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QUOTIENTS AND INVERSE LIMITS OF SPACES OF ORDERINGS

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0. Introduction. A connection between the theory of quadratic forms defined over a given field F, and the space X_F of all orderings of F is developed by A. Pfister in [**12**]. X_F can be viewed as a set of characters acting on the group $F^\times/\Sigma F^{\times 2}$, where $\Sigma F^{\times 2}$ denotes the subgroup of $F^\times$ consisting of sums of squares. Namely, each ordering $P \in X_F$ can be identified with the character

$$\sigma_P : F^\times/\Sigma F^{\times 2} \rightarrow \{1, -1\}$$

defined by

$$\sigma_P(a) = \begin{cases} 1 & \text{if } a \in P \\ -1 & \text{if } a \notin P \end{cases}.$$

It follows from Pfister's result that the Witt ring of F modulo its radical is completely determined by the pair $(X_F, F^\times/\Sigma F^{\times 2})$.

This result of Pfister's led the author to consider 'abstract' spaces of orderings. These are pairs (X, G) where G is an Abelian group satisfying $x^2 = 1$ for all $x \in G$, and X is a subset of the character group $\chi(G)$ satisfying some special properties. These properties are stated in detail in § 1. The idea at the time was that by removing the 'non-essentials' one might more easily determine the structure of such spaces, and hence of their corresponding Witt Rings. At the same time, it was hoped that the axioms defining a space of orderings would be rigid enough to eliminate all 'uninteresting' examples.

For finite spaces of orderings, this proved to be the case. In [**11**], finite spaces of orderings are classified and it is proved, using results from [**5**] or [**7**] that every finite space of orderings is equivalent to the space of orderings of a Pythagorian field.

Before these results were obtained, two papers [**1, 8**] appeared in which a theory parallel to Pfister's was developed, but over semi-local rings. It follows from results in these papers that the spaces of signatures of semi-local rings provide additional examples of spaces of orderings in the sense considered here.

In studying spaces of orderings, the concept of *subspace* has proved essential. In the case of the space of orderings of a field F, subspaces correspond bijectively to the *preorders* of F. A preorder of F is just a subset T of $F^\times$ satisfying $T + T \subseteq T$, $TT \subseteq T$, $F^{\times 2} \subseteq T$. The subspace of $(X_F, F^\times/\Sigma F^{\times 2})$ corresponding to the preorder T is the pair $(X_F(T), F^\times/T)$, where $X_F(T)$ denotes the set of all orderings of F which are positive on T. Especially important sub-

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spaces are obtained by considering valuations. To a real valuation ν of F one can associate the subspace $(X_\nu, F^\times/T_\nu)$. Here X_ν denotes the set of all orderings of F compatible with the valuation ν as discussed in [**3**] or [**13**].

This paper represents a departure from previous papers on the subject in that an attempt is made to study a space of orderings in terms of its quotient spaces. A space of orderings (X', G') is said to be a *quotient space* of the space of orderings (X, G) if G' is a subgroup of G, and X' is the set of restrictions of elements of X to G'. As an example, if ν is a real valuation of F, then the space of orderings of the corresponding residue field is a quotient of the space $(X_\nu, F^\times/T_\nu)$ mentioned above. A second example is obtained by taking a subfield K of F such that the degree $[F : K]$ is finite and odd. Then X_K is a quotient space of X_F. Generally speaking, quotients seem to be most easily and naturally discussed in the abstract setting.

Here is an outline of the content of this paper.

In sections 1, 2, and 3 the category of spaces of orderings is introduced and some basic notions (subspaces, direct sums, quotients, group extensions) are developed. Most of this material is implicit in [**11**]. In § 4 the *inverse limit* of an inverse system of spaces of orderings is defined, and it is proved in Theorem 4.7 that every space of orderings is the inverse limit of countable spaces. A space of orderings (X, G) is said to be *countable* if G is countable. In § 5 the following question is considered.

Question 1. Which spaces of orderings are inverse limits of finite spaces?

Although this question is not answered, the class of such spaces is shown to be quite large. The major results in this connection are Theorems 5.7, 5.8, and 5.11.

Another question considered in § 5 is:

Question 2. Suppose X is a space of orderings with Witt Ring W. Suppose $k \geqq 1$, and that $f \in W$ satisfies $\sigma f \equiv 0 \pmod{2^k}$ for all $\sigma \in X$. Is it true that $f \in M^k$? (Here M denotes the ideal of W consisting of even dimensional forms.)

Question 2 has been considered previously in [**7, 9, 10**]. Theorem 5.2 provides a connection between open questions 1 and 2.

Many results peripheral to the main theme of the paper are given as *Remarks* without proof.

1. Spaces of orderings. The concept of a space of orderings is defined in [**9, 11**]. For completeness the definition is given below.

Definition 1.1. A *space of orderings* is a pair (X, G) where G is an Abelian group such that $x^2 = 1 \quad \forall x \in G$, and where X is a subset of the character group $\chi(G) = \text{Hom}(G, \{1, -1\})$ satisfying:

0_1 : X is closed in $\chi(G)$.

0_2 : If $x \in G$ satisfies $\sigma(x) = 1 \quad \forall \sigma \in X$, then $x = 1$.

