CHARACTERIZATION OF FINITE AMENABLE TRANSFORMATION SEMIGROUPS

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Abstract. In this paper we develop necessary and sufficient conditions for a finite transformation semigroup to have a mean value which is invariant under the induced shift operators. The structure of such transformation semigroups is described and an explicit description of all possible invariant means given.

1. Introduction

A *transformation semigroup* (briefly, a τ-semigroup) is a pair (X, S), where X is a set and S a semigroup of transformations on X, i.e. functions s on X into itself. Each s in S induces a shift operator, T_s, on the Banach space $B(X)$ of all bounded real functions on X under the supremum norm, defined by

$$T_s f(x) = f(sx) \quad (x \in X, \ f \in B(X)).$$

A mean on $B(X)$ (i.e. an element μ of $B(X)^*$ such that $\|\mu\| = 1$, $\mu(1) = 1$, where $1(x) = 1$ for all x, and $\mu(f) \geqq 0$ whenever $f(x) \geqq 0$ for all x) is called *S-invariant* if

$$\mu(T_s f) = \mu(f) \text{ for all } s \in S, \ f \in B(X).$$

The notion of an S-invariant mean was introduced and a number of basic properties developed in [3].

Important special cases of τ-semigroups occur when X is in an abstract semigroup and S either the semigroup of transformations of X induced by left multiplication or that induced by right multiplication; these will be called l-semigroup and r-semigroups, respectively. In [2], Rosen obtained a complete characterization of the finite l- and r-semigroups which have invariant means. He proved that a finite semigroup has a left invariant mean if and only if each pair of right ideals has a nonvoid intersection. In this case the intersection of all right ideals, called the kernel, is a union of groups, is the smallest right ideal and also the smallest two-sided ideal, and is the union of the minimal left ideals of the original semigroup. The dual results hold for the r-semigroups.

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In this paper we determine necessary and sufficient conditions in order that X have an S-invariant mean in general. In obtaining the characterization, we establish for amenable τ-semigroups the existence of a "kernel" in X which has many of the properties of the kernel already known for l- and r-semigroups; in particular, the semigroup acts as a permutation group on the kernel. In the process of obtaining the characterization, we also obtain a complete description of all the S-invariant means on X.

The notation presented above will be used freely throughout the sequel.

2. Some preliminaries

We take as our point of departure the work of Rosen [2]. First we observe that the necessary and sufficient condition that Rosen finds for a finite semigroup to have a left invariant mean is equivalent to the condition that for any s, t from S we have $sS \cap tS \neq \varnothing$. Since it is easy to show in general that if X has an S-invariant mean, then

$$\cap \{R_s: s \in S\} \neq \varnothing,$$

where R_s denotes the range of s, (this will become clear later on), it is natural to conjecture that this condition is also sufficient. The following simple example shows this conjecture to be false.

2.1 Example. Let $X = \{a, b, c\}$, and let

$$S = \{(a,b,b), (a,c,c), (b,a,a), (c,a,a)\},$$

where the notation is the natural extension of the ordinary notation of permutations. Clearly $\cap \{R_s: s \in S\} = \{a\}$, however X has no S-invariant mean, as will be established below.

Although the natural extension of Rosen's theorem conjectured above fails, there is a beautiful extension available; our goal is to obtain this extension. Our development rests in part on a representation of means on $B(X)$. Let n denote the number of elements in X. Then $B(X)$ can be represented by E^n, ordinary (real) n-dimensional space, equipped with supremum norm. The representation can be accomplished as follows: let

$$X = \{x_1, \cdots, x_n\};$$

then the correspondence $B(X) \leftrightarrow E^n$ is given by

$$f \longleftrightarrow (a_1, \cdots, a_n),$$

where

$$f(x_i) = a_i \text{ for } i = 1, \cdots, n.$$

It is well known, then, that $B(X)^*$ also corresponds to E^n, only this time equipped with the l_1-norm. The correspondence is established via the evaluation mapping $q: X \to B(X)^*$ defined by

