu)$ for arbitrary discrete series representations $\omega\in\mathcal{E}_2^I(M)$.

Theorem 3.11. *Let $w\in W_{\mathrm{aff}}$. Then $f_w(1)$ is a rational function of q with poles drawn from a finite set of roots of unity depending only on W_{aff}. The numerator is a Laurent polynomial in $q^{1/2}$. The denominator depends only on the two-sided cell containing w. If w is in the lowest two-sided cell, then the denominator divides the Poincaré polynomial of $\mathbf{G}$.*

In the proof, we do not attempt to record any information about degrees of the numerators. It will therefore be necessary to control the possible numerators of $f_w(1)$ in a different manner than for $\mathbf{G}=\mathrm{GL}_n$ in order to prove Proposition 3.13 below.

Proof of Theorem 3.11. First, we note that the reasoning for the lowest cell used in the proof of Corollary 3.3 holds for arbitrary G. Therefore (3.3) proves the last claim just as for GL_n.

More generally, let $w\in W_{\mathrm{aff}}$ and M_P be a Levi subgroup corresponding to the two-sided cell containing w. Let $N=\mathrm{rk}\,A_P$. Given a coroot $\alpha^\vee$ and a basis of the cocharacter lattice as explained in Section 2.7, we write $\alpha^\vee = z_1^{e_1}\cdots z_n^{e_n}$ for integers $e_i=e_i(\alpha)$.

By Lemma 2.17 and Theorem 2.2, it suffices to show the conclusions of the theorem hold for integrals of the form

$$\left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}}\cdots\int_{\mathbb{T}} \prod_{\alpha\in R_{1,+}\setminus R_{P,1,+}} \cdot \frac{q_\alpha(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega)(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega^{-1})}{\left(z_1^{\frac{e_1}{2}}z_2^{\frac{e_2}{2}}\cdots z_n^{\frac{e_n}{2}}+q_\omega^{1/2}q_\alpha^{1/2}\right)\left(z_1^{\frac{e_1}{2}}z_2^{\frac{e_2}{2}}\cdots z_n^{\frac{e_n}{2}}+q_\omega^{-1/2}q_\alpha^{-1/2}\right)}$$
$$\cdot \frac{1}{\left(z_1^{\frac{e_1}{2}}z_2^{\frac{e_2}{2}}\cdots z_n^{\frac{e_n}{2}}-q_\omega^{1/2}q_\alpha^{1/2}q_{2\alpha}\right)\left(z_1^{\frac{e_1}{2}}z_2^{\frac{e_2}{2}}\cdots z_n^{\frac{e_n}{2}}-q_\omega^{1/2}q_\alpha^{1/2}q_{2\alpha}\right)} z_1^{f_1}\cdots z_n^{f_n}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}, \tag{3.10}$$

where $f_i\in\mathbb{Z}$ and $q_\omega = q_{\omega,\alpha}$ is the value of $\alpha^\vee$ on the Satake parameter of ω – this is a positive power of $q^{1/2}$ (of q if $\alpha^\vee/2$ is a coroot) by Sections 7 and 8 of [38]. We have $q_{2\alpha}=1$ and the resulting simplification (2.6) whenever $\alpha^\vee/2$ is not a coroot.

With notation fixed, the theorem is essentially an observation. Indeed, suppose that we integrate with respect to z_1, and wish to compute the contribution to the residue at a pole

$$z_1 = \xi q_\omega^{-\frac{1}{e_1}} q_\alpha^{-\frac{1}{e_1}} z_2^{\frac{e_2}{e_1}}\cdots z_n^{\frac{e_n}{e_1}}$$

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arising from the factor

$$\frac{q_\alpha(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega)(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega^{-1})}{(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega q_\alpha)(z_1^{e_1}z_2^{e_2}\cdots z_n^{e_n}-q_\omega^{-1}q_\alpha^{-1})}\frac{1}{z_1}$$
$$=\frac{q_\alpha(z_1^{e_1}-q_\omega z_2^{-e_2}\cdots,z_n^{-e_n})(z_1^{e_1}-q_\omega^{-1}z_2^{-e_2}\cdots,z_n^{-e_n})}{(z_1^{e_1}-q_\omega q_\alpha z_2^{-e_2}\cdots,z_n^{-e_n})\prod_\zeta(z_1-\zeta q_\omega^{-\frac{1}{e_1}}q_\alpha^{-\frac{1}{e_1}}z_2^{\frac{e_2}{e_1}}\cdots z_n^{\frac{e_n}{e_1}})}\frac{1}{z_1},$$

where ξ and the ζ are primitive e_1-st roots of unity. The contribution is then

$$\frac{q_\alpha(q_\omega^{-1}q_\alpha^{-1}z_2^{-e_2}\cdots z_n^{-e_n}-q_\omega z_2^{-e_2}\cdots z_n^{-e_n})}{(q_\omega^{-1}q_\alpha^{-1}z_2^{-e_2}\cdots z_n^{-e_n}-q_\omega q_\alpha z_2^{-e_2}\cdots z_n^{-e_n})}$$
$$\cdot\frac{(q_\omega^{-1}q_\alpha^{-1}z_2^{-e_2}\cdots z_n^{-e_n}-q_\omega^{-1}z_2^{-e_2}\cdots z_n^{-e_n})}{\prod_{\zeta\neq\xi}(\xi q_\omega^{\frac{-1}{e_1}}q_\alpha^{\frac{-1}{e_1}}z_2^{\frac{-e_2}{e_1}}\cdots z_n^{\frac{-e_n}{e_1}}-\zeta q_\omega^{\frac{-1}{e_1}}z_2^{\frac{-e_2}{e_1}}\cdots z_n^{\frac{-e_n}{e_1}})}\cdot\frac{1}{\xi q_\omega^{-\frac{1}{e_1}}q_\alpha^{-\frac{1}{e_1}}z_2^{\frac{e_2}{e_1}}\cdots z_n^{\frac{e_n}{e_1}}}, \tag{3.11}$$

which simplifies to

$$\frac{(1-q_\alpha q_\omega^2)(1-q_\alpha)}{(1-q_\alpha^2q_\omega^2)\prod_{\zeta\neq\xi}(1-\xi^{-1}\zeta)}. \tag{3.12}$$