0_3 : There exists a (necessarily unique) element $-1 \in G$ satisfying $\sigma(-1) = -1 \quad \forall \sigma \in X$.

0_4 : If f and g are forms over G, and if $f \oplus g$ is isotropic, then there exists $x \in G$ such that $x \in D_f$, $-x \in D_g$.

Terminology 1.2. A form over G is an n-tuple $f = \langle a_1, \ldots, a_n \rangle$, $n \geqq 1$ with $a_1, \ldots, a_n \in G$. n is referred to as the *dimension* of f. The *signature of f* at $\sigma \in X$ is $\sigma f = \sum_1^n \sigma(a_i) \in \mathbf{Z}$. Two forms of f, g are said to be *congruent* (modulo X) denoted $f \equiv g$ or $f \equiv g \pmod X$ if and only if $\dim f = \dim g$ and $\sigma f = \sigma g \quad \forall \sigma \in X$. A form f is said to *represent* $x \in G$ if there exist $x_2, \ldots, x_n \in G$ such that $f \equiv \langle x, x_2, \ldots, x_n \rangle$. We use D_f to denote the set of all elements of G represented by f. A form f is said to be *isotropic* if $f \equiv \langle 1, -1, x_3, \ldots, x_n \rangle$ for some $x_3, \ldots, x_n \in G$. Certain *operations* are defined in forms; namely if $f = \langle a_1, \ldots, a_n \rangle$, $g = \langle b_1, \ldots, b_m \rangle$, then

$$f \oplus g = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle \quad \text{and}$$
$$f \oplus g = \langle a_1 b_1, \ldots, a_1 b_m, \ldots, a_n b_1, \ldots, a_n b_m \rangle.$$

For $a \in G$ we define $af = \langle a \rangle \otimes f = \langle aa_1, \ldots, aa_n \rangle$. In particular

$$-f = (-1)f = \langle -a_1, \ldots, -a_n \rangle$$

(for $x \in G$, $-x$ is by definition equal to $(-1)(x)$.)

Definition 1.1 does not read quite the same as the definition of a space of orderings given in [**9**, **11**]. To prove both definitions are equivalent, we need the following lemma.

LEMMA 1.3. *Let G be an Abelian group satisfying $x^2 = 1$ for all $x \in G$. Let X be a subset of $\chi(G)$ satisfying 0_1, 0_2, and 0_3. Then (X, G) is a space of orderings if and only if the following property holds:*

$0_4'$: *For all forms f, g defined over G and for all $x \in G$, $x \in D_{f \oplus g}$ implies there exists $y \in D_f$, $z \in D_g$ such that $x \in D_{\langle y, z \rangle}$.*

Proof. Suppose first that (X, G) is a space of orderings and let $x \in D_{f \oplus g}$. Thus $f \oplus g \oplus \langle -x \rangle$ is isotropic, so by 0_4 there exists $y \in D_f$, $-y \in D_{g \oplus \langle -x \rangle}$. Thus $g \oplus \langle -x, y \rangle$ is isotropic, so by 0_4 there exists $z \in D_g$, $-z \in D_{\langle -x, y \rangle}$. Thus $\langle -x, y \rangle \equiv \langle -z, xyz \rangle$, so $\langle y, z \rangle \equiv \langle x, xyz \rangle$. Thus $x \in D_{\langle y, z \rangle}$.

Suppose now that $0_4'$ holds and let $f \oplus g \equiv \langle 1, -1 \rangle \oplus h$. Let $x \in D_f$, and write $f \equiv \langle x \rangle \oplus f'$. Since $\langle 1, -1 \rangle \equiv \langle x, -x \rangle$ we have $f \oplus g \equiv \langle x, -x \rangle \oplus h$ so $f' \oplus g \equiv \langle -x \rangle \oplus h$. Thus $-x \in D_{f' \oplus g}$ so there exists $y \in D_{f'}$, $z \in D_g$ such that $-x \in D_{\langle y, z \rangle}$. Thus $-z \in D_{\langle x, y \rangle} \subseteq D_f$.

Definition 1.4. Let (X_i, G_i), $\quad i = 1,2$ be spaces of orderings. A *morphism* ϕ from (X_1, G_1) to (X_2, G_2) is a continuous group homomorphism

$$\phi : \chi(G_1) \rightarrow \chi(G_2)$$

which carries X_1 into X_2.

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Note 1.5. By duality, specifying a continuous group homomorphism $\phi : \chi(G_1) \rightarrow \chi(G_2)$ is equivalent to specifying a group homomorphism $\phi^* : G_2 \rightarrow G_1$.

Note 1.6. If $\phi : (X_1, G_1) \rightarrow (X_2, G_2)$ is a morphism of spaces of orderings, then the dual $\phi^* : G_2 \rightarrow G_1$ carries -1 into -1.

Definition 1.7. A morphism $\phi : (X_1, G_1) \rightarrow (X_2, G_2)$ is called an *equivalence* (or *isomorphism*) if $\phi : \chi(G_1) \cong \chi(G_2)$ and $\phi(X_1) = X_2$. As in [**11**], we say two spaces (X_1, G_1) and (X_2, G_2) are *equivalent* (denoted $(X_1, G_1) \backsim (X_2, G_2)$) if there exists such an equivalence.

