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WEIGHTED L_P SOLVABILITY FOR PARABOLIC EQUATIONS WITH PARTIALLY BMO COEFFICIENTS AND ITS APPLICATIONS

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Abstract

We consider the weighted L_p solvability for divergence and nondivergence form parabolic equations with partially bounded mean oscillation (BMO) coefficients and certain positive potentials. As an application, global regularity in Morrey spaces for divergence form parabolic operators with partially BMO coefficients on a bounded domain is established.

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1. Introduction

We are concerned about the parabolic divergence and nondivergence equations of the form

$$Lu(t,x) = -u_t(t,x) + a^{ij}(t,x)u_{x_ix_j}(t,x) + b^i(t,x)u_{x_i}(t,x) + c(t,x)u(t,x), \tag{1.1}$$

and

$$\begin{aligned}\mathcal{L}u(t,x) = -u_t(t,x) &+ (a^{ij}(t,x)u_{x_i}(t,x) + a^i(t,x)u(t,x))_{x_j} \\ &+ b^i(t,x)u_{x_i}(t,x) + c(t,x)u(t,x).\end{aligned} \tag{1.2}$$

We assume that the coefficients of these operators are bounded and measurable, and a^{ij} are uniformly elliptic, i.e. for some $K > 0$ and $\delta \in (0,1]$,

$$|b^i| + |c| \le K, \quad |a^{ij}| \le \delta^{-1}, \quad \delta|\xi|^2 \le a^{ij}\xi_i\xi_j \le \delta^{-1}|\xi|^2. \tag{1.3}$$

For equations with uniformly continuous leading coefficients, the solvability is classical. The L_p theory of second-order equations with discontinuous coefficients was studied extensively in the last two decades. One important class of discontinuous coefficients contains functions with vanishing mean oscillation (VMO), the study of

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which was started in [6] about 20 years ago and continued in [2, 7, 9]. On the other hand, the Morrey space theory of second-order equations with discontinuous coefficients was also studied in [11, 12, 20, 30]. In addition Dintelmann *et al.* [10] studied the mixed weighted inequalities for higher-order parabolic systems with VMO coefficients independent of t, and Munckenhoupt weights independent of t. Recently, Tang [32] studied $W^{2,p}_{\omega}$-solvability of the Cauchy–Dirichlet problem for nondivergence parabolic equations with bounded mean oscillation (BMO) coefficients and parameter $\lambda > 0$.

Recently, Krylov [25, 26] gave a unified approach to investigate the L_p-solvability of both divergence and nondivergence form parabolic and elliptic equations with $a^{ij} \in$ VMO in the spatial variables (and measurable in the time variable in the parabolic case). This result was later improved and generalized in a series of papers [15–17].

In contrast, the $L_p(p > 2)$ theory of elliptic and parabolic equations with partially BMO coefficients is quite new, and was originated in [24]. Later, this result was improved and generalized in [13, 14].

The main purpose of this paper is to show the weighted L_p solvability for divergence and nondivergence form parabolic operators with partially BMO coefficients and certain positive potentials. Furthermore, as an application, we establish global regularity in Morrey spaces for divergence form parabolic operators with partially BMO coefficients on a bounded domain.

We now give a brief outline of this paper. In the next section, we introduce some notation and some definitions. In Section 3, we establish the weighted L_p solvability for divergence and nondivergence form parabolic equations with VMO_x coefficients and certain positive potentials. In Section 4, we establish the weighted L^p solvability for divergence and nondivergence form parabolic equations with partially BMO coefficients and certain positive potentials by using the main results in Section 3. In Section 5, we further establish the weighted L^p solvability for divergence and nondivergence form parabolic equations with hierarchically partially BMO coefficients and certain positive potentials by using the main results in Section 4. The weighted L^p solvability on half spaces for divergence with partial BMO coefficients and nondivergence form parabolic equations with VMO_x coefficients are obtained in Sections 6 and 7. In Section 8, by using the main results in Sections 3, 4, 6 and 7, we establish global regularity in Morrey spaces for divergence and nondivergence form parabolic equations in a bounded domain. It should be pointed out that we only consider parabolic equations without potential in Sections 6–8, unless more technical assumptions need to be imposed on the potentials. Finally, in Section 9, we obtain the boundedness for some Schrödinger-type operators by using the main results in Section 5.

Finally, it should be pointed out that our proof in this paper follows from [15–19, 25, 26], and our results generalize the corresponding results in [15–18, 25, 26], and generalize and improve some well-known results in [10–12, 20–22, 30] in some ways. In addition, our results in the elliptic equation case are also true.

2. Preliminaries

Let $d \geq 1$ be an integer. A typical point in $\mathbb{R}^{d+1}$ is denoted by $(t, x) = (t, x^1, \dots, x^d) = (t, x^1, x')$. We set

$$D_i u = u_{x^i}, \quad D_{ij} u = u_{x^i x^j}, \quad \partial_t u = u_t.$$

By Du and $D^2 u$ we mean the gradient and the Hessian matrix of u, with respect to the x variable. On many occasions we need to take these objects relative to only part of variables. We also use the following notation:

$$D_{x'} u = u_{x'}, \quad Du = u_x, \quad D_{x^1 x'} u = u_{x^1 x'}, \quad D_{xx'} u = u_{xx'}, \quad D^2 u = u_{xx}.$$

For a function $f(t, x)$ in $\mathbb{R}^{d+1}$, we set

$$(f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(t, x)\, dx\, dt,$$

and

$$\|f\|_{L_{p,\omega}(\mathcal{D})} = \left(\int_{\mathcal{D}} |f(t, x)|^p \omega(t, x)\, dx\, dt \right)^{1/p},$$

where $\mathcal{D}$ is open subset in $\mathbb{R}^{d+1}$ and $|\mathcal{D}|$ is the $d + 1$-dimensional Lebesgue measure of $\mathcal{D}$ and ω is a nonnegative function. For $-\infty \leq S < T \leq \infty$, we denote

$$\begin{aligned}
W^{1,2}_{p,\omega}((S, T) \times \mathbb{R}^d) &= \{u : \ u, u_t, Du, D^2 u \in L_{p,\omega}((S, T) \times \mathbb{R}^d)\},\\
\mathcal{H}^1_{p,\omega}((S, T) \times \mathbb{R}^d) &= (1 - \triangle + \partial_t)^{1/2} W^{1,2}_{p,\omega}((S, T) \times \mathbb{R}^d),\\
H^{-1}_{p,\omega}((S, T) \times \mathbb{R}^d) &= (1 - \triangle + \partial_t)^{1/2} L_{p,\omega}((S, T) \times \mathbb{R}^d).
\end{aligned}$$