We have used no special properties of the integers e_i or f_j, and thus after integrating with respect to z_1, (3.10) is equal to a sum of integrals with respect to $z_2,\ldots,z_n$ again of the form (3.10), except the factors in the denominator will now involve powers $q_\omega(\alpha)q_\omega(\alpha')^{1/2}$ and $q_\omega(\alpha)q_\omega(\alpha')^{-1/2}$, as for $\mathbf{G}=\mathrm{GL}_n$. The coefficients of this sum are the form

$$\frac{Q}{(1\pm q^{r_1})^{c_1}\cdots(1\pm q^{r_k})^{c_k}},$$

where Q is a Laurent polynomial in $q^{\frac{1}{2}}$, $r_i\in\frac{1}{2}\mathbb{N}$ with, *a priori*, complex coefficients, and $c_i\in\mathbb{N}$. Indeed, as we never required any cancellations with the numerator to extract rational functions of q of the required form, it is clear that the simplification of (3.11) to (3.12) works essentially the same way for any higher order poles that appear – again thanks to the quotient rule – and that when integrating with respect to subsequent variables, the additional rational functions of q appearing have the same shape as in the proof of Theorem 3.2. It is also clear that the factors corresponding to non-reduced roots behave similarly.

Therefore (3.10) has poles in q only at a finite number of roots of unity. Again by the quotient rule, exponents r_i and c_i depend only on M. Clearly there are only finitely many exponents r_i that appear for any M. The theorem now follows. □

3.3. Relating t_w and f_w

We will now relate the Schwartz functions f_w on G to the elements $\phi^{-1}(t_w)$ of a completion $\boldsymbol{\mathcal{H}}^-$ of $\mathbf{H}$, whose definition we will now recall. In this section $\mathbf{G}$ is general.

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3.3.1. Completions of H and $J\otimes_{\mathbb{C}}\mathcal{A}$. Let $\hat{\mathcal{A}}=\mathbb{C}((\mathbf{q}^{-1/2}))$ and $\hat{\mathcal{A}}^-=\mathbb{C}[[\mathbf{q}^{-1/2}]]$. Write $\boldsymbol{\mathcal{H}}^-$ for the $\hat{\mathcal{A}}$-algebra

$$\boldsymbol{\mathcal{H}}^- := \left\{ \sum_{x\in W_{\mathrm{aff}}} b_x T_x \,\middle|\, b_x\in\hat{\mathcal{A}},\ b_x\to 0 \text{ as } \ell(x)\to\infty \right\},$$

where we say that $b_x\to 0$ as $\ell(x)\to\infty$ if for all $N>0$, $b_x\in(\mathbf{q}^{-1/2})^N\hat{\mathcal{A}}^-$ for all x sufficiently long.

Consider also the completions

$$\boldsymbol{\mathcal{H}}^-_{C'} := \left\{ \sum_{x\in W_{\mathrm{aff}}} b_x C'_x \,\middle|\, b_x\in\hat{\mathcal{A}},\ b_x\to 0 \text{ as } \ell(x)\to\infty \right\}$$

and

$$\boldsymbol{\mathcal{H}}^-_{C} := \left\{ \sum_{x\in W_{\mathrm{aff}}} b_x C_x \,\middle|\, b_x\in\hat{\mathcal{A}},\ b_x\to 0 \text{ as } \ell(x)\to\infty \right\}$$

of **H** (note the difference between C'_x and C_x), as well as the completion

$$\boldsymbol{\mathcal{J}} := \left\{ \sum_{x\in W_{\mathrm{aff}}} b_x t_x \,\middle|\, b_x\in\hat{\mathcal{A}},\ b_x\to 0 \text{ as } \ell(x)\to\infty \right\}$$

of $J\otimes_{\mathbb{C}}\mathcal{A}$.

In [28], Lusztig shows that ϕ extends to an isomorphism of $\hat{\mathcal{A}}$-algebras $\boldsymbol{\mathcal{H}}^-_C\to\boldsymbol{\mathcal{J}}$. In this way the elements $t_w\in J\subset\boldsymbol{\mathcal{J}}$ may be identified with elements of $\boldsymbol{\mathcal{H}}^-_C$ via ϕ.

Lemma 3.12. *We have $\boldsymbol{\mathcal{H}}^-_{C'}\subset\boldsymbol{\mathcal{H}}^-$. The inclusion is continuous.*

Proof. Given an infinite sum $\sum_x b_x C'_x$, upon rewriting this sum in the standard basis, the coefficient of some T_y is

$$a_y := \sum_{x\geq y} b_x \mathbf{q}^{-\frac{\ell(x)}{2}} P_{y,x}(\mathbf{q}).$$

As $\deg P_{y,x}\leq\frac{1}{2}(\ell(x)-\ell(y)-1)$, we have that $\mathbf{q}^{-\frac{\ell(x)}{2}}P_{y,x}(\mathbf{q})$ is a polynomial in $\mathbf{q}^{-1/2}$. Therefore the above sum defines a formal Laurent series. Moreover, as $\ell(y)\to\infty$, it is clear that $a_y\to 0$. Continuity of the inclusion is clear from the formula for a_y. □

3.3.2. The functions f_w and the basis elements t_w. We shall now explain how the map $\tilde{\phi}\colon J\to\mathcal{C}(G)^I$ induces a map of $\mathcal{A}$-algebras $\hat{\phi}\colon J\otimes_{\mathbb{C}}\mathcal{A}\to\boldsymbol{\mathcal{H}}^-$.