Remark 1.8. If (X, G) is a given space of orderings, we can associate to (X, G) a ring $W(X, G)$ called the *Witt Ring* of the space (X, G). This ring can be described as the set of equivalence classes of forms under the equivalence relation $\approx$ defined by: $f \approx g$ if and only if $\dim f \equiv \dim g \pmod 2$ and

$$\sigma f = \sigma g \quad \forall \sigma \in X.$$

Addition and multiplication in $W(X, G)$ are understood to be those induced by $\oplus$ and $\otimes$. Further, a morphism $\phi : (X_1, G_1) \rightarrow (X_2, G_2)$ of spaces of orderings gives rise to ring homomorphism $\tilde{\phi} : W(X_2, G_2) \rightarrow W(X_1, G_1)$ via

$$\tilde{\phi}\langle a_1, \ldots, a_n\rangle = \langle \phi^*(a_1), \ldots, \phi^*(a_n)\rangle.$$

Thus there is a contravariant functor from the category $\mathscr{O}$ of spaces of orderings and morphisms into the category $\mathscr{W}$ of Witt rings of such spaces, and (unitary) ring homomorphisms. This functor is even an equivalence of categories, since a space is completely determined by its Witt Ring. (G is the unit group of $W(X, G)$, and X corresponds bijectively to the ring homomorphisms $\sigma : W(X, G) \rightarrow \mathbf{Z}$.) In what follows we state results only in the category $\mathscr{O}$. The analogous properties of $\mathscr{W}$ are only mentioned in passing in the remarks.

2. Subspaces and direct sums. Recall the definition of a subspace given in [**11**]:

Definition 2.1. Let (X, G) be a space of orderings. A *subspace* of (X, G) is a pair $(Y, G/\Delta)$ where $Y \subseteq X$, and $\Delta \subseteq G$ satisfy $Y^\perp = \Delta$, $\Delta^\perp \cap X = Y$.

Theorem 2.2 in [**11**] is valid for infinite spaces of orderings. This is the content of the following theorem.

THEOREM 2.2. *Every subspace of a space of orderings is a space of orderings.*

Proof. Let (X, G) be a space of orderings, and let $(Y, G/\Delta)$ be a subspace. Then 0_1, 0_2, and 0_3 are clearly satisfied by the pair $(Y, G/\Delta)$. Moreover, the proof in [**11**] shows the result is true if Y is a Harrison Basic clopen set, i.e. has the form $Y = X(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in G$. (In this case $\Delta = D_p$, where $p = \langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$.) Thus to show 0_1 in the general case it is enough to prove the following:

Claim. Let f and g be forms over G. Then $f \equiv g (\text{mod } Y)$ holds if and only if there exists $a_1, \ldots, a_n \in \Delta$ such that $f \equiv g \text{ mod } X(a_1, \ldots, a_n)$.

The proof of the non-trivial part of this claim is as follows: Let

$$U = \{\sigma \in X | \sigma f = \sigma g\}.$$

Thus U is open in X, and $Y \subseteq U$. Note that Y is the intersection of the sets $X(a_1, \ldots, a_n)$ where $\{a_1, \ldots, a_n\}$ runs through all finite subsets of Δ. It follows by compactness that there exists $a_1, \ldots, a_n \in \Delta$ such that $X(a_1, \ldots, a_n) \subseteq U$. Thus $f \equiv g \text{ mod } X(a_1, \ldots, a_n)$.

Example 2.3. Let Z be any subset of X. Let $\Delta = Z^\perp$ and let $Y = \Delta^\perp \cap X$. Then $(Y, G/\Delta)$ is a subspace of X. We will refer to this subspace of (X, G) as *the subspace generated by Z*.

Remark 2.4. Let $(Y, G/\Delta)$ be a subspace of (X, G). Then there is an obvious morphism $\phi : (Y, G/\Delta) \rightarrow (X, G)$. The corresponding ring homomorphism $\tilde{\phi} : W(X, G) \rightarrow W(Y, G/\Delta)$ is clearly surjective.

Notation 2.5. For $(Y, G/\Delta)$ a subspace of (X, G) we will denote by $[Y]$ the closed subgroup of $\chi(G)$ generated by Y. Thus $[Y] = \Delta^\perp = Y^{\perp\perp}$.

Definition 2.6. Let $(X_i, G/\Delta_i)$ be subspaces of (X, G), $i = 1, \ldots, k$ and suppose $X = \bigcup_1^k X_i$ and that the product $\prod_1^k [X_i] = \chi(G)$ is a direct product. Then we will say (X, G) is the *direct sum* of the subspaces $(X_i, G/\Delta_i)$, $i = 1, \ldots, k$ and will write $X = X_1 \oplus \ldots \oplus X_k$ (or more precisely $(X, G) = (X_1, G/\Delta_1) \oplus \ldots \oplus (X_k, G/\Delta_k)$).

Remark 2.7. The condition that the product $\prod_1^k [X_i] = \chi(G)$ be direct is equivalent to the condition that the natural injective homomorphism $G \rightarrow \prod_1^k G/\Delta_i$ be surjective.