We also use the abbreviations $L_{p,\omega} = L_{p,\omega}(\mathbb{R}^{d+1})$, $\mathcal{H}^1_{p,\omega} = \mathcal{H}^1_{p,\omega}(\mathbb{R}^{d+1})$ and so on. For any $T \in (-\infty, \infty]$, we denote

$$\mathbb{R}_T = (-\infty, T), \quad \mathbb{R}^{d+1}_T = \mathbb{R}_T \times \mathbb{R}^d.$$

For any integer $k \geq 1$ and $x \in \mathbb{R}^k$, we denote by $B^k_r(x)$ the k-dimensional cube

$$\left\{ y \in \mathbb{R}^k : \ \max_i |y^i - x^i| < r \right\}.$$

Set

$$Q^k_r(t, x) = (t - r^2, t) \times B^k_r(x), \quad B^k_r = B^k_r(0), \quad Q^k_r = Q^k_r(0, 0).$$

In case $k = d$ or $d = 1$, we use the abbreviations

$$\begin{gathered}
B_r(x) = B^d_r(x), \quad Q_r(t, x) = Q^d_r(t, x),\\
B'_r(x') = B^{d-1}_r(x'), \quad Q'_r(t, x') = Q^{d-1}_r(t, x'),\\
\mathbb{Q} = \{Q_r(t, x) : (t, x) \in \mathbb{R}^{d+1}, \ r \in (0, \infty)\}.
\end{gathered}$$

For a function g defined on $\mathbb{R}^{d+1}$, we denote its parabolic maximal and sharp function, respectively, by

$$Mg(t,x) = \sup_{Q\in\mathcal{Q}:(t,x)\in Q} \frac{1}{|Q|}\int_Q |g(s,y)|\,dy\,ds,$$

$$g^{\sharp}(t,x) = \sup_{Q\in\mathcal{Q}:(t,x)\in Q} \frac{1}{|Q|}\int_Q |g(s,y)-(g)_Q|\,dy\,ds.$$

We now introduce weight classes A_p from [31]. We define the weight class $A_p(\mathbb{R}^{d+1})$ $(1<p<\infty)$ as consisting of all nonnegative locally integrable functions ω on $\mathbb{R}^{d+1}$ for which

$$A_p(\omega) := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^p}\int_Q \omega(t,x)dxdt\Big(\int_Q \omega^{-p'/p}(t,x)\,dx\,dt\Big)^{p/p'} < \infty, \tag{2.1}$$

where $1/p+1/p'=1$. The function ω is said to belong to the weight class of $A_1(\mathbb{R}^{d+1})$ on $\mathbb{R}^{d+1}$ for which

$$A_1(\omega) := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_Q \omega(t,x)\,dx\,dt\Big(\sup_{(t,y)\in Q}[\omega(t,y)]^{-1}\Big) < \infty. \tag{2.2}$$

In what follows, we will write $\lambda Q_r(t,x) = Q_{\lambda r}(t,x)$ for any $\lambda>0$. Given a Lebesgue measurable set E and a weight ω, let $\omega(E) = \int_E \omega\,dx$.

Now, we recall some properties for the classical $A_p(\mathbb{R}^{d+1})$ Muckenhoupt weights $\omega \in A_\infty(\mathbb{R}^{d+1}) := \bigcup_{p\ge 1} A_p(\mathbb{R}^{d+1})$.

Lemma 2.1. *If $\omega\in A_p(\mathbb{R}^{d+1})$, then there exists a positive constant c_ω such that*

(i) *if $\omega\in A_p$ for $1\le p<\infty$, then $\omega(2Q)\le c_\omega\omega(Q)$;*
(ii) *if $\omega\in A_p$ for $1<p<\infty$, then there exists $\epsilon>0$ such that $\omega\in A_{p-\epsilon}(\mathbb{R}^{d+1})$ for $p-\epsilon>1$;*
(iii) *if $1\le p_1<p_2<\infty$, then $A_{p_1}\subset A_{p_2}$;*
(iv) *$\omega\in A_p$ if and only if $\omega^{-1/(p-1)}\in A_{p'}$;*
(v) *the Hardy–Littlewood maximal operator M is bounded on $L_{p,\omega}$ if $\omega\in A_p$ with $p\in(1,\infty)$.*

Lemma 2.1 was proved in [22, 31].

3. VMO$_x$ coefficients

We first give the definition of VMO$_x$ function introduced by Krylov in [25, 26]. Denote

$$\mathrm{osc}_x(a,Q_r(x,t)) = r^{-2}|B_r(x)|^{-2}\int_{t-r^2}^{t}\int_{y,z\in B_r(x)} |a(s,y)-a(s,z)|\,dy\,dz\,ds,$$

$$a_R^{\sharp(x)} = \sup_{(x,t)\in\mathbb{R}^{d+1}}\sup_{r<R} \mathrm{osc}_x(a,Q_r(x,t)), \quad a^{\sharp(x)} = a_\infty^{\sharp(x)}.$$

This definition is either naturally modified if a is independent of t as in the elliptic operators or is kept as is.

ASSUMPTION 3.1. We assume that $a \in VMO_x$, that is

$$\lim_{R\to 0} a_R^{\sharp(x)} = 0.$$

For convenience of stating our results we take any continuous function $\eta(R)$ on $[0,\infty)$, such that $\eta(0) = 0$ and $a_R^{\sharp(x)} \le \eta(R)$ for all $R \in (0,\infty)$. Obviously, $a \in \mathrm{VMO}_x$ if a depends only on t. In this section, we always assume that Assumption 3.1 holds.

Krylov [25, 26] obtained an L_p theory of divergence and nondivergence form parabolic equations with the main coefficients belonging to the class VMO_x. In this section, we will study the weighted L_p spaces theory for parabolic type equations with VMO coefficients and certain positive potentials V satisfying the following conditions

$$|\nabla_x^2(V(t,x)^{1/2})| + |\nabla_x V(t,x)| + |\partial_t V(t,x)| \le \frac{C_0}{\lambda^{\delta_0}} V(t,x) \tag{3.1}$$

or

$$|\nabla_x V(t,x)| + |\partial_t V(t,x)| \le \frac{C_0}{\lambda^{\delta_0}} V(x,t) \tag{3.2}$$

holds for all $(x,t) \in \mathrm{R}^{d+1}$ and the positive constants δ_0, C_0 are independent of V, λ, and $\lambda := \inf_{(t,x)\in\mathrm{R}^{d+1}} V(t,x) \ge 1$.

