SOME PROJECTIVE REPRESENTATIONS OF FINITE ABELIAN GROUPS

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1. In this paper, we continue the work initiated by Morris [**5**] and Saeed-ul-Islam [**6**, **7**] and determine complete sets of inequivalent irreducible projective representations (which we shall write as i.p.r.) of finite Abelian groups with respect to some additional factor sets.

We consider an Abelian group

$$A = \langle w_1, \ldots, w_m : w_i^{a_i} = 1, 1 \leq i \leq m, a_i \mid a_{i+1}, 1 \leq i \leq m-1 \rangle$$

which will be referred to as an Abelian group of type $(a_1, \ldots, a_m)$.

The irreducible ordinary representations of A are well-known and are given as follows.

Let ω_i be a primitive a_ith root of unity for $i = 1, \ldots, m$. Then a complete set of inequivalent irreducible ordinary representations of A is given by

$$\{\chi_{(\lambda_1,\ldots,\lambda_m)} : \lambda_i \in \{0, 1, \ldots, a_i - 1\}, \quad i = 1, \ldots, m\}$$

where

$$\chi_{(\lambda_1,\ldots,\lambda_m)}(w_1^{\alpha_1} \ldots w_m^{\alpha_m}) = \prod_{i=1}^{m} \omega_i^{\lambda_i \alpha_i}$$

for all $\alpha_i \in \{0, 1, \ldots, a_i - 1\}$, $i = 1, \ldots, m$.

Let α be a factor set of A (see Morris [**4**] and Karpilovsky [**2**] for definitions and other properties of factor sets and projective representations). Let $\mathbb{C}$ be the complex field and let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Define $\alpha' : A \times A \to \mathbb{C}^*$ by

$$\alpha'(a, b) = \alpha(a, b)\alpha(b, a)^{-1}$$

for all $a, b \in A$. Then, easy calculations using the definition of a factor set show that:

(i) α' is a bilinear mapping, that is,

$$\alpha'(ab, c) = \alpha'(a, c)\alpha'(b, c)$$
$$\alpha'(a, bc) = \alpha'(a, b)\alpha'(a, c)$$

for all $a, b, c \in A$;

(ii) the factor set α may be chosen, up to equivalence, such that $\alpha'(w_i, w_j) = \theta_{ij}$ (say) satisfy the following relations:

$$\theta_{ij}^{a_j} = 1, \quad \theta_{ii} = 1, \quad \theta_{ji} = \theta_{ij}^{-1} \qquad 1 \leq i, j \leq m \ (i \neq j), \tag{1.1}$$

and

$$\prod_{j=1}^{a_i} \alpha(w_i^j, w_i) = 1 \qquad 1 \leq i \leq m. \tag{1.2}$$

(iii) We shall call the matrix (θ_{ij}) the matrix associated with α and we shall write $\alpha \in (\theta_{ij})$.

In this paper we determine complete sets of inequivalent i. p. r. of A with factor sets belonging to the special classes given as follows.

In Section 2 we take $\alpha \in (\theta_{ij})$ where θ_{st} is a primitive a_sth root of unity for a fixed pair of indices (s, t), $s<t$ and $\theta_{ij}=1$ for $(i, j) \neq (s, t)$. In Section 3, we generalise the results of Section 2 and consider a set of indices $1 \leqslant s_1 < s_2 < \ldots < s_{2r} \leqslant m$ and take $\alpha \in (\theta_{ij})$ where $\theta_{s_i s_{i+1}}$ is a primitive a_{s_i}th root of unity for $i = 1, 3, \ldots, 2r-1$ and $\theta_{ij}=1$ otherwise. Finally, in Section 4, we consider the factor set $\alpha \in (\theta_{ij})$ such that each θ_{ij} is a primitive a_ith root of unity.

Let $T: A \to \mathrm{GL}(n, \mathbb{C})$ be a projective representation of A with factor set $\alpha \in (\theta_{ij})$ which satisfies (1.1) and (1.2). Put $T_i = T(w_i)$, $i = 1, \ldots, m$. Then $T_1, \ldots, T_m$ satisfy the following relations:

$$\left.\begin{array}{ll} T_i^{a_i} = I, & i = 1, \ldots, m \\ T_i T_j = \theta_{ij} T_j T_i, & i, j = 1, \ldots, m. \end{array}\right\} \tag{2}$$