Remark 2.8. Here is an external characterization of direct sum: Let (X_i, G_i) $i = 1, \ldots, k$ be spaces of orderings. Let $G = \prod_1^k G_i$ and for each i, $1 \leqq i \leqq k$, let $Y_i = \{\sigma \in \chi(G) | \quad \sigma|_{G_i} \in X_i, \sigma|_{G_j} = 1 \text{ if } j \neq i\}$. (Here, for H a subgroup of G, $\sigma|_H$ denotes the restriction of σ to H.) Let $X = \bigcup_1^k Y_i$. Then one can easily verify (X, G) is a space of orderings. (This is basically because every n-dimensional form f on G determines and is determined by k n-dimensional forms $f_1, \ldots, f_k$ on $G_1, \ldots, G_k$ respectively, and $f \equiv g (\text{mod } X)$ holds if and only if $f_i \equiv g_i (\text{mod } X_i)$ holds for each $i = 1, \ldots, k$.) Moreover, if we identify the spaces (X_i, G_i) $i = 1, \ldots, k$ as subspaces of (X, G) in the natural way, we see that $X = X_1 \oplus \ldots \oplus X_k$ (internal direct sum).

Remark 2.9. Suppose (X, G) is the direct sum of the spaces

$$(X_i, G_i), \quad i = 1, \ldots, k.$$

Then there is a natural injective ring homomorphism from $W(X, G)$ into the product ring $\prod_1^k W(X_i, G_i)$. The image consists of those k-tuples

$$(f_1, f_2, \ldots, f_k)$$

such that $\dim f_i \equiv \dim f_1 (\text{mod } 2)$, $i = 2, \ldots, k$. This description of the direct sum is also found in [**7**].

One is especially interested in spaces of the following type:

Definition 2.10. A space of orderings (X, G) is said to be *indecomposible* if $X = X_1 \oplus X_2$, X_1, X_2 subspaces implies either $X_1 = \emptyset$ or $X_2 = \emptyset$.

Remark 2.11. Many interesting questions about spaces of orderings can be reduced to the indecomposible case. Two examples of this are found in Theorem 5.8 and Remark 5.9.

Remark 2.12. The best criterion for indecomposibility known to date appears to be the following: Let X be a space of orderings. Then X is indecomposible if and only if for each clopen subset $U \subseteq X$, $U \neq X$, $U \neq \emptyset$, there exists a 4-element fan $F \subseteq X$ such that $F \not\subseteq U$, $F \cap U \neq \emptyset$. (A *fan* [**4**] is just a subspace $F \subseteq X$ satisfying $\alpha, \beta, \gamma \in F$ implies $\alpha\beta\gamma \in F$. This result follows from results in [**2**] in case X is the space of orderings of a field, but it can be proved in general.)

Example 2.13. Here is a trivial application of the criterion for indecomposibility given in Remark 2.12. Consider a 1-stable space (X, G). One characterization of such a space is that the natural injection $G \rightarrow \text{Cont}(X, \{1, -1\})$ is onto. Another is that (X, G) has no 4-element fans. Thus we see that (up to equivalence) the only indecomposible 1-stable space is the space with a single ordering.

3. Quotient spaces and group extensions. Let (X, G) be a space of orderings. Fix a subgroup $G' \subseteq G$, and let X' denote the set of all restrictions of elements of X to G'. Then the pair (X', G') clearly satisfies 0_1 and 0_2. Moreover, 0_3 will be satisfied if and only if $-1 \in G'$. Simple conditions which will ensure that (X', G') satisfies 0_4 are not easy to obtain. In any case, we make the following definition.

Definition 3.1. A *quotient space* of a space of orderings (X, G) is a pair (X', G') obtained in the above fashion which is itself a space of orderings.

Remark 3.2. If (X', G') is a quotient space of (X, G) then there is a natural morphism from (X, G) to (X', G'). The corresponding ring homomorphism on the Witt Rings identifies $W(X', G')$ with a subring of $W(X, G)$.

The following result is useful in the proof of Theorem 4.3.

LEMMA 3.3. *Let* (X, G) *be a space of orderings, let* G' *be a subgroup of* G *containing* -1, *and let* $X' = \{\sigma|_{G'} \mid \sigma \in G\}$. *Suppose the following condition holds:*

(*) *If f, g are forms defined over G' and if $D_f \cap D_g \neq \emptyset$, then*

$D_f \cap D_g \cap G' \neq \emptyset$.

Then (X', G') is a quotient space of (X, G).

Proof. We must verify that (X', G') satisfies 0_4. For f a form defined over G', let D_f' denote the set of elements of G' represented by f over G'. That is, $x \in D_f'$ if and only if $x \in G'$, and there exists $x_2, \ldots, x_n \in G'$ such that $f \equiv \langle x, x_2, \ldots, x_n \rangle$.

Claim. For any form f defined over G', $D_f' = D_f \cap G'$.

This is clear if $\dim f = 1$ or 2. In general, let $x \in D_f \cap G'$ and write $f \equiv \langle a \rangle \oplus g$ where $a \in G'$, and g is a form defined over G'. Thus $\langle a, -x \rangle \oplus g$ is isotropic over G, so $D_{\langle a,-x \rangle} \cap D_{-g} \neq \emptyset$. Thus $D_{\langle a,-x \rangle} \cap D_{-g} \cap G' \neq \emptyset$. Let $y \in D_{\langle a,-x \rangle} \cap D_{-g} \cap G'$. Thus, by induction on the dimension, $y \in D_{-g}'$. Also $y \in D_{\langle a,-x \rangle} \cap G' = D'_{\langle a,-x \rangle}$ so $x \in D'_{\langle a,-y \rangle} \subseteq D_f'$. This completes the proof of the claim.