We remark that a typical example is $V(t,x) = (\lambda + |x|^2 + t^2)^\alpha$ with $\alpha > 0$ and $\lambda \ge 1$ or $V(t,x) = (\lambda + |x|^2 + t^2)^\alpha + \lambda$ with $\alpha \le 0$ and $\lambda \ge 1$. Another interesting example is $V(t,x) = \lambda e^{c(\sqrt{1+|x|^2+t^2}/\lambda^2)}$ with $\lambda \ge 1$ and $c \in \mathrm{R}$. Obviously, in the two examples above, both V satisfy (3.1) and (3.2).

We first state the result for nondivergence form parabolic equations.

THEOREM 3.2. *Let V satisfy (3.1), $\omega \in A_p(\mathrm{R}^{d+1})$ with $1 < p < \infty$. Then for any $T \in (-\infty, +\infty]$ the following holds.*

(i) *For any $u \in W^{1,2}_{p,\omega}(\mathrm{R}^{d+1}_T)$,*

$$\begin{aligned}\|Vu\|_{L_{p,\omega}(\mathrm{R}^{d+1}_T)} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathrm{R}^{d+1}_T)} + \|u_{xx}\|_{L_{p,\omega}(\mathrm{R}^{d+1}_T)}\\ + \|u_t\|_{L_{p,\omega}(\mathrm{R}^{d+1}_T)} \le N\|(L-V)u\|_{L_{p,\omega}(\mathrm{R}^{d+1}_T)}.\end{aligned} \tag{3.3}$$

provided that $V \ge \lambda_0$, where λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \eta$ and ω.

(ii) *For any $V(t,x) \equiv \lambda > \lambda_0 = \lambda_0(p,K,d,\delta,\eta,\omega)$ and $f \in L_{p,\omega}(\mathrm{R}^{d+1}_T)$, there exists a unique solution $u \in W^{1,2}_{p,\omega}(\mathrm{R}^{d+1}_T)$ of equation $Lu - \lambda u = f$ in R^{d+1}_T.*

(iii) *In the case that $a^{ij} = a^{ij}(t)$, $b^i \equiv c \equiv 0$ and $V(t,x) \equiv \lambda$, we can take $\lambda_0 = 0$ in* (i) *and* (ii).

We remark that Bramanti *et al.* [3] obtained the global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition.

To prove Theorem 3.2, we need the following Lemma.

Lemma 3.3. *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^{d+1})$. There exists a constant N depending only on p, q, d, δ, K and $A_p(\omega)$, such that for any $u \in C_0^\infty(\mathbb{R}^{d+1})$*

$$\|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N(\|Lu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}). \tag{3.4}$$

Proof. Note that we included $\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})}$ and $\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}$ on the right-hand side. Therefore, while (3.4) holds we may certainly assume that $b^i \equiv c \equiv 0$. Since $u_t = Lu - a^{ij}u_{ij}$, we only need to estimate u_{xx}.

Since $\omega \in A_p(\mathbb{R}^{d+1})$, then there exist $q, v > 1$ such that $\omega \in A_{\frac{p}{qv}}(\mathbb{R}^{d+1})$ by Lemma 2.1(ii). If $u \in C_0^\infty(Q_R)$, then by Lemma 5.3 in [16], using the Fefferman–Stein theorem on sharp functions, and the Hardy–Littlewood maximal function theorem

$$\begin{aligned}\|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N\|u^\sharp_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} &\le N_1 k^{(d+2)/q}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\\ &\quad + N_2(k^{-1} + k^{(d+2)/q}\eta^{1/(\mu q)}(R))\|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})},\end{aligned}$$

where $k \ge 4$ and $1/\mu + 1/v = 1$, where N_i are determined by $p, q, K, v, d, \delta, A_p(\omega)$ and the function η. We choose a large $k = k(N_2, d)$ and small $R = R(N_2, d, p, K, \eta)$ so that

$$N_2(k^{-1} + k^{d+2/q}\eta^{1/(\mu q)}(R)) \le 1/2.$$

Hence, we have

$$\|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})}$$

provided that k is large enough and R is small enough.

After that (3.4) is derived by a standard procedure using partitions of unity. The proof is finished. □

Proof of Theorem 3.2. First we assume $T = \infty$. We now prove (3.3). We follow the same pattern as in the proof of Theorem 4.1 of [25]. To prove (3.3) observe that

$$\|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le \|Lu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + C\|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + C\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u\|_{L_{p,\omega}(\mathbb{R}^{d+1})},$$

so

$$\|Lu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le \|Lu - Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})}.$$

Hence, from Lemma 3.3, we only need to prove that for large λ_0

$$\|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le C\|(L - V)u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}. \tag{3.5}$$

We will use a method introduced by Agmon. Consider the space $\mathbb{R}^{d+2} = \{(t, z) = (t, x, y) : t, y \in \mathbb{R}, x \in \mathbb{R}^n\}$ and the function

$$\widetilde{u}(t, z) = u(t, x)\xi(y)\cos(v(t, x)y), \tag{3.6}$$

where $v(t, x) = \sqrt{V(t, x)}$ and ξ is a $C_0^\infty(\mathbb{R})$-function, $\xi \not\equiv 0$. Also introduce the operator

$$\widetilde{L}u(t, z) = L(x, t)u(t, z) + u_{yy}(t, z).$$

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Finally, set

$$\widetilde{B}_r(z_0) = \{|z - z_0| < r\}, \quad \widetilde{Q}_r(z_0, t_0) = (t_0 - r^2, t_0) \times \widetilde{B}_r(z_0).$$

For any $r \in (0, \infty)$, $(z_0, t_0) \in \mathbb{R}^{d+2}$, set $\bar{Q}_r = \bar{Q}_r(z_0, t_0)$. For appropriate $\bar{a}(t)$ we have

$$\begin{aligned}\int_{\bar{Q}_r} |a(t,x) - \bar{a}(t)|\, dz\, dt &\le \int_{(t_0 - r^2, t_0)} \int_{|x - x_0| < r, |y - y_0| < r} |a(x,t) - \bar{a}(t)|\, dz\, dt \\ &= 2r \int_{Q_r(x_0, t_0)} |a(t,x) - \bar{a}(t)|\, dz\, dt \le C r^{d+3} a_R^{\sharp(x)}.\end{aligned} \tag{3.7}$$

Since $a \in VMO_x(\mathbb{R}^{d+2})$ and $\omega \in A_p(\mathbb{R}^{d+2})$, it follows that (3.4) holds with $\tilde{u}$, $\tilde{L}$, and $\mathbb{R}^{d+2}$, respectively. Now, since $\nu(x,t) \ge 1$, we then have

$$\int_{\mathbb{R}} |\xi(y) \sin(\nu(t,x)y)|^p\, dy \ge C_1 > 0,$$

where the constant C_1 is independent of x, t and ν.