Conversely, if $T_1, \ldots, T_m$ are non-singular $n \times n$ matrices satisfying equations (2) and (θ_{ij}) is an $m \times m$ matrix whose entries satisfy (1.1) then these $n \times n$ matrices define a projective representation T of A with factor set $\alpha \in (\theta_{ij})$ defined by

$$T(w_1^{\alpha_1} \ldots w_m^{\alpha_m}) = T_1^{\alpha_1} \ldots T_m^{\alpha_m}.$$

It is well known (see Karpilovsky [**2**]) that the number of inequivalent i. p. r. of a finite Abelian group A is equal to the number of α-regular elements of the group. Let $\alpha(A)$ be the set of α-regular elements of A and let $n_\alpha(A) = |\alpha(A)|$. Furthermore, since all the i. p. r. of an Abelian group with a fixed factor set α are of the same degree $d_\alpha(A)$ (Frucht [**1**]) we have $n_\alpha(A)(d_\alpha(A))^2 = |A|$ and thus the degrees of the i. p. r. are known as soon as the numbers $n_\alpha(A)$ are known.

2. For $1 \leqslant s < t \leqslant m$, let $B_{st} = (\theta_{ij})$ where

$$\theta_{ij} = \begin{cases} \omega_{a_s}, & \text{a primitive } a_s\text{th root of unity if } (i, j) = (s, t), \\ 1 & \text{otherwise.} \end{cases}$$

In this section we determine complete sets of inequivalent i. p. r. of the Abelian group A with factor set $\alpha \in B_{st}$ for all $1 \leqslant s < t \leqslant m$.

THEOREM 2.1. *The number $n_\alpha(A)$ of inequivalent i. p. r. of A with factor set $\alpha \in B_{st}$ is $|A|/a_s^2$ and $d_\alpha(A) = a_s$.*

Proof. We need only find the number of α-regular elements of A for this factor set α.

Now, $w = w_1^{\alpha_1} \ldots w_m^{\alpha_m} \in A$ is α-regular if and only if

$$\alpha'(w_s, w_1^{\alpha_1} \ldots w_m^{\alpha_m}) = \alpha'(w_t, w_1^{\alpha_1} \ldots w_m^{\alpha_m}) = 1$$

(since $\alpha'(w_i, w_1^{\alpha_1} \ldots w_m^{\alpha_m}) = 1$ for all $i \neq s, t$). Thus w is α-regular if $\alpha'(w_s, w_t^{\alpha_t}) = 1$ and $\alpha'(w_t, w_s^{\alpha_s}) = 1$; that is, $\omega_{a_s}^{\alpha_t} = \omega_{a_s}^{\alpha_s} = 1$. Thus we have $\alpha_t \equiv 0 \pmod{a_s}$ and $\alpha_s \equiv 0 \pmod{a_s}$, which implies that $\alpha_s = 0$ and $\alpha_t = za_s$, $z = 0, 1, \ldots, (a_t/a_s) - 1$. Thus, the total number of α-regular elements of A is equal to

$$a_1 a_2 \ldots a_{s-1} a_{s+1} \ldots (a_t/a_s) a_{t+1} \ldots a_r = |A|/a_s^2$$

and the theorem follows.

We now construct a set of $k \times k$ matrices which are used not only to give an explicit construction of the i. p. r. corresponding to this factor set, but also for the other factor sets considered later in this paper.

Let ω_k be a primitive kth root of unity and ζ_k a primitive $2k$th root of unity such that $\zeta_k^2 = \omega_k$. Then, if k is odd, let P_k and $Q_k(\omega_k)$ be the $k \times k$ matrices defined by

$$P_k = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad Q_k(\omega_k) = \begin{bmatrix} 0 & \omega_k & 0 & \ldots & 0 \\ 0 & 0 & \omega_k^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_k^{k-1} \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

If k is even, let P_k be defined as above and

$$Q_k(\omega_k) = \begin{bmatrix} 0 & \zeta_k & 0 & \ldots & 0 \\ 0 & 0 & \zeta_k^3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \ldots & \zeta_k^{2k-3} \\ \zeta_k^{2k-1} & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then, in both cases, it can be readily verified (see Morris [**3**]) that

$$P_k^k = (Q_k(\omega_k))^k = I_k; \qquad P_k Q_k(\omega_k) = \omega_k Q_k(\omega_k) P_k$$

where I_k is the identity matrix of order $k \times k$. The matrices P_k and $Q_k(\omega_k)$ are sometimes referred to as generalised Pauli matrices.