Now suppose f, g are forms over G'. Then $f \oplus g$ isotropic over G' implies $f \oplus g$ isotropic over G implies $D_f \cap D_{-g} \neq \emptyset$ implies $D_f \cap D_{-g} \cap G' \neq \emptyset$ implies $D_f' \cap D_{-g}' \neq \emptyset$. Thus (X', G') satisfies 0_4.

Remark 3.4. Suppose (X', G') is already known to be a quotient space of (X, G). Then one verifies that condition (*) of Lemma 3.3 is equivalent to the following condition:

(**): If a form f defined over G' is isotropic over G, then it is isotropic over G'.

The type of quotient space mentioned in the following example is described in [**11**]. This type of quotient is very special, but also very useful. The motivation for the construction comes from the theory of valuations on fields. See, for example, [**13**, section 7].

Example 3.5. Let (X, G) be a space of orderings. Let T denote the set of all characters $\alpha \in \chi(G)$ such that $\alpha X = X$. T is clearly a closed subgroup of $\chi(G)$. Let $G' = T^\perp$, and let $X' = \{\sigma|_{G'} \ |\sigma \in X\}$. Exactly as in [**11**], one may verify that (X', G') is a space of orderings. Note that X consists of *all* $\sigma \in \chi(G)$ such that $\sigma|_{G'} \in X'$. In terms of the definition to follow, this says that (X, G) is a group extension of (X', G'). Of course, it may happen that $T = 1$.

Definition 3.6. Let (X', G') be a space of orderings and let G be a group containing G' as a subgroup and satisfying $x^2 = 1 \quad \forall x \in G$. Let

$$X = \{\sigma \in \chi(G) | \quad \sigma|_{G'} \in X'\}.$$

Then (X, G) is a space of orderings. We refer to a space of orderings (X, G) obtained in this way as a group *extension* of (X', G'). We refer to such a group extension as *proper* if $G \neq G'$.

Remark 3.7. The fact that the pair (X, G) in Definition 3.6 is indeed a space of orderings is a consequence of the following:

Let f be a form over G, and suppose f is expressed in the form $f \equiv x_1f_1 \oplus \ldots \oplus x_sf_s$ where $x_1, \ldots, x_s \in G$ lie in distinct cosets modulo G', and where $f_1, \ldots, f_s$ are forms defined over G'. Then f is isotropic (over G) if and only if at least one of $f_1, \ldots, f_s$ is isotropic over G'. The proof of this is not given here.

Remark 3.8. Suppose (X, G) is a group extension of (X', G'). Then the Witt Rings of these two spaces are related by

$$W(X, G) \cong W(X', G')[G/G'].$$

(Here $R[H]$ denotes the group ring extension of the ring R by the group H). The isomorphism is not canonical.

Remark 3.9. The above construction provides additional examples of indecomposible spaces. Namely, if (X, G) is any proper group extension of any space (X', G'), and if $|X| > 2$, then (X, G) is indecomposible. This is easily verified.

4. Inverse limits. In this section we define inverse limits, and prove that every space of orderings is an inverse limit of countable spaces.

Definition 4.1. An *inverse system* of spaces of orderings is a triple consisting of (a) a directed set $(I, \geqq)$ (*b*) spaces of orderings (X_i, G_i), one for each $i \in I$ and (c) morphisms $\phi_{ij}(X_i, G_i) \rightarrow (X_j, G_j)$ for all $i, j \in I$ satisfying $i \geqq j$. It is assumed that each morphism ϕ_{ij} satisfies $\phi_{ij}(X_i) = X_j$. This implies in particular that $\phi^*_{ij} : G_j \rightarrow G_i$ is injective. It is further assumed for each $i \geqq j \geqq k$, $i, j, k \in I$, that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$.

Definition 4.2. Let $(I, (X_i, G_i), \phi_{ij})$ be a given inverse system of spaces of orderings. Let $G = \varinjlim G_i$, and $X = \varprojlim X_i \subseteq \varprojlim \chi(G_i) = \chi(G)$. The pair (X, G) thus obtained is referred to as the *inverse limit* of the given inverse system. This is denoted by writing $(X, G) = \varprojlim (X_i, G_i)$, the limit being taken with respect to the directed set $(I, \geqq)$.

THEOREM 4.3. *The inverse limit of a given inverse system of spaces of orderings is a space of orderings.*

Proof. Use the notations of Definition 4.2. Let $\phi_i^* : G_i \rightarrow G$ denote the canonical injection. Then the dual $\phi_i : \chi(G) \rightarrow \chi(G_i)$ is a continuous surjective group homomorphism, so it follows that $X = \bigcap_i \phi_i^{-1}(X_i)$ is closed. Thus 0_1 holds.

Claim. For each $i \in I$, $\phi_i(X) = X_i$. For let $\sigma \in X_i$. Since I is directed and $\phi_{kj}(X_k) = X_j$ for $k \geqq j$, it follows that any finite intersection of the sets $\phi_j^{-1}(X_j) \cap \phi_i^{-1}(\sigma)$, $j \in I$, is not empty. Thus, by compactness,

$$X \cap \phi_i^{-1}(\sigma) \neq \emptyset.$$

This proves the claim.