$$qx_i(f) = f(x_i),$$

so that

$$qx_i \leftrightarrow (0, \cdots, 0, 1, 0, \cdots, 0),$$

where the 1 occurs as the ith component. The correspondence $B(X)^* \leftrightarrow E^n$ is then given by:

$$\mu \leftrightarrow (\alpha_1, \cdots, \alpha_n),$$

where

$$\mu(f) = \sum_{i=1}^{n} \alpha_i q x_i(f) = \sum_{i=1}^{n} \alpha_i : f(x_i).$$

Then μ is a mean if and only if $\alpha_i \geqq 0$, $i = 1, \cdots, n$, and $\sum \alpha_i = 1$. For ease of notation we drop the q and write μ as a formal sum,

$$\mu = \sum_{i=1}^{n} \alpha_i x_i,$$

where the notation means that for all f in $B(x)$

$$\mu(f) = \sum_{i=1}^{n} \alpha_i f(x_i);$$

that is, a mean μ simply denotes integration with respect to a convex combination of unit masses concentrated at the points of X.

We can now prove the assertion made in example 2.1. Assume that μ is an S-invariant mean on $B(X)$. Let $\mu = \alpha_1 a + \alpha_2 b + \alpha_3 c$. The condition

$$\mu(T_s f) = \mu(f) \text{ for all } s \text{ in } S \text{ and all } f \text{ in } B(X)$$

is then equivalent to the system of equations

$$\alpha_1 f(a) + \alpha_2 f(b) + \alpha_3 f(c) = \alpha_1 f(sa) + \alpha_2 f(sb) + \alpha_3 f(sc), \text{ all } s \text{ in } S,$$

for every f in $B(X)$. Thus for $s = (a, b, b)$, $t = (a, c, c)$, we must have

$$\alpha_1 f(a) + \alpha_2 f(b) + \alpha_3 f(b) = \alpha_1 f(a) + \alpha_2 f(c) + \alpha_3 f(c)$$

for all f, which implies $\alpha_2 = \alpha_3 = 0$. Then using the remaining two elements in S, we obtain

$$\alpha_1 f(b) = \alpha_1 f(c)$$

for all f, which implies $\alpha_1 = 0$, a contradiction since $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

3. The main results

We begin with a characterization of finite amenable τ-semigroups which will lead to further characterizations and structure theorems.

3.1 THEOREM. *Let (X,S) be a finite τ-semigroup. Then X has an S-invariant mean if and only if there exists a nonempty subset M of S such that $s[M] = M$ for all s in S.*

PROOF. $\Leftarrow$ Let $M = \{x_1, \cdots, x_m\}$, with the x_i's distinct, and put

$$\mu = \frac{1}{m} \Sigma x_i .$$

Then μ is a mean on $B(X)$ clearly, and if $s \in S$, $f \in B(X)$, then

$$\mu(T_s f) = \frac{1}{m} \Sigma T_s f(x_i) = \frac{1}{m} \Sigma f(sx_i) = \frac{1}{m} \Sigma f(x_i) = \mu(f).$$

so that μ is S-invariant.

$\Rightarrow$ Let μ be an S-invariant mean, and M the carrier of μ. Then $M \subseteq X$, $M \neq \varnothing$, and

$$\mu = \sum_{i=1}^{m} \alpha_i x_i ,$$

where

$M = \{x_1, \cdots, x_m\}$, $x_i \neq x_j$ if $i \neq j$, $\alpha_i > 0$ for $i = 1, \cdots, m$, and $\Sigma \alpha_i = 1$.