Now if we let $T_s = P_{a_s}$, $T_t = Q_{a_s}(\omega_{a_s})$ and $T_i = I_{a_s}$ for all $i \neq s, t$ then it can be easily verified that the T_i, $i = 1, \ldots, m$ satisfy equations (2) of Section 1 for $(\theta_{ij}) = B_{st}$ and therefore generate a projective representation T_{st} of A with factor set $\alpha \in B_{st}$ which is clearly irreducible because its degree is equal to $d_\alpha(A) = a_s$.

Let $\Lambda(s, t) = \{(\lambda_1, \ldots, \lambda_m) : \lambda_s = 0,\ a_s \mid \lambda_t,\ 0 \leq \lambda_k < a_k,\ 1 \leq k \leq m\}$ and let $\chi_{(\lambda_1, \ldots, \lambda_m)}$ be an irreducible ordinary representation of A associated with the sequence $(\lambda_1, \ldots, \lambda_m) \in \Lambda(s, t)$. Then $\chi_{(\lambda_1, \ldots, \lambda_m)} \neq \chi_{(\lambda'_1, \ldots, \lambda'_m)}$ on $\alpha(A)$ if and only if $(\lambda_1, \ldots, \lambda_m) \neq (\lambda'_1, \ldots, \lambda'_m)$ for $(\lambda_1, \ldots, \lambda_m), (\lambda'_1, \ldots, \lambda'_m) \in \Lambda(s, t)$.

If T_{st} is the i. p. r. of A with factor set $\alpha \in B_{st}$ as defined above, then $\chi_{(\lambda_1, \ldots, \lambda_m)} \otimes T_{st}$ is also an i. p. r. of A with factor set $\alpha \in B_{st}$. It can be verified, by comparing the values of the projective characters on $\alpha(A)$, that $\{\chi_{(\lambda_1,\ldots,\lambda_m)} \otimes T_{st} : (\lambda_1, \ldots, \lambda_m) \in \Lambda(s, t)\}$ is a set of inequivalent i. p. r. of A whose number is equal to the number of α-regular elements of A. We have thus proved the following theorem.

THEOREM 2.2. *Let A be a finite Abelian group of type $(a_1, \ldots, a_m)$. Then, in the above notation,*

$$\{\chi_{(\lambda_1,\ldots,\lambda_m)} \otimes T_{st} : (\lambda_1, \ldots, \lambda_m) \in \Lambda(s, t)\}$$

is a complete set of inequivalent i. p. r. of A with factor set $\alpha \in B_{st}$.

3. In this section let $\{s_1, \ldots, s_{2r}\}$ be such that $1 \leq s_1 < s_2 < \ldots < s_{2r} \leq m$ and consider the factor set $\alpha \in (\theta_{ij})$ where

$$\theta_{s_i s_{i+1}} = \omega_{s_i} \quad \text{if} \quad i = 1, 3, \ldots, 2r-1,$$

and

$$\theta_{ij} = 1 \quad \text{if} \quad (i, j) \neq (s_i, s_{i+1})$$

where ω_{s_i} denotes a primitive a_{s_i}th root of unity. We prove the following theorem.

THEOREM 3.1. *Let $\alpha \in (\theta_{ij})$ be as above. Then A has $|A|/a_{s_1}^2 \ldots a_{s_{2r-1}}^2$ inequivalent i. p. r. with degree $d_\alpha(A) = \prod_{i=1}^{r} a_{s_{2i-1}}$.*

Proof. As in Theorem 2.1, it can be easily seen that $w_1^{\alpha_1} \ldots w_m^{\alpha_m} \in A$ is α-regular if and only if α_i are solutions of the congruences

$$\alpha_{s_{i+1}} \equiv 0 \,(\text{mod}\, a_{s_i}) \quad \text{and} \quad \alpha_{s_i} \equiv 0 \,(\text{mod}\, a_{s_i})$$

for all $i = 1, 3, \ldots, 2r-1$. The number of solutions of these congruences is clearly equal to

$$\left(\prod_{j=1}^{m} a_j\right) / (a_{s_1}^2 a_{s_3}^2 \ldots a_{s_{2r-1}}^2)$$

and thus is the number of inequivalent i. p. r. of A with factor set α. Furthermore, $d_\alpha(A) = a_{s_1} a_{s_3} \ldots a_{s_{2r-1}}$.