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Now identify each G_i with its image under ϕ_i^*. By the claim, this identifies X_i with the restriction of X to G_i. 0_2 and 0_3 are now clear. If f, g are forms defined over G, then f, g are both defined over some G_i, $i \in I$. It is clear that $f \equiv g(\text{mod } X)$ if and only if $f \equiv g(\text{mod } X_i)$. Thus 0_4 is clear.

Remark 4.4. Let (X, G) be the inverse limit of the inverse system $(I, (X_i, G_i), \phi_{ij})$. The morphisms $\phi_{ij} : (X_i, G_i) \rightarrow (X_j, G_j)$ determine injective ring homomorphisms $\tilde{\phi}_{ij} : W(X_j, G_j) \rightarrow W(X_i, G_i)$. $W(X, G)$ is just the *direct* limit of the system $(I, W(X_i, G_i), \tilde{\phi}_{ij})$.

Definition 4.5. A space of orderings (X, G) is said to be *countable* if G is countable.

Remark 4.6. If a space (X, G) is countable, then as a topological space X has a countable base, for clearly the Harrison basic sets $X(a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in G$ will be a countable base for X in this case.

THEOREM 4.7. *Each space of orderings is an inverse limit of countable spaces.*

Proof. Let (X, G) be a given space of orderings. To show (X, G) is an inverse limit of countable spaces, it is enough to show that for each countable subset S of G there exists a countable quotient space (X', G') of (X, G) such that $S \subseteq G'$.

Let S be a given countable subset of G. Define a sequence of subgroups $G_1 \subseteq G_2 \subseteq G_3 \subseteq \ldots$ of G as follows: Let G_1 denote the smallest subgroup of G containing S and -1. Now supposing G_n is defined, define G_{n+1} as follows: For each pair of forms f, g defined over G_n such that $D_f \cap D_g \neq \emptyset$ pick an element $x_{f,g} \in D_f \cap D_g$ and let G_{n+1} be the smallest subgroup of G containing G_n and the elements $x_{f,g}$ obtained in this way. Finally, let $G' = \bigcup_1^\infty G_n$. Then clearly $S \subseteq G'$, $-1 \in G'$ and G' is countable (since each G_n is countable). Also G' has the property (*) of Lemma 3.3. Thus, by that Lemma, (X', G') is a quotient space of (X, G).

5. Spaces which are inverse limits of finite spaces.

In this section the following question is considered:

Question 1. Which spaces of orderings are inverse limits of finite spaces?

One motivation for studying this question is obtained by considering the following question, and the remark and theorem following it.

Question 2. Let (X, G) be a space of orderings, and $k \geqq 1$. Let $f \in W(X, G)$ be such that $\sigma f \equiv 0 \bmod 2^k$ holds for all $\sigma \in X$. Then is it true that $f \in M^k(X, G)$? ($M(X, G)$ denotes the ideal of even dimensional forms in $W(X, G)$.)

Remark 5.1. Question 2 is known to have an affirmative answer for many spaces, e.g. see [**7**, **9**, **10**]. In particular the result is known to be true for all finite spaces of orderings. From the next theorem, it follows that Question 2 has an affirmative answer for all spaces of orderings which are inverse limits of finite spaces.

THEOREM 5.2. *Suppose* $(X, G) = \varprojlim (X_i, G_i)$ *and that Question 2 has an affirmative answer for each* (X_i, G_i). *Then it has an affirmative answer for* (X, G).

Proof. Let $f \in W(X, G)$ satisfy $\sigma f \equiv 0 \bmod 2^k \quad \forall \sigma \in X$. Since $W(X, G) = \varinjlim W(X_i, G_i)$ we may as well assume $f \in W(X_i, G_i)$ for some i. Since the map $X \to X_i$ is surjective it follows that $\sigma f \equiv 0 \bmod 2^k \quad \forall \sigma \in X_i$. Thus, by assumption $f \in M^k(X_i, G_i)$. But clearly $M^k(X_i, G_i) \subseteq M^k(X, G)$.

Remark 5.3. Here is additional motivation for studying Question 1. Denote by $\mathscr{B}$ the category whose objects are the compact totally disconnected topological spaces, and whose morphisms are the continuous maps. There is a natural identification of $\mathscr{B}$ with a subcategory of the category $\mathscr{O}$ of all spaces of orderings, namely the subcategory of $\mathscr{O}$ consisting of all 1-stable spaces. Under this identification $X \in \mathscr{B}$ is identified with the 1-stable space (X, G) where $G = \text{Cont}(X, \{1, -1\})$. It follows from results in [**6**] (also see [**13**]) that every space of orderings in this subcategory is equivalent to the space of orderings of a Pythagorian field satisfying S.A.P. (strong approximation property) and conversely. The point to be made here is that a well-known result in topology asserts that *every* $X \in \mathscr{B}$ *is the inverse limit of finite spaces*. Thus it is natural to ask how far this familiar property of $\mathscr{B}$ extends into the larger category.

Notation 5.4. From now on $\mathscr{F}$ will denote the subcategory of $\mathscr{O}$ consisting of all finite spaces of orderings, and pro$\mathscr{F}$ the subcategory of $\mathscr{O}$ consisting of inverse limits of finite spaces.