Hence, by (3.1), we have

$$\begin{aligned}\|(\nu u_x)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} &\le C_1^{-1} \int_{\mathbb{R}^{d+1}} |u_x(t,x)\nu(t,x)\xi(y)\sin(\nu(t,x)y)|^p \omega(t,x)\, dz \\ &\le C \int_{\mathbb{R}^{d+1}} |u_x(t,x)[(\xi(y)\cos(\nu(t,x)y))' \\ &\quad - \xi'(y)\cos(\nu(t,x)y)]|^p \omega(t,x)\, dz \\ &\le C \int_{\mathbb{R}^{d+1}} |\widetilde{u}_{zz}(t,z)|^p \omega(t,x)\, dz + C \int_{\mathbb{R}^{d+1}} |u_x(t,x)\xi'(y)|^p \omega(t,x)\, dz \\ &\quad + C \int_{\mathbb{R}^{d+1}} |u(t,x)\nu_x(t,x)\nu(x,t)y^2\xi(y)|^p \omega(t,x)\, dz \\ &\quad + C \int_{\mathbb{R}^{d+1}} |u(t,x)\nu_x(t,x)y\xi'(y)|^p \omega(t,x)\, dz \\ &\le C \int_{\mathbb{R}^{d+1}} |\widetilde{u}_{zz}(t,z)|^p \omega(t,x)\, dz + C \int_{\mathbb{R}^d} |u_x(t,x)|^p \omega(t,x)\, dx \\ &\quad + C \int_{\mathbb{R}^d} |u(t,x)\nu^{3/2}(t,x)|^p \omega(t,x)\, dx,\end{aligned}$$

where C is a positive constant independent of x, t and ν.

From this, taking $\lambda_0 > 2C$, then

$$\|\nu u_x(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} \le C\|\widetilde{u}_{zz}(t,\cdot)\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu^{3/2} u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}, \tag{3.8}$$

here and in what follows, we write

$$\begin{aligned}\|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} &:= \int_{\mathbb{R}^d} |u(t,x)|^p \omega(t,x)\, dx, \\ \|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})} &:= \int_{\mathbb{R}^{d+1}} |u(t,x,y)|^p \omega(t,x)\, dx\, dy.\end{aligned}$$

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Similarly,

$$\begin{aligned}\|(\nu^2 u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} &\le C\int_{\mathbb{R}^{d+1}} |\widetilde{u}_{yy}(t,z) - u(t,x)[2\xi'(y)\sin(\nu(t,x)y) \\ &\qquad + \xi''(y)\cos(\nu(t,x)y)]|^p\omega(t,x)\,dz + C\|(\nu u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} \\ &\le C\|\widetilde{u}_{zz}(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})} + C\|(\nu u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} \\ &\le C\|\widetilde{u}_{zz}(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})},\end{aligned} \tag{3.9}$$

taking $\lambda_0 > 2C$, where C is a positive constant independent of x, t and ν.

Combining (3.8) and (3.9), we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \|(\nu^2 u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}\,dt + \int_{\mathbb{R}} \|(\nu u_x)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}\,dt \\ &\qquad \le C\int_{\mathbb{R}} \|\widetilde{u}_{zz}(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})}\,dt.\end{aligned} \tag{3.10}$$

Thus, the left-hand side of (3.5) is estimated through the left-hand side (3.4) written for $\widetilde{u}$, $\widetilde{L}$, and $\mathbb{R}^{d+2}$ in place of u, L, and $\mathbb{R}^{d+1}$, respectively. Hence, by (3.1) and Lemma 3.3, we obtain

$$\begin{aligned}\|\widetilde{u}_{zz}\|_{L_{p,\omega}(\mathbb{R}^{d+2})} &\le N\Big(\|\widetilde{L}\widetilde{u}\|_{L_{p,\omega}(\mathbb{R}^{d+2})} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu_x u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\Big) \\ &\le N\Big(\|(L-V)u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \\ &\qquad + \|\nu_x u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu_x u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \\ &\qquad + \|\nu_{xx} u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\nu_t u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\Big) \\ &\le N\Big(\|(L-V)u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \frac{1}{\lambda_0^{\delta_0}}\|\nu u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \\ &\qquad + \|\nu u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \frac{1}{\lambda_0^{\delta_0}}\|\nu^2 u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\Big).\end{aligned}$$

By this and (3.10), for large λ_0, we prove (3.5).

Thus, (i) is proved for $T = \infty$. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W^{1,2}_{p,\omega}$ solves $(\mathcal{L} - V)v = \chi_{t<T}(\mathcal{L} - V)u$. Assertion (ii) is established from assertion (i) by the method of continuity. Finally, we prove assertion (iii). If $\lambda = 0$, assertion (iii) is easily proved by Theorem 5.1 in [26]. In the case $\lambda > 0$, adapting the same proof of (3.3), and using Theorem 5.1 in [26], we can prove that

$$\begin{aligned}&\lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \\ &\quad \le N\|(L-\lambda)u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}\end{aligned}$$

provided that $\lambda \ge \lambda_0$, where N, λ_0 depending only on p, K, d, δ, η and ω. Let $\bar{w}(t,x) = \omega(t\lambda_0/\lambda, x\sqrt{\lambda_0/\lambda})$, it is easy to see that $\bar{\omega}$ satisfies the same properties in Lemma 2.1 as ω. So,

$$\begin{aligned}&\lambda\|u\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \|u_t\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} \\ &\quad \le N\|(L-\lambda)u\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)}\end{aligned} \tag{3.11}$$

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provided that $\lambda \geq \lambda_0$, where N, λ_0 depending only on p, K, d, δ, η and ω. Using a scaling and by (3.11)

$$x \to x\sqrt{\lambda_0/\lambda}, \quad t \to t\lambda_0/\lambda$$

we obtain the estimate for any $\lambda > 0$. The proof is finished. □

Now, we consider the divergence form parabolic equations.

THEOREM 3.4. *Let V satisfy (3.2), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then for any $T \in (-\infty, +\infty]$ the following assertions hold.*

(i) *For $u \in W^1_{p,\omega}(\mathbb{R}^{d+1}_T)$, $f = (f^1, \ldots, f^d) \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, $g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, and*

$$\mathcal{L}u - Vu = \operatorname{div} f + g.$$

Then there exist constants λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \eta$ and ω, such that

$$\|\sqrt{V}u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \leq N\biggl(\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \biggl\|\frac{g}{\sqrt{V}}\biggr\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}\biggr) \quad (3.12)$$

provided that $V \geq \lambda_0$.