Now define a relation on M by:

$$x_i \sim x_j \text{ if } \alpha_i = \alpha_j .$$

Then $\sim$ is clearly an equivalence relation which partitions M into subsets $M_1, \cdots, M_k$, $k \leqq m$. We show that $s[M_i] = M_i$ for $i = 1, \cdots, k$. So fix i, and assume that there exists an s in S and an x_0 in X such that $x_0 \in s[M_i] - M_i$. Define f by:

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 . \end{cases}$$

Let $\beta_1, \cdots, \beta_k$ denote the distinct values of the α's for $M_1, \cdots, M_k$, respectively. Then

$$\mu(f) = \sum_{j=1}^{m} \alpha_j f(x_j) = \begin{cases} 0 & \text{if } x_0 \notin M \\ \beta_l \text{ for some } l \neq i, \text{ if } x_0 \in M , \end{cases}$$

while

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$$\mu(T_s f) = \sum_{j=1}^{m} \alpha_j f(sx_j) = \beta_i .$$

Since $\beta_1 > 0$ and $\beta_i \neq \beta_j$ when $i \neq j$, the assumption that $s[M_i] - M_i \neq \emptyset$ leads to a contradiction. Similarly, the assumption that there exists $x_0 \in M_i - s[M_i]$ leads to the contradiction that $\mu(f) = \beta_i \neq 0$ while $\mu(T_s f) = 0$.

Thus $s[M_i] = M_i$ for $i = 1, \cdots, k$, and it follows immediately that $s[M] = M$.

The proof of theorem 3.1 provides the motivation for the following definition.

3.2 DEFINITION. Let (X, S) be a τ-semigroup. A subset F of X is called a *fixed set for* S if $s[F] = F$ for all s in S; F is called a *minimal fixed set for* S if F is a fixed set for S and no proper subset of F is. The union of the minimal fixed sets for S (they are pairwise disjoint) will be called the *kernel of* X *relative to* S and denoted by K_S.

3.3 THEOREM. *Let* (X, S) *be a* τ-*semigroup.*

(i) *Then* X *has an* S-*invariant mean* μ if and only if $K_S \neq \emptyset$.

(ii) *When case* (i) *attains,* μ *is concentrated on* K_S; *moreover* μ *can be represented in the form*

$$\mu = \sum_{i=1}^{K} \alpha_i \mu_i , \ \alpha_i \geqq 0, \quad i = 1, \cdots, K, \ \sum_{i=1}^{K} \alpha_i = 1,$$

where μ_i *is obtained as follows:* $M_1, \cdots, M_K$ *are the distinct nonempty minimal fixed sets for* S, *and* μ_i *is the unweighted average over* M_i,

$$\mu_i = \frac{1}{|M_i|} \sum_{x \in M_i} x .$$

PROOF. (i) Both implications follow immediately from theorem 3.1 in view of definition 3.2.

(ii) We first obtain the asserted representation under the assumption that $K_S = X$. In this case μ is trivially concentrated on K_S, hence μ can be represented in the form

$$\mu = \sum \beta_i x_i , \ \beta_i \geqq 0 \ , \ \sum \beta_i = 1,$$

where the sum extends over all of X. Let $\gamma_1, \cdots, \gamma_k$ denote the distinct values of the β's, and define L_i, $i = 1, \cdots, k$, by

$$L_i = \{x_j : \beta_j = \gamma_i\} .$$

Then $L_i \neq \emptyset$ for each i, and the collection $\{L_1, \cdots, L_k\}$ forms a partition of X. Now put

$$\delta_i = |L_i| \gamma_i \text{ and } \nu_i = \frac{1}{|L_i|} \sum_{x_j \in L_i} x_j , \qquad i = 1, \cdots, k .$$