Remark 5.5. To show that a space of orderings (X, G) belongs to pro$\mathscr{F}$ it is necessary and sufficient to show that for each given finite subset

$$a_1, \ldots, a_n \in G,$$

there exists a finite quotient space (X', G') of (X, G) such that $a_1, \ldots, a_n \in G'$.

Remark 5.6. If we carry through the proof of Theorem 4.7 in the case S is finite we can (by a proper choice of the elements $x_{f,g}$ at each stage) construct the groups $G_1, G_2, G_3, \ldots$ in such a way that each is finite. Of course this still doesn't imply the finiteness of $G' = \bigcup_1^\infty G_n$.

The following Theorem summarizes some obvious properties of pro$\mathscr{F}$.

THEOREM 5.7. (1) *If* $X_i \in$ pro$\mathscr{F}$, $i = 1, 2$, *then* $X_1 \oplus X_2 \in$ pro$\mathscr{F}$.
(2) *If* X *is a group extension of* $X' \in$ pro$\mathscr{F}$, *then* $X \in$ pro$\mathscr{F}$.
(3) *If* $X = \varprojlim X_i$, *and if* $X_i \in$ pro$\mathscr{F}$ *for each* i, *then* $X \in$ pro$\mathscr{F}$.

Proof. All these results are fairly elementary. Here is the proof of (2). Let $a_1, \ldots, a_n \in G$. Let H denote the subgroup of G generated by $a_1, \ldots a_n$, and let $H' = H \cap G'$. By assumption there exists a finite quotient space (X_1', G_1') of (X', G') such that $H' \subseteq G_1'$. Let G_1 denote the subgroup of G generated by G_1' and H, and let X_1 denote the restriction of X to G_1. Thus $a_1, \ldots, a_n \in G_1$. Moreover (X_1, G_1) is a space of orderings. In fact it is a group extension of (X_1', G_1').

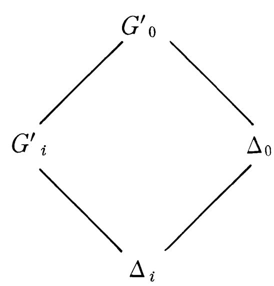
A slightly deeper result is now given.

THEOREM 5.8. *Let (X, G) be a space of orderings. Suppose every indecomposible subspace of X belongs to* pro$\mathscr{F}$. *Then $X \in$* pro$\mathscr{F}$.

Proof. It is enough to show that for each finite subset $a_1, \ldots, a_n \in G$ there exists a finite quotient space (X', G') of (X, G) such that $a_1, \ldots, a_n \in G'$. We assume that for some finite set $a_1, \ldots, a_n$ no such quotient exists, and obtain a contradiction. First we need the following:

Claim. Let X_i, $i \in I$ be a set of subspaces of X linearly ordered by inclusion, and let $X_0 = \bigcap_I X_i$. X_0 is clearly a subspace of X. Let $\Delta_i = X_i^\perp$ for each $i \in I \cup \{0\}$. Suppose there is a finite quotient of $(X_0, G/\Delta_0)$ containing $a_1\Delta_0, \ldots, a_n\Delta_0$. Then there exists $i \in I$ such that $(X_i, G/\Delta_i)$ has a finite quotient containing $a_1\Delta_i, \ldots, a_n\Delta_i$.

To prove this claim, first note that $\Delta_0 = \bigcup_I \Delta_i$. Let $(X_0', G_0'/\Delta_0)$ be a finite quotient of $(X_0, G/\Delta_0)$ such that $a_1, \ldots, a_n \in G_0'$. For each $i \in I$, let G_i' denote the subgroup of G generated by Δ_i and $b_1, \ldots, b_k$ where $b_1, \ldots, b_k$ is a fixed basis of G_0' modulo Δ_0.



It is clear that $G_i'/\Delta_i \cong G_0'/\Delta_0$ cannonically and that the dual isomorphism $\chi(G_0'/\Delta_0) \cong \chi(G_i'/\Delta_i)$ carries X_0' *into* X_i' (X_i' is the restriction of X_i to G_i'). To simplify notation, identify these groups and consider

$$\sigma' \in \bigcap_I X_i' \subseteq \chi(G_0'/\Delta_0).$$

Suppose $\sigma'(b_j) = \epsilon_j \in \{1, -1\}$, $j = 1, \ldots, k$. It follows that

$$X_i(b_1\epsilon_1, \ldots, b_k\epsilon_k) \neq \emptyset$$

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for each $i \in I$. By compactness,

$$X_0(b_1\epsilon_1, \ldots, b_k\epsilon_k) = \bigcap_I X_i(b_1\epsilon_1, \ldots, b_k\epsilon_k) \neq \emptyset.$$

This implies $\sigma' \in X_0'$. Thus $X_0' = \bigcap_I X_i'$. Since the spaces X_i', $i \in I$ are linearly ordered by inclusion, and each is finite, it follows that we have $X_i' = X_0'$, and hence $(X_i', G_i'/\Delta_i) \backsim (X_0, G_0'/\Delta_0)$ for $i \in I$ sufficiently large. In particular, this implies $(X_i', G_i'/\Delta_i)$ is a space of orderings for $i \in I$ sufficiently large. On the other hand $\bigcup_I G_i' = G_0'$, $a_1, \ldots, a_n \in G_0'$, and the groups G_i', $i \in I$ are linearly ordered by inclusion. Thus $a_1, \ldots, a_n \in G_i'$ for $i \in I$ sufficiently large. This completes the proof of the claim.