Proposition 3.13. *There is a map of $\mathcal{A}$-algebras $\hat{\phi}\colon J\otimes_{\mathbb{C}}\mathcal{A}\to\boldsymbol{\mathcal{H}}^-$ such that if $\hat{\phi}(t_w)=\sum_x a_{x,w}T_x$, then $a_{x,w}(q)=f_w(x)$. Moreover, there is a constant N depending only on W_{aff} such that $a_{x,w}\in(\mathbf{q}^{1/2})^N\hat{\mathcal{A}}^-$ for all $x,w\in W_{\mathrm{aff}}$.*

The most difficult part of the proof of the proposition is showing that $a_{x,w}\to 0$ as $\ell(x)\to\infty$. To this end we have

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Lemma 3.14. *Let $w \in W_{\mathrm{aff}}$. Then the degree in q of the numerator of $f_w(1)$ is bounded uniformly in w by some N depending only on W_{aff}, and hence in the notation of Proposition 3.13 above, we have $a_{1,w} \in (\mathbf{q}^{1/2})^N \hat{\mathcal{A}}^-$ for all $w \in W_{\mathrm{aff}}$.*

Remark 3.15. In type A, the Lemma follows immediately from the order of integration given after the proof of Lemma 3.6.

Proof of Lemma 3.14. Let $\pi_\omega = \mathrm{Ind}_P^G(\nu \otimes \omega)$ be a tempered I-spherical representation of G arising by induction from a parabolic P such that $\mathrm{rank}\, A_P = k$. Let $\mathbb{T}_\omega^k$ be the compact torus parametrizing twists of ω, and let t_w be given. By Lemma 2.17, $\mathrm{trace}(\pi, f_w)$ is a regular function on $\mathbb{T}^k$, *i.e.* a Laurent polynomial in the coordinates $z_1, \ldots, z_k$ on $\mathbb{T}^k$. The coefficients of this Laurent polynomial are independent of q, as J and its representation theory are independent of q. As we have

$$|1-q^r| \leq |z_i - q^r z_1^{e_1} \cdots z_k^{e_k}| \leq |1+q^r|$$

for all $z_i \in \mathbb{T}$ and r, we may bound, for large q, the absolute value $|\mu_{M_P}(z)|$ of the Plancherel density for M_P by a rational function $U_{M_P,\omega}(q)$. By Theorem 2.2, we may do the same for formal degrees $d(\omega)$, bounding $|d(\omega)|$ by a rational function $D_\omega(q)$. Thus we define a rational function of q by

$$\frac{h_1(q)+h_2(q^{-1})}{k_1(q)+k_2(q^{-1})} = \sum_\omega U_{M_P,\omega}(q) D_\omega(q) \max_{z \in \mathbb{T}_\omega^k} |\mathrm{trace}(\pi_\omega, f_w)|,$$

where $h_1, k_1 \in \mathbb{C}[q]$ and $h_2, k_2 \in q^{-1}\mathbb{C}[q^{-1}]$, and the sum is over the finitely many, up to unitary twist, pairs (M, ω), for $\omega \in \mathcal{E}_2(M)$, such that $\mathrm{trace}(\pi_\omega, f_w) \neq 0$.

Fix $w \in W_{\mathrm{aff}}$. Then

$$|f_w(1)| = \left| \sum_{(P,\omega)} \int_{\mathcal{O}(\omega)} \mathrm{trace}(\pi_\omega, f_w)\, d(\omega)\, \mathrm{d}\mu_{M_P}(z) \right| \leq \frac{h_1(q)+h_2(q^{-1})}{k_1(q)+k_2(q^{-1})}.$$

Crucially, this expression holds for all $q \gg 1$.

On the other hand, by Corollary 3.7 and Theorem 3.11, $f_w(1)$ is a rational function of q, and for q sufficiently large, we may write

$$|f_w(1)| = \frac{f_1(q)+f_2(q^{-1})}{d_1(q)+d_2(q^{-1})}$$

where $f_1, d_1 \in \mathbb{C}[q]$ and $f_2, d_2 \in q^{-1}\mathbb{C}[q^{-1}]$. Therefore for all $q \gg 1$ we have

$$|(f_1(q)+f_2(q^{-1}))||(k_1(q)+k_2(q^{-1}))| \leq |(d_1(q)+d_2(q^{-1})||(h_1(q)+h_2(q^{-1}))|. \tag{3.13}$$

We claim that this implies

$$|f_1(q)k_1(q)| \leq |d_1(q)h_1(q)| \tag{3.14}$$

for q sufficiently large. Indeed, let $\epsilon > 0$ be given and choose $q \gg 1$ such that

$$\big| |(f_1(q)+f_2(q^{-1}))||(k_1(q)+k_2(q^{-1}))| - |f_1(q)k_1(q)| \big| \leq \epsilon$$

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and

$$\big||(d_1(q)+d_2(q^{-1}))||(h_1(q)+h_2(q^{-1}))|-|d_1(q)h_1(q)|\big|\leq\epsilon,$$

and (3.13) holds. Then we have

$$\begin{aligned}|f_1(q)||k_1(q)| &\leq |(f_1(q)+f_2(q^{-1}))||(k_1(q)+k_2(q^{-1}))|+\epsilon\\ &\leq |(d_1(q)+d_2(q^{-1}))||(h_1(q)+h_2(q^{-1}))|+\epsilon\\ &\leq |d_1(q)||h_1(q)|+2\epsilon,\end{aligned}$$

which proves (3.14).