Then each v_j is an unweighted mean over L_i, and

$$\mu = \sum_{i=1}^{k} \delta_i v_i .$$

We next show that each L_i can be expressed in the form

$$L_i = \bigcup_{j=1}^{k_i} M_{ij},$$

where the M_{ij}'s are the distinct nonempty minimal fixed sets for S. Fix i, and for each s in S, denote by p_s the restriction of s to L_i. It follows from the proof of theorem 3.1 that each p_s is a permutation of L_i. Let $G = \{p_s: s \in S\}$; then G is a semigroup which is contained in the symmetric group on $|L_i|$ letters, hence G is a group. It is well-known (and in any case easy to prove by introducing on L_i the equivalence relation $x \sim y$ if there exists $p_s \in G$ such that $p_s x = y$) that L_i is partitioned by a collection of sets $\{M_{ij}\}$ as required above. Now

$$v_i = \frac{1}{|L_i|} \sum_{x_l \in L_i} x_l = \sum_{j=1}^{k_i} \frac{1}{|L_i|} \sum_{x_l \in M_{ij}} x_l,$$

hence

$$\delta_i v_i = \sum_{j=1}^{k} \gamma_i \sum_{x_l \in M_{ij}} x_l = \sum_{j=1}^{k_i} \alpha_{ij}\mu_{ij},$$

where

$$\mu_{ij} = \frac{1}{|M_{ij}|} \sum_{x_l \in M_{ij}} x_l \quad \text{and} \quad \alpha_{ij} = \gamma_i |M_{ij}|.$$

Thus

$$\mu = \sum_{i=1}^{k} \delta_i v_i = \sum_{i=1}^{k} \sum_{j=1}^{k_i} \alpha_{ij}\mu_{ij} = \sum_{i=1}^{K} \alpha_i \mu_i,$$

as desired.

Now suppose that $K_S \neq X$, and let $x_j \in X - K_S$. Again let

$$\mu = \sum \beta_i x_i ,$$

where the sum extends over all of X, and assume that $\beta_j > 0$. It follows from the proof of theorem 3.1 that $\{x_k: \beta_k = \beta_j\}$ is a fixed set for S. It now follows from the proof just completed in this theorem that x_j belongs to a minimal fixed set for S, i.e. $x_j \in K_S$, a contradiction. Thus μ is concentrated on K_S. The argument completed above for the case $K_S = X$ now applies here to yield the desired result.

Thus the structure of any finite amenable τ-semigroup (X, S) is clearly revealed; namely, there is in X a largest fixed set for S, which we called the kernel and denoted by K_S, which is the union of the minimal fixed sets for S, and on which S acts as a group of permutations. Moreover the kernel completely deter-

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mines the invariant mean structure. The extension from the known results for l-semigroups is now strongly suggested, and in the next theorem we reconcile our results with this special case, thereby justifying our use of the term "kernel".

3.4 THEOREM. *Let (X,S) be a finite l-semigroup, i.e. $X = S$, and the action of s on X is defined by: $s(t) = st$. Suppose that S has a left invariant mean. If K' denotes the intersection of all right ideals of S, then $K_S = K'$.*

PROOF. Since K' is a left ideal, $sK' \subseteq K'$ for all $s \in S$. But K' is a right ideal for any $s \in S$, hence $sK' \supseteq K'$. It follows that K' is a fixed set for S, hence $K' \subseteq K_S$.

To obtain the reverse inclusion, let μ denote the unweighted average over K_S. It follows from the proof of theorem 3.1 that μ is left invariant. Now in an l-semigroup the shift operator T_s coincides with the left translation l_s, defined by $l_s f(t) = f(st)$, and if R is any right ideal in S, then we have that for $s \in R$, $t \in S$

$$l_s \chi_R(t) = \chi_R(st) = 1,$$

so that

$$l_s \chi_R = 1,$$

hence

$$\mu(\chi_R) = \mu(l_s \chi_R) = \mu(1) = 1.$$

Hence $R \supseteq K_S$. Since R was an arbitrary right ideal of S, $K_S \subseteq K'$.

The way in which our results extend those of Rosen should now be clear— in an l-semigroup, the minimal left ideals correspond to the minimal fixed sets. There are important contrasts, however, in the general case. It is known [1] that in an amenable l-semigroup the minimal left ideals are all mutually isomorphic, while in the general case the minimal fixed sets may be in a variety of sizes.