(ii) *For any $V(t, x) \equiv \lambda > \lambda_0 = \lambda_0(p, K, d, \delta, \eta, \omega)$ and $f, g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, then there exists an unique solution $u \in \mathcal{H}^1_{p,\omega}(\mathbb{R}^{d+1}_T)$ of equation $\mathcal{L}u - \lambda u = \operatorname{div} f + g$ in $\mathbb{R}^{d+1}_T$ and satisfying*

$$\|u_t\|_{H^{-1}_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}$$
$$\leq N(\sqrt{\lambda}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}).$$

(iii) *In the case that $a^{ij} = a^{ij}(t)$, $a^i \equiv b^i \equiv c \equiv 0$ and $V(t, x) \equiv \lambda$, we can take $\lambda_0 = 0$ in assertions* (i) *and* (ii).

To prove Theorem 3.4, we need the following result.

LEMMA 3.5. *Let $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0, c = 0$, $\mathcal{L}u = \operatorname{div} f$, where $f = (f^1, \ldots, f^d)$. There exists a constants $\epsilon > 0$ and $N < \infty$ depending only on p, d, δ and $A_p(\omega)$, such that if $a^{\sharp(x)}_R < \epsilon$ for some $R > 0$, then for any $u \in C^\infty_0(Q_R)$ we have*

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \leq N\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})}.$$

PROOF. Similar to the proof of Lemma 3.3, by Lemma 7.3 in [16], using the Fefferman–Stein theorem on sharp functions, and the Hardy–Littlewood maximal function theorem, we can obtain the desired result. □

PROOF OF THEOREM 3.4. First we assume $T = \infty$. We now prove (3.10). We follow the same pattern as in the proof of Theorem 4.4 of [26]. Similar to the proof of Theorem 3.2, we use a method introduced by Agmon. We first assume that

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$u \in C_0^\infty(Q_{R/2})$, where R is the same as in Lemma 3.5. Consider the space $\mathbb{R}^{d+2} = \{(t,z) = (t,x,y) : t, y \in \mathbb{R}, x \in \mathbb{R}^d\}$ and the function

$$\widetilde{u}(t,z) = u(t,x)\xi(y)\cos(\nu(t,x)y),$$

where $\nu(t,x) = \sqrt{V(t,x)}$ and ξ is an odd $C_0^\infty(-R/2, R/2)$ function, $\xi \not\equiv 0$. Also introduce the operator

$$\widetilde{\mathcal{L}}u(t,z) = u_t(t,z) + (a^{ij}(t,x)u_{x_i}(t,z))_{x_j} + u_{yy}(t,z).$$

As in the proof of Theorem 3.2 one checks that $a^{\sharp(z)}$ is small enough.

Set

$$\begin{aligned}\widetilde{f^i}(t,z) = (f^i(t,x) - a^j(t,x)u(t,x))\xi(y)\cos(\nu(t,x)y)\\ + a^{ji}(t,x)u(t,x)\nu_{x_j}(t,x)y\xi(y)\sin(\nu(t,x)y) \quad \text{for } i = 1,\dots,d,\end{aligned}$$

and

$$\begin{aligned}\widetilde{f^{d+1}}(t,z) = (g(t,x) - c(t,x)u(t,x))\xi_1^t(y) - 2u(t,x)\xi_2^t(y) + u(t,x)\xi_3^t(y)\\ + (-\nu_t(t,x)u(t,x) + [f^i(t,x) - a^j(t,x)u(t,x)\\ - a^{ji}(t,x)u_{x_j}(t,x)]\nu_{x_i}(t,x))\,\xi_4^t(y),\end{aligned}$$

where

$$\xi_1^t(y) = \int_{-\infty}^{y} \xi(s)\cos(\nu(t,x)y)\,ds, \quad \xi_3^t = \int_{-\infty}^{y} \xi''(s)\cos(\nu(t,x)y)\,ds,$$

$$\xi_2^t(y) = \nu(x,t)\int_{-\infty}^{y} \xi'(s)\sin(\nu(t,x)y)\,ds = -\xi'(y)\cos(\nu(t,x)y) + \xi_3^t(y),$$

and

$$\xi_4^t(y) = \int_{-\infty}^{y} s\xi(s)\sin(\nu(t,x)s)\,ds.$$

Observe that $\xi_i \in C_0^\infty(\mathbb{R})$ since ξ is odd and has compact support. Furthermore, it is easy to check that

$$\widetilde{\mathcal{L}}\widetilde{u}(t,z) = ((\widetilde{f^i}(t,z))_{x_i} + \cdots + (\widetilde{f^d}(t,z))_{x_d} + (\widetilde{f^{d+1}}(t,z))_y).$$

We denote by $\widetilde{L}_{p,\omega}$ the $L_{p,\omega}$ space of functions of $z(x,y)$, note that ω is the $A_p(\mathbb{R}^{d+1})$ of weighted function with (t,x) variable and by Lemma 3.3,

$$\int_{-\infty}^{\infty} \|\widetilde{u}_z(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}}\,dt \le N\Big(\sum_{i=1}^{n+1}\int_{-\infty}^{\infty} \|\widetilde{f^i}(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}}\,dt + \int_{-\infty}^{\infty} \|\widetilde{u}(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}}\,dt\Big). \tag{3.13}$$

Since $\nu(t,x) \ge 1$, then there exist constants C_2 and C_3 independent of x, t, ν such that

$$\int_{\mathbb{R}^d} |\xi(y)\sin(\nu(x,t)y)|^p\,dy \ge C_2 > 0, \quad \int_{\mathbb{R}^d} |\xi(y)\cos(\nu(t,x)y)|^p\,dy \ge C_3 > 0.$$

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From these, we get for each t and $\nu(t,x) \geq 1$ that

$$\|u_x(t,\cdot)\|_{L_{p,\omega}(\mathbb{R}^d)} \leq C_2^{-1} \int_{\mathbb{R}^{d+1}} |u_x(t,x)\xi(y)\cos(\nu(t,x)y)|^p \omega(t,x)\,dz \\ \leq N\|\widetilde{u}_z(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}},$$

and

$$\|(\nu u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} \leq \frac{1}{C_3} \int_{\mathbb{R}^{d+1}} |\widetilde{u}_y(t,z) - u(t,x)\xi'(y)\cos(\nu(x,t)y)|^p \omega(t,x)\,dz \\ \leq N(\|\widetilde{u}_z(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} + \|u\|^p_{L_{p,\omega}(\mathbb{R}^d)}).$$

It follows that if λ_0 is large enough, then

$$\|(\nu u)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} \leq N\|\widetilde{u}_z(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}}.$$

Hence, by (3.13) for large λ_0

$$\|\nu u\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_x\|^p_{L_{p,\omega}(\mathbb{R}^{d+1})} \\ \leq N\Big(\sum_{i=1}^{d+1} \int_{-\infty}^{\infty} \|\widetilde{f}^i(t,\cdot)\|^q_{\widetilde{L}_{p,\omega}}\,dt + \int_{-\infty}^{\infty} \|\widetilde{u}(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}}\,dt\Big). \tag{3.14}$$