Therefore

$$\deg f_1 \leq \deg f_1+\deg k_1 \leq \deg d_1+\deg h_1.$$

Now, the denominator of $f_w(1)$, and hence $\deg d_1$, depends only on the two-sided cell containing w, again by Corollary 3.7 and Theorem 3.11. We can also bound $\deg h_1$ uniformly in terms of W_{aff}, as it depends only on the Plancherel measure and the finitely many possible formal degrees appearing in the parametrization of the I-spherical part of the tempered dual of G. This proves the lemma. □

If $t_w \in t_d J t_{d'}$ for $d\neq d'$, then t_w is a commutator. We record this observation as

Lemma 3.16. *We have that $f_w(1)=0$ unless $f_d\star f_w\star f_d\neq 0$ for some distinguished involution d.*

Now we can prove the proposition.

Proof of Proposition 3.13. Let $w\in W_{\mathrm{aff}}$. Write $\tilde{\phi}(t_w)=f_w=\sum_x A_{x,w}T_x$ as Schwartz functions on G so that $A_{x,w}=f_w(x)$. We must show that there is a unique element $a_{x,w}\in\hat{\mathcal{A}}$ such that $a_{x,w}(q)=A_{x,w}$ as complex numbers. We will then check that $a_{x,w}\to 0$ rapidly enough as $\ell(x)\to\infty$ for $\sum_x a_{x,w}T_x$ to define an element of $\boldsymbol{\mathcal{H}}^-$.

By Corollary 3.7 and Theorem 3.11, there is a formal power series in $\hat{\mathcal{A}}^-$ with constant term equal to 1 that specializes to the denominator of $A_{1,w}$ when $\mathbf{q}=q$. Moreover, there is a unique formal Laurent series $a_{1,w}\in\hat{\mathcal{A}}$ such that $a_{1,w}(q)=A_{1,w}$ for all prime powers. Indeed, $a_{1,w}$ is convergent for $\mathbf{q}=q$, and the difference of any two such series defines a meromorphic function of $\mathbf{q}^{-1/2}$ outside the unit disk with zeros at $q=p^r$ for every $r\in\mathbb{N}$. As these prime powers accumulate at ∞, such a meromorphic function must be identically zero.

If $f\in\mathcal{C}^{I\times I}$ is a Harish-Chandra Schwartz function, then

$$\begin{aligned}q^{-\ell(x)}(f\star T_{x^{-1}})(1) &= q^{-\ell(x)}\int_G f(g)T_{x^{-1}}(g^{-1})\,\mathrm{d}\mu_I(g)\\ &= q^{-\ell(x)}\int_{IxI} f(g)\,\mathrm{d}\mu_I(g)=q^{-\ell(x)}\mu_I(IxI)f(x)=f(x).\end{aligned}$$

By definition, $f_w(x)=A_{x,w}$.

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On the other hand, according to Lemma 2.9, we have, for $\omega(x^{-1})_f$ as defined above Lemma 1.10,

$$\begin{aligned}
&q^{-\ell(x)}(f_w \star T_{x^{-1}})(1)\\
&\quad = q^{-\ell(x)}\left(\tilde\phi(t_w)\star\tilde\phi\left(\phi_q({}^{\dagger}T_{x^{-1}})\right)\right)(1)\\
&\quad = q^{-\ell(x)}\tilde\phi\left(t_w\phi_q({}^{\dagger}T_{x^{-1}})\right)(1)\\
&\quad = q^{-\ell(x)}\tilde\phi\left(t_w\phi_q\left(\sum_{y\le x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})-\ell(y)}Q_{y,x^{-1}}(q)^{\dagger}C'_y\right)\right)(1) \qquad (3.15)\\
&\quad = q^{-\ell(x)}\tilde\phi\left(t_w\phi_q\left(\sum_{y\le x^{-1}} q^{\frac{\ell(y)}{2}}(-1)^{\ell(x^{-1})}(-1)^{\ell(\omega(x^{-1})_f)}Q_{y,x^{-1}}(q)(C_y)\right)\right)(1) \qquad (3.16)\\
&\quad = (-1)^{\ell(x^{-1})}(-1)^{\ell(\omega(x^{-1})_f)}q^{-\ell(x)}\tilde\phi\left(t_w\sum_{y\le x^{-1}} q^{\frac{\ell(y)}{2}}Q_{y,x^{-1}}(q)\sum_{\substack{r\\ d\in\mathcal{D}\\ a(d)=a(r)}} h_{y,d,r}t_r\right)(1)\\
&\qquad\qquad (3.17)\\
&\quad = (-1)^{\ell(x^{-1})+\ell(\omega(x^{-1})_f)}q^{-\ell(x)}\tilde\phi\left(\sum_{y\le x^{-1}} q^{\frac{\ell(y)}{2}}Q_{y,x^{-1}}(q)\sum_{\substack{r\sim_L d\\ d\in\mathcal{D}\\ a(d)=a(r)=a(w)}} h_{y,d,r}t_wt_r\right)(1)\\
&\quad = (-1)^{\ell(x^{-1})+\ell(\omega(x^{-1})_f)}q^{-\ell(x)}\sum_{y\le x^{-1}} q^{\frac{\ell(y)}{2}}Q_{y,x^{-1}}(q)\sum_{\substack{r\sim_L d_w\\ a(r)=a(w)}} h_{y,d_w,r}(f_w\star f_r)(1), \qquad (3.18)
\end{aligned}$$

where d_w is the unique distinguished involution in the right cell containing w.