For $i = 1, 3, \ldots, 2r-1$, define an i. p. r. $T_{s_i s_{i+1}}$ of A with factor set belonging to the class $B_{s_i s_{i+1}}$ as in the previous section and let

$$T_s = T_{(s_1,\ldots,s_{2r})} = T_{s_1 s_2} \otimes T_{s_3 s_4} \otimes \ldots \otimes T_{s_{2r-1} s_{2r}}.$$

Then T_s is an i. p. r. of A with the required factor set. Furthermore, if $\lambda_{s_i} = 0$, $a_{s_i} \mid \lambda_{s_{i+1}}$, $i = 1, 3, \ldots, 2r-1$ and $1 \leq \lambda_k < a_k$ for $1 \leq k \leq m$, then $F_{(\lambda_1,\ldots,\lambda_m)} = \chi_{(\lambda_1,\ldots,\lambda_m)} \otimes T_s$ is an i. p. r. of A with factor set α and

$$\{F_{(\lambda_1,\ldots,\lambda_m)} : \lambda_{s_i} = 0,\, a_{s_i} \mid \lambda_{s_{i+1}},\, i = 1, 3, \ldots, 2r-1 \text{ and } 1 \leq \lambda_k < a_k \text{ for } 1 \leq k \leq m\}$$

gives a complete set of inequivalent i. p. r. of A with factor set α because $\chi_{(\lambda_1,\ldots,\lambda_m)}$ and hence the projective character of $F_{(\lambda_1,\ldots,\lambda_m)}$ are distinct when restricted to $\alpha(A)$, the set of all the α-regular elements of A, and the number of sequences $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_{s_i} = 0$, $a_{s_i} \mid \lambda_{s_{i+1}}$, $i = 1, 3, \ldots, 2r-1$ and $1 \leq \lambda_k < a_k$ for $1 \leq k \leq m$ is equal to the number of i. p. r. of A with factor set α as determined above.

4. We now determine the i. p. r. for Abelian groups of type $(a_1, \ldots, a_m)$ for the factor set $\alpha \in (\theta_{ij})$ where each θ_{ij} is a primitive a_ith root of unity. We prove the following result.

THEOREM 4.1. *Let A be a finite Abelian group of type $(a_1, \ldots, a_m)$ where $a_i \mid a_{i+1}$, $i = 1, \ldots, m-1$. If $m = 2v$ is even and α is the factor set defined above, then A has $\prod_{i=1}^{v} (a_{2i}/a_{2i-1})$ inequivalent i. p. r. of degree $\prod_{i=1}^{v} a_{2i-1}$ with factor set α.*

Proof. Proceeding as in Theorem 2.1 an arbitrary element $w = w_1^{\alpha_1} \ldots w_m^{\alpha_m}$ of A is α-regular if and only if the α_i are solutions of the following congruences

$$\sum_{i=1}^{k-1} (a_{m-1}/a_i)\alpha_i - (a_{m-1}/a_k)\left[\sum_{i=k+1}^{m} \alpha_i\right] \equiv 0 \pmod{a_{m-1}} \tag{3}$$

for all $k = 1, \ldots, m$. These congruences in matrix form are equivalent to $P(\alpha) \equiv 0 \pmod{a_{m-1}}$ where $(\alpha)' = (\alpha_1, \ldots, \alpha_m)$ and P is an $m \times m$ skew-symmetric integer matrix with entries $p_{ij} = c_{ij}a_{m-1}/(a_i, a_j)$ where

$$c_{ij} = \begin{cases} -1 & \text{if} \quad i<j, \\ 0 & \text{if} \quad i=j, \\ 1 & \text{if} \quad i>j. \end{cases}$$

An easy matrix calculation shows that the matrix P is row equivalent to the matrix $Q = E_1 \oplus E_3 \oplus \ldots \oplus E_{m-1}$ where

$$E_i = \begin{pmatrix} 0 & -a_{m-1}/a_i \\ a_{m-1}/a_i & 0 \end{pmatrix}$$

for $i = 1, 3, \ldots, m-1$. Thus the linear congruences (3) are equivalent to $Q(\alpha) \equiv 0 \pmod{a_{m-1}}$; that is

$$\left.\begin{aligned} -\alpha_{i+1}a_{m-1}/a_i &\equiv 0 \pmod{a_{m-1}} \\ \alpha_i a_{m-1}/a_i &\equiv 0 \pmod{a_{m-1}} \end{aligned}\right\} \quad i = 1, 3, \ldots, m-1,$$

which are equivalent to the following:

$$\left.\begin{aligned} \alpha_{i+1} &\equiv 0 \pmod{a_i} \\ \alpha_i &\equiv 0 \pmod{a_i} \end{aligned}\right\} \quad i = 1, 3, \ldots, m-1.$$