Now consider the set $\mathscr{S}$ of all subspaces $(Y, G/\Delta)$ of (X, G) such that there does not exist a finite quotient of $(Y, G/\Delta)$ containing $a_1\Delta, \ldots, a_n\Delta$. Order this set by inclusion, i.e. $(Y_1, G/\Delta_1) \geqq (Y_2, G/\Delta_2)$ if and only if $Y_1 \supseteq Y_2$. By assumption, $(X, G) \in \mathscr{S}$, so $\mathscr{S} \neq \emptyset$. By the claim, and Zorn's Lemma, $\mathscr{S}$ has a minimal element. Say $(Y, G/\Delta)$ is a minimal element of $\mathscr{S}$.

If Y decomposes as $Y = Y_1 \oplus Y_2$, $Y_i \neq \emptyset$, $i = 1, 2$, then by the minimality of Y, there would exist a finite quotient Y_i' of Y_i containing the cosets of $a_1, \ldots, a_n$, for $i = 1, 2$. But then $Y_1' \oplus Y_2'$ would be a quotient of Y with the same properties. This is a contradiction. Thus, Y is indecomposible, so by assumption, $Y \in \text{pro}\mathscr{F}$. But this is also a contradiction.

Remark 5.9. The technique used here also serves to reduce many other problems about spaces of orderings to the indecomposible case. For example, Question 2 reduces to the indecomposible case by this technique.

Definition 5.10. An indecomposible space of orderings (X, G) is said to be of *elementary type* if either $|X| = 1$, or if (X, G) is a proper group extension of some space of orderings (X', G').

THEOREM 5.11. *Let (X, G) be a space of orderings, each of whose indecomposible subspaces is of elementary type. Then $X \in \text{pro}\mathscr{F}$.*

Proof. Proceed as in Theorem 5.8. That is, suppose there is a finite set $a_1, \ldots, a_n \in G$ which is not contained in any finite quotient of (X, G). Define $\mathscr{S}$ and $(Y, G/\Delta)$ exactly as in Theorem 5.8. Then $(Y, G/\Delta)$ is indecomposible and hence of elementary type. Clearly Y is not singleton, so there exists a non-trivial character $\gamma \in \chi(G/\Delta)$ such that $\gamma Y = Y$. Thus $(Y, G/\Delta)$ is a group extension of $(Y', G'/\Delta)$ where $G'/\Delta = \text{kern } \gamma$.

Fix an element $x \in G$, $x \notin G'$. Then $Y(x)^\perp = \Delta \cup \Delta x$. The subspace $(Y(x), G/\Delta \cup \Delta x)$ of $(Y, G/\Delta)$ is equivalent to the space $(Y', G'/\Delta)$ in a canonical way. By the minimality of Y, there exists a finite quotient of $Y(x)$ containing the cosets of $a_1, \ldots, a_n$. Let $(Y_1', G_1'/\Delta)$ denote the corresponding quotient of $(Y', G'/\Delta)$ under the equivalence. Write each a_i in the form $a_i = b_i x^{\epsilon_i}$, $b_i \in G'$, $\epsilon_i = 0$ or 1, $i = 1, \ldots, n$. It should be clear that $b_1, \ldots, b_n \in G_1'$. Now let $G_1 = G_1' \cup G_1'x$, and let Y_1 denote the restriction of Y to G_1/Δ. Then $(Y_1, G_1/\Delta)$ is a group extension of $(Y_1', G_1'/\Delta)$ so it is a space of orderings and hence is a finite quotient of $(Y, G/\Delta)$. Also $a_1, \ldots, a_n \in G_1$. This contradiction completes the proof.

Remark 5.12. The subcategory $\mathscr{E}$ of $\mathscr{O}$ consisting of all spaces of orderings satisfying the hypothesis of Theorem 5.11 is fairly extensive. For example, it follows from results in [**11**] that $\mathscr{F} \subseteq \mathscr{E}$. Also

(1) $\mathscr{E}$ contains all 1-stable spaces.

(2) If $X_1, X_2 \in \mathscr{E}$, then $X_1 \oplus X_2 \in \mathscr{E}$.

(3) If X is a group extension of $X' \in \mathscr{E}$, then $X \in \mathscr{E}$.

In particular, $\mathscr{E}$ contains all spaces of the type discussed in [**7**].

Remark 5.13. There is an abstract classification of the category of spaces $\mathscr{C}$ considered in [**7**]. Namely, a space of orderings belongs to $\mathscr{C}$ if and only if it is generated by a finite number of fans. The proof of this assertion uses essentially only the theorems and techniques used in [**11**]. Note that to say a space X is generated by subspaces $F_1, \ldots, F_k$ means simply that

$$(F_1 \cup \ldots \cup F_k)^\perp = F_1^\perp \cap \ldots \cap F_k^\perp = 1.$$

The connection between the category $\mathscr{C}$ and the category of spaces satisfying the chain condition [see **9**, **10**] is not clear, although it is clear that every space $X \in \mathscr{C}$ does satisfy the chain condition. One might guess that these categories are equal.

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