In line (3.15), we rewrote $T_{x^{-1}}$ in terms of the C'-basis of H, using the inverse Kazhdan-Lusztig polynomials $Q_{y,x^{-1}}$. In line (3.16), we applied the involution ${}^{\dagger}(-)$ (see Lemma 1.10). In line (3.17) we applied Lusztig's map ϕ_q, and then in line (3.18), we applied the map $\tilde\phi$. Also in line (3.18), we used that left (respectively right) cells give left (respectively right) ideals of J, and so t_wt_r is an integral linear combination of $t_{z^{-1}}$, with $d\sim_L z^{-1}\sim_R d_w$. By lemma 3.16, $f_{z^{-1}}(1)\neq 0$ only if $d=d_w$, that is, $z^{-1}\sim_L d_w$.

We use (3.18) to define $a_{x,w}\in\hat{\mathcal{A}}$. By the same arguments as above, $a_{x,w}$ is unique and defines a meromorphic function of $\mathbf{q}^{-1/2}$. It remains to show that as $\ell(x)\to\infty$, $a_{x,w}\to 0$ in the $(\mathbf{q}^{-1/2})$-adic topology. This follows in fact from (3.18). Indeed, the product $f_w\star f_r$ is an $\mathbb{N}$-linear combination of functions f_z, and the values $f_z(1)$ are rational functions of q, the numerators of which have uniformly bounded degree in q by Lemma 3.14. The polynomials $h_{y,d_w,r}$ have bounded degree in q (for example in terms of the a-function). Finally, the degree in q of

$$q^{-\ell(x)}q^{\frac{\ell(y)}{2}}Q_{y,x^{-1}}(q)$$

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is at most

$$q^{-\ell(x)}q^{\frac{\ell(y)}{2}}q^{\frac{\ell(x^{-1})-\ell(y)-1}{2}} = q^{-\ell(x)}q^{\frac{\ell(y)}{2}}q^{\frac{\ell(x)-\ell(y)-1}{2}} = q^{\frac{-\ell(x)-1}{2}} \to 0 \tag{3.19}$$

as $\ell(x) \to \infty$. This completes the definition of $\hat{\phi}$ as a map of $\mathcal{A}$-modules.

It is easy to see that $\hat{\phi}$ is a morphism of rings, essentially because $\tilde{\phi}$ is. Indeed, we have

$$\hat{\phi}(t_w t_{w'}) = \sum_z \gamma_{w,w',z^{-1}} \sum_x a_{x,z} T_x \tag{3.20}$$

while on the other hand

$$\hat{\phi}(t_w)\hat{\phi}(t_{w'}) = \sum_x a_{x,w} T_x \cdot \sum_y a_{y,w'} T_y \tag{3.21}$$

and when $\mathbf{q} = q$, we have that (3.21) becomes by definition

$$\tilde{\phi}(t_w) \star \tilde{\phi}(t_{w'}) = \tilde{\phi}\left(\sum_z \gamma_{w,w',z^{-1}} t_z\right) = \sum_z \sum_x \gamma_{w,w',z^{-1}} A_{x,z} T_x.$$

Hence for infinitely many prime powers we have that the specializations of (3.20) agrees with those of (3.21), and hence (3.20) is equal to (3.21) in $\boldsymbol{\mathcal{H}}^-$. A similar argument shows that $\hat{\phi}$ preserves units. □

Remark 3.17. The proof, specifically (3.19), gives a necessary condition for an element of $\boldsymbol{\mathcal{H}}^-$ to belong to the image of $\hat{\phi}$: the coefficients must decay asymptotically at least as fast as $\mathbf{q}^{-\frac{\ell(x)}{2}}$.

Proposition 3.18. *There is a commutative diagram*

$$\begin{array}{ccccc} \mathbf{H} & \xrightarrow{\phi} & J \otimes_{\mathbb{C}} \mathcal{A} & \xrightarrow{\hat{\phi}} & \boldsymbol{\mathcal{H}}^- \\ & & \downarrow{\scriptstyle \phi^{-1}} & & \uparrow \\ & & \boldsymbol{\mathcal{H}}_C^- & \xrightarrow{{}^\dagger(-)} & \boldsymbol{\mathcal{H}}_{C'}^-, \end{array}$$

and we have $\hat{\phi} = {}^\dagger(-) \circ \phi^{-1}$ *as morphisms of* $\mathcal{A}$*-algebras* $J \otimes_{\mathbb{C}} \mathcal{A} \to \boldsymbol{\mathcal{H}}^-$. *In particular,* $a_{x,w}$ *has integer coefficients for all* $x, w \in W_{\mathrm{aff}}$.

Proof. The second claim follows from the first if we show that $\hat{\phi}$ extends to a continuous morphism $\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{H}}^-$, by density of $\phi(\mathbf{H})$ in $\boldsymbol{\mathcal{J}} \simeq \boldsymbol{\mathcal{H}}_C^-$, and the third claim follows from the second and the fact that the completions we consider are actually defined over $\mathbb{Z}$ [28, Thm. 2.8].