Hence the α-regular elements are given by $w_2^{\alpha_2} w_4^{\alpha_4} \ldots w_m^{\alpha_m}$, where $\alpha_{2i} \equiv 0 \pmod{a_{2i-1}}$,

$i = 1, \ldots, \nu = \frac{1}{2}m$. Clearly

$$|\alpha(A)| = (a_2/a_1) \times (a_4/a_3) \times \ldots \times (a_m/a_{m-1}) = \prod_{i=1}^{\nu} (a_{2i}/a_{2i-1})$$

and

$$d_\alpha(A) = \left[\left(\prod_{i=1}^{m} a_i\right) \Big/ \prod_{i=1}^{\nu} (a_{2i}/a_{2i-1})\right]^{\frac{1}{2}} = \left[\prod_{i=1}^{\nu} a_{2i-1}^2\right]^{\frac{1}{2}} = \prod_{i=1}^{\nu} a_{2i-1}$$

as required.

Let P_{a_i}, $Q_{a_i}(\omega_{a_i})$ and I_{a_i} be the generalised Pauli matrices of order $a_i \times a_i$ for $i = 1, 3, \ldots, m-1$. We will henceforth refer to these matrices as P_i, Q_i and I_i respectively. Define

$$R_i = \begin{cases} P_i^{a_i - 1} Q_i & \text{if } a_i \text{ is odd,} \\ \zeta_{a_i} P_i^{a_i - 1} Q_i & \text{if } a_i \text{ is even.} \end{cases}$$

Then $R_i^{a_i} = I_i$ for $i = 1, 3, \ldots, m-1$ and

$$P_i R_i = \omega_{a_i} R_i P_i, \qquad Q_i R_i = \omega_{a_i} R_i Q_i.$$

Now form the tensor products

$$E_{2j-1} = R_1 \otimes R_3 \otimes \ldots \otimes R_{2j-3} \otimes P_{2j-1} \otimes I_{2j+1} \otimes \ldots \otimes I_{m-1}$$
$$E_{2j} = R_1 \otimes R_3 \otimes \ldots \otimes R_{2j-3} \otimes Q_{2j-1} \otimes I_{2j+1} \otimes \ldots \otimes I_{m-1}$$

for $j = 1, 2, \ldots, \nu$.

Clearly the E_i, $i = 1, \ldots, m$, satisfy equations (2) for $\theta_{ij} = \omega_{a_i}$, $i < j = 1, \ldots, m$ and therefore generate a projective representation T of A with factor set α. Also, the degree of this representation being equal to $d_\alpha(A) = \prod_{i=1}^{\nu} a_{2i-1}$, this is an i. p. r. of A with factor set α.

It can now easily be seen that a complete set of inequivalent i. p. r. of A with factor set α is given by

$$\{\chi_{(\lambda_1, \ldots, \lambda_m)} \otimes T : \lambda_{2i-1} = 0, a_{2i-1} \mid \lambda_{2i}, i = 1, \ldots, \nu \text{ and } 1 \leq \lambda_i \leq a_i \text{ for } i = 1, \ldots, m\}.$$

THEOREM 4.2.

(a) *Let A be a finite Abelian group of type $(a_1, \ldots a_m)$ where $m = 2\nu + 1$ is odd. Let $\alpha \in (\theta_{ij})$ where $\theta_{ij} = \omega_{a_i}$, $1 \leq i < j \leq m$ and each ω_{a_i} is a primitive a_i*th *root of unity. Then A has $\left(\prod_{i=0}^{\nu} a_{2i+1}\right) \Big/ \left(\prod_{i=1}^{\nu} a_{2i}\right)$ inequivalent i. p. r. of degree $\prod_{i=1}^{\nu} a_{2i}$ with factor set α.*

(b) *If $E_1, \ldots, E_{2\nu}$ are the matrices defined as in the case m even and*

$$E_m = R_1 \otimes R_3 \otimes \ldots \otimes R_{2\nu-1}$$

then $E_1, \ldots, E_m$ generate an i. p. r. T of A with factor set α and a complete set of *inequivalent i. p. r. of A with factor set α is given by*

$$\{\chi_{(\lambda_1,\ldots,\lambda_m)} \otimes T : \lambda_{2i} = 0,\ a_{2i} \mid \lambda_{2i+1},\ i = 1, \ldots, v \text{ and } 1 \leq \lambda_i \leq a_i \text{ for } i = 1, \ldots, m\}.$$

Proof. The proof is similar to the case when m is even and is therefore omitted.

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