Now we estimate the right-hand side of (3.14). By (3.2), we have for $i = 1, \ldots, d$,

$$\|\widetilde{f}^i(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} \leq N(\|f^i(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \lambda_0^{-\delta_0}\|\nu u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}),$$

and

$$\|\widetilde{u}(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} \leq N\|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}.$$

Furthermore,

$$\xi_1^t = \nu(t,x)^{-1}\Big[\xi(y)\sin(\nu(t,x)y) - \int_{-\infty}^{y} \xi'(s)\sin(\nu(t,x)s)\,ds\Big],$$

which shows that ξ_1 equals ν^{-1} times a uniformly bounded function with support not wider than that of ξ in the coordinate y. Hence,

$$\|c(t,\cdot)u(t,\cdot)\xi_1\|^p_{\widetilde{L}_{p,\omega}} \leq N\|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}, \quad \|g(t,\cdot)\xi_1^t\|^p_{\widetilde{L}_{p,\omega}} \leq N\|(g/\nu)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}.$$

Similarly,

$$\xi_4^t = -\nu(t,x)^{-1}\Big[y\xi(y)\cos(\nu(x,t)y) - \int_{-\infty}^{y} (s\xi(s))'\cos(\nu(t,x)s)\,ds\Big],$$

which shows that ξ_4^t equals ν^{-1} times a uniformly bounded function with support not wider than that of ξ in the coordinate y. Hence, by (3.2), we obtain

$$\|(\nu_t u - [f^i + a^j u - a^{ji}u_{x_j}]\nu_{x_i})\xi_4(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} \\ \leq N(\|f^i(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \lambda_0^{-\delta_0}\|u_x(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}).$$

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Also ξ_2^t and ξ_3^t are uniformly bounded with support not wider than that of ξ. Therefore,

$$\|(2u\xi_2 - u\xi_3)(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} \le N\|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}.$$

From these, we have

$$\|\widetilde{f}^{d+1}(t,\cdot)\|^p_{\widetilde{L}_{p,\omega}} \le N(\|f^i(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \|u(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \lambda_0^{-\delta_0}\|u_x(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)} + \|(g/\nu)(t,\cdot)\|^p_{L_{p,\omega}(\mathbb{R}^d)}). \tag{3.15}$$

Combining (3.14) and (3.15), we prove (3.12) if $u \in C_0^\infty(B_{R/2})$. For general u, we adapt the same proof of Theorem 5.7 in [25], we can obtain the desired result.

Thus, assertion (i) is proved for $T = \infty$. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W^1_{p,\omega}$ solves $(\mathcal{L} - V)v = \chi_{t<T}(\mathcal{L} - V)u$.

Assertion (ii) is established from assertion (i) by the method of continuity once we prove the following inequality

$$\|u_t\|_{H^{-1}_\omega(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}). \tag{3.16}$$

In fact, it suffices to observe

$$(1 - \triangle + \partial_t)^{-1/2}u_t = -(1 - \triangle + \partial_t)^{-1/2}D_j(a^{ij}u_{x^i} - f^j) + (1 - \triangle + \partial_t)^{-1/2}(\lambda u + g),$$

hence, by the weighted $L_{p,\omega}(\mathbb{R}^{d+1}_T)$ of $(1 - \triangle + \partial_t)^{-1/2}$ and $(1 - \triangle + \partial_t)^{-1/2}D_j$, we have

$$\|(1 - \triangle + \partial_t)^{-1/2}u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1})}).$$

From this and (3.12), we prove (3.16).

For assertion (iii). If $\lambda = 0$, assertion (iii) is easily proved by Theorem 7.1 in [26]. In the case $\lambda > 0$, adapting the same proof of (3.3), and using Theorem 7.1 in [26], we can prove that there exists a constant N, λ_0 depending only on p, K, d, δ, η and ω, such that

$$\lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)})$$

provided that $\lambda \ge \lambda_0$. Let $\bar{w}(t,x) = \omega(t\lambda_0/\lambda, x\sqrt{\lambda_0/\lambda})$, clearly, $\bar{\omega}$ satisfies the same properties in Lemma 2.1 as ω. So, there exists a constant N, λ_0 depending only on p, K, d, δ, η and ω, such that

$$\lambda\|u\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|f\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)}) \tag{3.17}$$

provided that $\lambda \ge \lambda_0$. Using a scaling and by (3.17)

$$x \to x\sqrt{\lambda_0/\lambda}, \quad t \to t\lambda_0/\lambda$$

we obtain the estimate for any $\lambda > 0$. The proof of Theorem 3.4 is complete. □

As a consequence of Theorem 3.4, we have the following result.

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COROLLARY 3.6. *Let V satisfy (3.2), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exist constants λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \eta$ and ω, such that*

$$\|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \leq C\|h\|_{L_{p,\omega}(\mathbb{R}^{d+1})}, \tag{3.18}$$

provided that $V \geq \lambda_0$, where $u = (\mathcal{L} - V)^{-1}h$.

PROOF. Let $v(x,t) = \sqrt{V(x,t)}$. Note that $\mathcal{L}u(x,t) - v^2(x,t)u(x,t) = h(x,t)$, then

$$\mathcal{L}U(x,t) - v^2(x,t)U(x,t) = \operatorname{div}(a^{ij}uv_{x_i})(x,t) + \widetilde{h}(x,t), \tag{3.19}$$

where $U(x,t) = u(x,t)v(x,t)$ and

$$\begin{aligned}\widetilde{h}(x,t) &= h(x,t)v(x,t) + u(x,t)v_t(x,t) + a_{ij}(x,t)u_{x_i}(x,t)v_{x_j}(x,t)\\ &\quad + \hat{b}^i(x,t)u(x,t)v_{x_j}(x,t) + b^i(x,t)u(x,t)v_{x_i}(x,t).\end{aligned}$$

Applying (3.12) to (3.19) with $u = U, f = (a^{1j}uv_{x_1}, \ldots, a^{nj}uv_{x_n})$ and $g = \widetilde{h}$, by (3.2), we obtain

$$\begin{aligned}\|vU\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|U_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} &\leq C\|a^{ij}uv_{x_i}\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\widetilde{h}/v\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\\ &\leq C(\|vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|h\|_{L_{p,\omega}(\mathbb{R}^{d+1})}).\end{aligned} \tag{3.20}$$

Observe that by (3.2) again

$$\begin{aligned}\|v^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|vu_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} &\leq \|vU\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|U_x\|_{L^p_\omega(\mathbb{R}^{d+1})} + \|v_xu\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\\ &\leq \|vU\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|U_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + C\|vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})}.\end{aligned}$$

From this and (3.20), we have

$$\|v^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|vu_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \leq C\|h\|_{L_{p,\omega}(\mathbb{R}^{d+1})},$$

if λ_0 is large enough. Thus, (3.18) is proved. □

4. Partially BMO coefficients

We first recall the definition of partially BMO function introduced by [13, 23].