We conclude our study with another interesting contrast between the l-semigroup and the general τ-semigroup. In the special case considered in theorem 3.4, it is easy to see that K' coincides with the intersection of all the principal right ideals of S. That is, $K' = \cap \{s[X] : s \in S\} = \cap \{R_s : s \in S\}$. In the light of theorem 3.4, it is tempting to conjecture that $K_S = \cap \{R_s : s \in S\}$ in general. Our final theorem provides an answer to this question.

3.5 THEOREM. *Let (X,S) be a finite τ-semigroup with $|X| = n$, and let $R = \cap \{R_s : s \in S\}$.*

(i) *Then $K_S \subseteq R$.*

(ii) *If $|R| \leqq n-2$, then R may or may not coincide with K_S.*

(iii) *If $|R| = n-1$ or if $|R| = n$, then $K_S = R$.*

PROOF. (i) For each $s \in S$ we have

$$T_s \chi_{s[x]} = 1.$$

Hence if $K_S \neq \varnothing$, let μ be an S-invariant mean; then

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$$\mu(\chi_{s[X]}) = \mu(T_s\chi_{s[X]}) = \mu(1) = 1,$$

hence $K_S \subseteq R_s$ for all $s \in S$, i.e. $K_S \subseteq R$.

(ii) The interesting part of this problem is to find an example where $K_S \neq R$ under the hypothesis. Let $X = \{x_1, \cdots, x_n\}$, and define maps s, t, u, v on X by:

$$s(x_i) = \begin{cases} x_i, & i = 1, \cdots, n-2 \\ x_{n-1}, & i = n-1, n \end{cases} \qquad t(x_i) = \begin{cases} x_i, & i = 1, \cdots, n-2 \\ x_n, & i = n-1, n, \end{cases}$$

$$u(x_i) = \begin{cases} x_i, & i = 1, \cdots, n-3 \\ x_{n-1}, & i = n-2 \\ x_{n-2}, & i = n-1, n \end{cases} \qquad v(x_i) = \begin{cases} x_i, & i = 1, \cdots, n-3 \\ x_n, & i = n-2 \\ x_{n-2}, & i = n-1, n \end{cases}$$

It is easy to check that (X, S) forms a τ-semigroup, in fact the multiplication table is given by:

	s	t	u	v
s	s	s	u	u
t	t	t	v	v
u	u	u	s	s
v	v	v	t	t

It is easy to check also that $K_S = \{x_1, \cdots, x_{n-3}\}$ and $\cap \{R_w \colon w \in S\} = \{x_1, \cdots, x_{n-2}\}$.

(iii) Let $\mathfrak{S}_n$ denote the symmetric group on n letters, and for notation let $\cap R_s \equiv \{x_1, \cdots, x_{n-1}\} \equiv R$. If $S \cap \mathfrak{S}_n = \emptyset$, then $s \colon R \to R$ for each $s \in S$, and if $s[R] \neq R$, we would have $R_{s^2} \subseteq R$, a contradiction. Hence in this case $R = K_S$. It remains only to show that the case $S \cap \mathfrak{S}_n \neq \emptyset$ does not attain. Assume there exists a permutation p of X in S. Then $S - \mathfrak{S}_n \neq \emptyset$, so let $s \in S - \mathfrak{S}_n$. It is easy to see then that $R_{ps} \cap R| = n - 2$, a contradiction.

References

[1] E. E. Granirer. 'On amenable groups with a finite-dimensional set of invariant means I', Ill. *J. Math.* 7 (1963), 32–48.

[2] W. G. Rosen, 'On invariant means over compact semigroups', *Proc. Amer. Math. Soc.* 7 (1956), 1076–1082.

[3] C. Wilde, T. Jayachandran, 'Amenable transformation Semigroups', *J. Australian Math. Soc.*, v. XII (1971), 502–510.

[4] C. Wilde, K. Witz, 'Invariant means and the Stone-Cech compactification,' *Pacific J. Math.* 21 (1967), 577–586.

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