Note that as $\tilde{\phi} \circ \phi_q = {}^\dagger(-)$ on H for all q, we have that $\hat{\phi} = {}^\dagger(-) \circ \phi^{-1}$ on $\phi(\mathbf{H})$. This says that the diagram commutes.

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We now show that $\hat{\phi}$ extends to a continuous map $\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{H}}_C^-$. Let $\sum_w b_w t_w$ define an element of $\boldsymbol{\mathcal{J}}$ and define

$$\hat{\phi}\left(\sum_w b_w t_w\right) = \sum_y b'_y T_y,$$

where

$$b'_y = \sum_w b_w a_{y,w}.$$

We must first show that this infinite sum of elements of $\hat{\mathcal{A}}$ is well-defined. By Lemma 3.14 and (3.18), we have that there is $M \in \mathbb{N}$ such that $a_{y,w} \in (\mathbf{q}^{1/2})^M \hat{\mathcal{A}}^-$ for all y, w. Therefore b'_y is well-defined, and as $a_{y,w} \to 0$ as $\ell(y) \to \infty$, we have $b'_y \to 0$ as $\ell(y) \to \infty$. Therefore $\hat{\phi}$ extends to $\boldsymbol{\mathcal{J}}$.

To show continuity, it suffices to show that if $\{\sum_w b_{w,n} t_w\}_n$ is a sequence of elements of $\boldsymbol{\mathcal{J}}$ tending to 0 as $n \to \infty$, then

$$\sum_w b_{w,n} \hat{\phi}(t_w) = \sum_y b'_{y,n} T_y \to 0$$

as $n \to \infty$ in $\boldsymbol{\mathcal{H}}^-$, where $b'_{y,n} = \sum_w b_{w,n} a_{y,w}$. For all $R > 0$, there is $N > 0$ such that $n > N$ implies $b_{x,n} \in (\mathbf{q}^{-1/2})^R \hat{\mathcal{A}}^-$ for all x. We have seen that there is M depending only on W_{aff} such that $a_{y,w} \in (\mathbf{q}^{1/2})^M \hat{\mathcal{A}}^-$ for all w, y. Therefore $b'_{y,n} \to 0$ as $n \to \infty$, because $b_{w,n} \to 0$ as $n \to \infty$. □

Note that Proposition 3.18 means in particular that $\hat{\phi}(J) \subset \boldsymbol{\mathcal{H}}_{C'}^-$.

Corollary 3.19. *We have*

$$(\phi \circ^\dagger (-))^{-1}(t_1) = f_1(1) \sum_{w \in W_{\mathrm{aff}}} (-1)^{\ell(w)} q^{-\ell(w)} T_w.$$

Proof. If $w = 1$, everything in (3.18) reduces to $r = d_w = y = 1$, and we need only recall that $Q_{1,x}(\mathbf{q}) = 1$ for any $x \in W_{\mathrm{aff}}$. This follows from unicity of the inverse Kazhdan-Lusztig polynomials and the identity $\sum_{x \le w} (-1)^{\ell(x)} P_{x,w}(\mathbf{q}) = 0$ for $w \neq 1$ [9, Exercise 5.17]. □

Corollary 3.20. *The map $\tilde{\phi}$ defined in [11] and recalled in diagram (1.2) is injective.*

Proof. Let $q > 1$ and let $j \neq 0$ be an element of J. We must show that $\tilde{\phi}(j) \neq 0$. By injectivity of $\hat{\phi}$, we have $\hat{\phi}(j) \neq 0$. By definition of the map $\hat{\phi}$, this means that there exists $q_0 > 1$, $u, s \in G^\vee$ such that $us = su$ with s compact, and a representation ρ of $\pi_0(Z_{G^\vee}(u,s))$ such that $jK(u,s,\rho,q_0) \neq 0$. But then $jK(u,s,\rho,q) \neq 0$, $K(u,s,\rho,q)$ being a different specialization of the restriction of the same J-module $E(u,s,\rho)$ as for $K(u,s,\rho,q_0)$, and $K(u,s,\rho,q)$ is also tempered. It follows that $\tilde{\phi}(j) \neq 0$. □

3.4. The case of GL_n

Theorem 3.21. *Let W_{aff} be of type $\tilde{A}_n$. Then statements 1 and 2 in the statement of Theorem 1.2 are true, together with a stronger version of statement 3: Let $u = (r_1, \ldots, r_k)$*

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be a unipotent conjugacy class in $\mathrm{GL}_n(\mathbb{C})$*. Let* d *be a distinguished involution in the two-sided cell* $\mathbf{c}(u) \subset W_{\mathrm{aff}}(\mathrm{GL}_n)$ *corresponding to* u*. Let* π *be the unique family of parabolic inductions that* t_d *does not annihilate. Then* $\operatorname{rank}\pi(t_d) = 1$.

Proof. By Corollary 3.7 and Propositions 3.13 and 3.18, $a_{1,x}$ is a rational function of $\mathbf{q}$ for all w. Then equation (3.18) implies that $a_{x,w}$, being a sum of rational functions with Laurent polynomial coefficients, is a rational function of $\mathbf{q}$ for all x. The same equation, together with the fact that $J_{\mathbf{c}}$ is a two-sided ideal for each cell $\mathbf{c}$ shows that the denominator of $a_{x,w}$ depends only on the two-sided cell containing w. This proves the first claim. The second claim now follows from the first claim and the first statement of Corollary 3.7 and the fact that W_{aff} has finitely many two-sided cells.