We assume that $a^{ij}, ij > 1$ are measurable in x^1 and t, and have locally small mean oscillations in the other variables. In addition, we assume that a^{11} are measurable in t and have locally small mean oscillations in the others. To state the assumptions on a^{ij} precisely, for $R > 0$, we denote

$$a_R^{11} = \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r\leq R} \frac{1}{|Q_r(t_0,x_0)|} \int_{Q_r(t_0,x_0)} |a^{11}(t,x) - \bar{a}^{11}(t)|\,dx\,dt,$$

$$a_R^{\sharp} = \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r\leq R} \sup_{(i,j)\neq(1,1)} \frac{1}{|Q_r(t_0,x_0)|} \int_{Q_r(t_0,x_0)} |a^{ij}(t,x) - \bar{a}^{ij}(t,x^1)|\,dx\,dt,$$

where for each $Q_r(t_0,x_0)$,

$$\bar{a}^{11}(t) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} a^{11}(t,x)\,dx,$$

$$\bar{a}^{11}(t,x^1) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} a^{11}(t,x^1,y')\,dy', \quad (i,j) \neq (1,1).$$

We shall impose part of the following assumption on the leading coefficients.

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Assumption 4.1 (γ). There exists a positive constant R_0 such that $a^{11}_{R_0} + a^{\sharp}_{R_0} \le \gamma$.

Now, we first consider the divergence form parabolic equations.

Theorem 4.2. *Let V satisfy (3.2), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exists a constant $\gamma > 0$ such that under Assumption 4.1(γ), for any $T \in (-\infty, +\infty]$ the following holds.*

(i) *For $u \in W^1_{p,\omega}(\mathbb{R}^{d+1}_T)$, $f = (f^1, \dots, f^d) \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, $g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, and*

$$\mathcal{L}u - Vu = \operatorname{div} f + g.$$

Then there exist constants λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \gamma$ and ω, such that

$$\|\sqrt{V}u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g/\sqrt{V}\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)})$$

provided that $V \ge \lambda_0$.

(ii) *For any $V(t,x) \equiv \lambda > \lambda_0 = \lambda_0(p, K, d, \delta, \gamma, \omega)$ and $f, g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, there exists a unique solution $u \in \mathcal{H}^1_{p,\omega}(\mathbb{R}^{d+1}_T)$ of the equation $\mathcal{L}u - \lambda u = \operatorname{div} f + g$ in $\mathbb{R}^{d+1}_T$ and satisfying*

$$\|u_t\|_{\mathbb{H}^{-1}_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}).$$

(iii) *In the case that $a^{11} = a^{11}(t)$, $a^{ij} = a^{ij}(t, x^1)$, $ij > 1$, $b^j \equiv c \equiv 0$ and $V(t,x) \equiv \lambda$, we can take $\lambda_0 = 0$ in assertions* (i) *and* (ii).

To prove Theorem 4.2, we need the following results.

Lemma 4.3. *Let $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0$, $c = 0$, $\mathcal{L}u = \operatorname{div} f$, where $f = (f^1, \dots, f^d)$. Also let $\sigma > 1$ and $\tau > 1$ such that $1/\tau + 1/\sigma = 1$ and $\omega \in A_{p/(q\tau)}(\mathbb{R}^{d+1})$ for some $q > 1$ and $q\tau < p$. Then there exists a constant $\gamma_1 > 0$ such that under Assumption 4.1(γ_1), such that*

$$(|u_{x'} - (u_{x'})_{Q_r(t,x)}|^q)_{Q_r(t,x)} \le Nk^{d+2}(|f|^q)_{Q_{kr}(t,x)} + N(k^{-q} + k^{d+2}(\gamma_1)^{1/\sigma})(|Du|^{q\tau})^{1/\tau}_{Q_{kr}(t,x)} \tag{4.1}$$

for any $k \ge 4$, $r \in (0, \infty)$, and $(t,x) \in \mathbb{R}^{d+1}$, there exists a constant N depending only on $q, p, d, \delta, C_0, \delta_0, \sigma$ and $A_p(\omega)$.

Proof. By Proposition 6.4 in [17], and by using the technique of freezing coefficients (see the proof of Lemma 7.3 of [16]), we can prove (4.1). □

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LEMMA 4.4. *Let* $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0, c = 0$, $\mathcal{L}u = \operatorname{div} f$, *where* $f = (f^1, \dots, f^d)$. *Then exists a constant* $\gamma > 0$ *such that* $a^{11}_{R_0} \le \gamma$ *for some* $R_0 \in (0, 1]$, *there exist* $\mu \in [1, \infty)$ *and* N *depending only on* $p, K, d, \delta, \delta_0$ *and* ω, *for* $u \in C_0^\infty$ *vanishing outside* $Q_{\mu^{-1}R_0}$, *we have*

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N(\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_{x'}\|_{L_{p,\omega}(\mathbb{R}^{d+1})}).$$

PROOF. We can obtain the desired result in the same way as in the proof of Lemma 3.6 in [17]. We omit the details. □

As the consequence of Lemmas 4.3 and 4.4.

PROPOSITION 4.5. *Let* $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0, c = 0$, $\mathcal{L}u = \operatorname{div} f$, *where* $f = (f^1, \dots, f^d)$. *Then exists a constant* $\gamma > 0$ *such that* $a^{11}_{R_0} \le \gamma$ *for some* $R_0 \in (0, 1]$, *there exists* N *depending only on* p, K, d, δ *and* ω *for* $u \in C_0^\infty(Q_{\mu^{-1}R_0})$, *we have*

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1})}.$$

Next, we set

$$\bar{\mathcal{L}}u = -u_t + D_i(a^{ij} D_j u),$$

where the coefficient $a^{11} = a^{11}(t)$ and $a^{ij} = a^{ij}(t, x^1)$ for $ij > 1$.

LEMMA 4.6. *Let* $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0, c = 0$, $k \ge 8$, $\bar{\mathcal{L}}u = \operatorname{div} f$, *where* $f = (f^1, \dots, f^d)$. *Then the following assertion holds for any* $T \in (-\infty, \infty]$,

$$\|u_{x'}\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(k^{(d+2)/p}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + k^{-1/2}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}), \tag{4.2}$$

where N *depends only on* p, d, δ *and* ω.