Finally, let $u = (r_1,\dots,r_k)$. We have $\pi = \mathrm{Ind}_P^G(\mathrm{St}_{M_P} \otimes \nu)$, for the unique Levi M_P such that u is distinguished in $M_P^\vee$, and as $\dim \mathrm{St}_{M_P}^I = 1$, clearly we have

$$\dim \pi^I = \frac{n!}{r_1!\cdots r_k!}.$$

This is also the number of distinguished involutions in $\mathbf{c}(u)$, by [45]. The claim follows from Corollary 3.20. □

3.5. Proof of Theorem 1.2

Proof of Theorem 1.2. Theorem 3.11 together with Propositions 3.13 and 3.18 show that $a_{1,w}$ is a rational function of $\mathbf{q}$ with denominator depending only on the two-sided cell containing w. Equation (3.18) again shows that $a_{x,w}$ is a rational function of $\mathbf{q}$ with denominator depending only on the two-sided cell containing w; up to twists the set $\mathcal{E}_2(M)^I$ is finite for every Levi subgroup M, so we may multiply through to include the denominators of all required formal degrees, which are in fact rational of the correct form by Theorem 2.2. Therefore there is a polynomial $P^1_{\mathbf{G}}(\mathbf{q})$ that clears denominators of all $a_{x,w}$. This proves the first statement of the Theorem.

Now, by Proposition 9 and Remark 2 of [7], ϕ induces a surjection

$$\bar{\phi}\colon \mathbf{H}/[\mathbf{H},\mathbf{H}]\left[\frac{1}{P_W(\mathbf{q})}\right] \twoheadrightarrow J/[J,J]\otimes\mathcal{A}.$$

Therefore for every $j \in J$, there is $N = N(j) \in \mathbb{N}$ and $h \in \mathbf{H}$ such that

$$j \equiv h\frac{1}{P_W(\mathbf{q})^N}$$

in $J/[J,J]\otimes\mathcal{A}$. Considering traces and invoking Proposition 3.18, we see that the denominator of every $a_{x,w}$ divides a power of $P_W(\mathbf{q})$. Therefore there is $N = N_{W_{\mathrm{aff}}}$ depending only W_{aff} such that we can take $P^1_{\mathbf{G}}(\mathbf{q}) = P_W(\mathbf{q})^N$. This proves the second statement of the Theorem. The third statement was proven in Corollary 3.3. □

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4. Representations with fixed vectors under parahoric subgroups

In this section we will give an application of the J-action on the tempered H-modules. The first statement is an immediate corollary of the existence of this action, but the second statement relies on Corollary 3.20.

Theorem 4.1. *Let $\mathcal{P}$ be a parahoric subgroup of G. Let π be a simple tempered representation of G with I-fixed vectors with Kazhdan-Lusztig parameter (u,s,ρ). Let $w_{\mathcal{P}}$ be the longest element in the parabolic subgroup of W_{aff} defined by $\mathcal{P}$ and $\mathcal{B}_u^\vee$ be the Springer fibre for u.*

1. *If*
$$\ell(w_{\mathcal{P}}) > a(u) = \dim_{\mathbb{C}} \mathcal{B}_u^\vee,$$
then
$$\pi^{\mathcal{P}} = \{0\}.$$
2. *Conversely, let $u_{\mathcal{P}}$ be the unipotent conjugacy class corresponding to the two-sided cell containing $w_{\mathcal{P}}$. Then there exists $s \in Z_{G^\vee}(u_{\mathcal{P}})$, a Levi subgroup $M^\vee$ of $G^\vee$ minimal such that $(u_{\mathcal{P}}, s) \in M^\vee$, and a discrete series representation $\omega \in \mathcal{E}_2(M)$ such that*
$$\pi^{\mathcal{P}} = i_{P_M}^G(\omega \otimes \nu)^{\mathcal{P}} \neq \{0\}$$
for all ν non-strictly positive and the parameter of π is $(u_{\mathcal{P}}, s)$.

Proof. Let $\mathcal{P}$ and $w_{\mathcal{P}}$ be as in the statement. Then $C'_{w_{\mathcal{P}}}$ is proportional by a power of q to the indicator function $1_{\mathcal{P}}$ in H. Moreover, $w_{\mathcal{P}}$ is a distinguished involution, with $a(w_{\mathcal{P}}) = \ell(w_{\mathcal{P}})$. By Proposition 3.18 and the fact that ϕ is 'upper-triangular' with respect to the a-function, we have $(\phi \circ^\dagger (-))(C'_{w_{\mathcal{P}}})J_{\mathbf{c}} = 0$ for $\mathbf{c}$ corresponding to u if $a(w_{\mathcal{P}}) > a(u)$.

For the second statement, by Corollary 3.20, there is a tempered representation $\pi = \mathrm{Ind}_P^G(\omega \otimes \nu)$ with $\pi(t_{w_{\mathcal{P}}}) \neq 0$; the unipotent part of its parameter is $u_{\mathcal{P}}$. In particular, there is a vector $v \in \pi$ such that $\pi(t_{w_{\mathcal{P}}})v = v$. We have

$$\begin{aligned}\left(\phi \circ^\dagger (-)\right)(C'_{w_{\mathcal{P}}})t_{w_P} &= (-1)^{\ell(w_{\mathcal{P}})} \sum_{\substack{d \in \mathcal{D} \\ z \sim_L d}} h_{w_{\mathcal{P}},d,z} t_z t_{w_{\mathcal{P}}} \\ &= (-1)^{\ell(w_{\mathcal{P}})} \sum_z h_{w_{\mathcal{P}},w_{\mathcal{P}},z} t_z = \mathrm{vol}(\mathcal{P}) t_{w_P},\end{aligned}$$

as $t_z t_{w_{\mathcal{P}}} \neq 0$ only if $z \sim_L w_{\mathcal{P}}$, and $C_{w_{\mathcal{P}}} C_{w_{\mathcal{P}}} = (-1)^{\ell(w_{\mathcal{P}})} \mathrm{vol}(\mathcal{P}) C_{w_{\mathcal{P}}}$. Thus $v \in \pi^{\mathcal{P}}$. As $\mathrm{trace}(\pi, t_d)$ is constant with respect to ν, the last part of the claim follows. □

A version of this statement for enhanced parameters in the case $\mathcal{P} = G(\mathcal{O})$ appears in [2, Prop. 10.1].

Example 4.2. The principal series representations have $u = \{1\}$, maximal a-value, and fixed vectors under every maximal compact subgroup of G. On the other extreme, the Steinberg representation has u regular, and does not have fixed vectors under any proper parahoric.

TABLE 1. Tempered Iwahori-spherical representation of $G = \mathrm{SO}_5(F) = \mathrm{PGSp}_4(F)$.

Cell\Levi	$\mathrm{GL}_1 \times \mathrm{GL}_1$	$\mathrm{GL}_1 \times \mathrm{SO}_3$	GL_2	SO_5
$(1,\ldots,1)$	ν			
$(2,1,1)$		$\nu \otimes \pm\mathrm{St}_{\mathrm{SO}_3}$		
$(2,2)$			$\xi\mathrm{St}_{\mathrm{GL}_2}$	$\pm\tau_2$
(4)				$\pm\mathrm{St}$

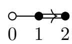
TABLE 2. Parahoric fixed vectors for $G = \mathrm{SO}_5(F) = \mathrm{PGSp}_4(F)$.

$t_{w_{\mathcal{P}}} \backslash\ \pi = \mathrm{Ind}_P^G(-)$	a	ν	$\nu \otimes \pm\mathrm{St}_{\mathrm{SO}_3}$	τ_{triv}	τ_{sgn}	$\pm\tau_2$	$\pm\mathrm{St}$	$\mathcal{P}$
t_{w_0}	4	**1**						$\mathrm{PGSp}_4(\mathcal{O})$
$t_{s_0 s_2}$	2	2	**1**					K
t_{s_0}	1	4	2	**1**		**1**		Kl
t_{s_1}	1	4	1	**1**	**1**			Si
t_1	0	8	4	3	1	4	**1**	I

In addition to the interpretation given in Section 1.1.3, in the examples below, the J-action also detects which direct summands of a reducible $\mathcal{P}$-spherical tempered representation are themselves $\mathcal{P}$-spherical.

Example 4.3. Let $G = \mathrm{SL}_2(F)$. As remarked in [14], the distinguished involutions in the lowest two-sided cell are the simple reflections s_0 and s_1, and each t_{s_i} is invariant under one of the two conjugacy classes of maximal parahoric subgroup of G. For unitary principal series representations π, one has $\mathrm{trace}(\pi, t_{s_0}) = \mathrm{trace}(\pi, t_{s_1}) = 1$. At the quadratic character, the corresponding principal series representation is reducible, and each summand contains fixed vectors under precisely one of the maximal parahorics. Indeed, in [14] this computation is carried out at the level of the Schwartz space of the basic affine space.

Example 4.4. Let $G = \mathrm{SO}_5(F) = \mathrm{PGSp}_4(F)$, with affine Dynkin diagram labelled as



.

There are five conjugacy classes of parahoric subgroups, each obtained by projection from $\mathrm{GSp}_4(F)$: the maximal parahoric $\mathrm{PGSp}_4(\mathcal{O})$, the image of the paramodular group K corresponding to $\{0,2\}$, the image of the Siegel parahoric subgroup Si corresponding to $\{1\}$, the image of the Klingen parahoric Kl corresponding to $\{0\}$, and the Iwahori subgroup.

The columns of table 1 give all the I-spherical tempered representations of G by denoting the representation $\mathrm{Ind}_P^G(\omega)$ by ω. We recall the few cases of reducibility of these inductions immediately below. The rows list unipotent conjugacy classes in $G^\vee = \mathrm{Sp}_4(\mathbb{C})$

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such that all tempered standard modules $K(u,s,\rho)$ are in row u. That is, the rows record which summand of J acts on each representation.

The discrete series are as in [41]. The only reducibility, by [34, Prop. 3.3], is the reducibility

$$\mathrm{Ind}_P^G(\xi\mathrm{St}_{\mathrm{GL}_2}) = \tau_{\mathrm{triv}} \oplus \tau_{\mathrm{sgn}}$$

for $\xi^2 = 1$.

We can now compute the traces of some elements t_d using the description of the simple J-modules given in [50]. For the cells $(1,\ldots,1)$, $(2,1,1)$, and (4), we have $\mathrm{trace}(\pi,t_d) = 1$ for all π. We have

$$\mathrm{trace}(\tau_{\mathrm{triv}},t_{s_0}) = \mathrm{trace}(\tau_2,t_{s_0}) = 1 = \mathrm{trace}(\tau_{\mathrm{sgn}},t_{s_1}) = \mathrm{trace}(\tau_{\mathrm{triv}},t_{s_1}).$$

In table 2, $\mathrm{trace}(\pi,t_d)$ is recorded in bold face, whereas the dimension of $\pi^{\mathcal{P}}$, taken from [42, Table A.15], is recorded in normal face. Representations attached to the same cell but belonging to different packets are separated with a dotted line.

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