PROOF. By Proposition 6.4 in [17], and using Fefferman–Stein theorem on sharp functions, and the Hardy–Littlewood maximal function theorem, we can prove (4.2). □

LEMMA 4.7. *Let* $T \in (-\infty, \infty]$, $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0$, $c = 0$, $\bar{\mathcal{L}}u - \lambda u = \operatorname{div} g + f$ *in* $\mathbb{R}^{d+1}_T$, *where* $\lambda \ge 0$ *and* $f, g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$. *Then there exists a constant* N *depending only on* p, K, d, δ *and* ω, *such that*

$$\sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|D_{x'}u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}).$$

In particular, if $\lambda = 0$ *and* $f = 0$, *then*

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\|D_{x'}u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}). \tag{4.3}$$

PROOF. The case when $\lambda = 0$ and $f = 0$ follows by just letting $\lambda \to 0$ after the estimate λ is proved. □

As the proof of Lemma 3.4 in [17], we can obtain the desired result. We omit the details. Applying Lemma 4.4 and (4.3) of Lemma 4.7, we have the following result.

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Proposition 4.8. *Let $T \in (-\infty, \infty]$, $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $\bar{\mathcal{L}}u - \lambda u = \operatorname{div} g + f$ in $\mathbb{R}^{d+1}_T$, where $\lambda \geq 0$ and $f, g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$. Then there exists a constant N depending only on p, K, d, δ and ω, such that*

$$\sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \leq N(\sqrt{\lambda}\|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}).$$

Proof. In fact, in the case $a^i = b^i = 0, c = 0$, by (4.2) and (4.3), we have

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \leq N\|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}. \tag{4.4}$$

Adapting the same arguments in the proof of Lemma 3.3, using (4.4), we obtain

$$\sqrt{\lambda}\|u_x\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} \leq N(\sqrt{\lambda}\|g\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)} + \|f\|_{L_{p,\bar{\omega}}(\mathbb{R}^{d+1}_T)}) \tag{4.5}$$

provided that $\lambda \geq \lambda_0$, where the constants N, λ_0 depend only on p, δ, d and ω, and $\bar{w}(t, x) = \omega(t\lambda_0/\lambda, x\sqrt{\lambda_0/\lambda})$.

Using a scaling and by (4.5)

$$x \to x\sqrt{\lambda_0/\lambda}, \quad t \to t\lambda_0/\lambda$$

we obtain the estimate for any $\lambda > 0$. The proof of Proposition 4.5 is complete. □

Now we are ready to prove Theorem 4.9.

Proof of Theorem 4.2. To prove assertion (i), for $T = \infty$ and $u \in C_0^\infty$, this in turn is obtained from Theorem 4.2 and an idea from Agmon; see also the proof of Lemma 3.3. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W^1_{p,\omega}$ solves $(\mathcal{L} - V)v = \chi_{t<T}(\mathcal{L} - V)u$. Assertion (ii) is established from assertion (i) by the method of continuity. Assertion (iii) is proved by Proposition 4.5.

Next, we consider the nondivergence form parabolic equations.

Theorem 4.9. *Let V satisfy (3.1), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exists a constant $\gamma > 0$ depending only on p, K, d, δ and ω such that under Assumption 4.1(γ), for any $T \in (-\infty, +\infty]$ the following assertions hold.*

(i) *For any $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$,*

$$\|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \leq N\|(L - V)u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}$$

provided that $V \geq \lambda_0$, where λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \gamma$ and ω.

(ii) *For any $V(t, x) \equiv \lambda > \lambda_0 = \lambda_0(p, K, d, \delta, \gamma, \omega)$ and $f \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, there exists a unique solution $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$ of the equation $Lu - \lambda u = f$ in $\mathbb{R}^{d+1}_T$.*

(iii) *In the case that $a^{11} = a^{11}(t)$, $a^{ij} = a^{ij}(t, x^1)$, $b^j \equiv c \equiv 0$ and $V(t, x) \equiv \lambda$, we can take $\lambda_0 = 0$ in assertions* (i) *and* (ii).

To prove Theorem 4.9, we need the following results.

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Lemma 4.10. *Let* $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1}),\ b^j = 0, c = 0$. *Then there exists a constant* $\gamma_1 > 0$ *depending only on* p, K, d, δ *and* ω *such that under Assumption 4.1*(γ_1) *for any* $u \in C_0^\infty(Q_{R_0})$ *and for any* $T \in (-\infty, +\infty]$, *we have*

$$\|D^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \le N(\|Lu\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|D_{x'}^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}), \tag{4.6}$$

where there exists a constant N *depending only on* p, K, d, δ *and* ω. *In particular, in the case* $d = 1$, *we have*

$$\|D^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \le N\|Lu\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}.$$

Proof. We write

$$-u_t + a^{11}D_1^2u + \triangle_{d-1}u = Lu + \sum_{ij>1}(\delta_{ij} - a^{ij})D_{ij}u.$$

By Theorem 3.2, we have

$$\|D^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \le N(\|Lu\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|D_{xx'}u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}).$$

Finally, to conclude the proof of (4.6) it suffices to note that $\omega^\epsilon(t, x) = \omega(\epsilon^2 t, \epsilon x^1, x') \in A_p(\mathbb{R}^{d+1})$ and $A_p(\omega^\epsilon) \le 8^p A_p(\omega)$ for any $\epsilon > 0$,

$$\begin{aligned}\|D_{x^1x'}u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \le{}& \epsilon(\|u_t\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|D_1^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})})\\ &+ N(d, p, A_p(\omega))\epsilon^{-1}\|D_{x'}^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})},\end{aligned} \tag{4.7}$$

where we use the notation

$$\begin{aligned}\|D_{x^1x'}u\|_{L_{p,\omega^\epsilon}(\mathbb{R}_T^{d+1})} \le N\|(\partial_t - \triangle)u\|_{L_{p,\omega^\epsilon}(\mathbb{R}_T^{d+1})} \le{}& N(\|u_t\|_{L_{p,\omega^\epsilon}(\mathbb{R}_T^{d+1})}\\ &+ \|D_1^2u\|_{L_{p,\omega^\epsilon}(\mathbb{R}_T^{d+1})} + \|D_{x'}^2u\|_{L_{p,\omega^\epsilon}(\mathbb{R}_T^{d+1})})\end{aligned}$$

by scaling in x^1 with ϵ and t with ϵ^2. □

Lemma 4.11. *Let* $1 < p < \infty,\ \ \omega \in A_p(\mathbb{R}^{d+1}),\ \ b^i = 0,\ \ c = 0$. *Then there exists a constant* $\gamma > 0$ *such that under Assumption 4.1*(γ) *for any* $u \in C_0^\infty(Q_{R_0})$ *and for any* $T \in (-\infty, +\infty]$, *we have*

$$\|D^2u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \le N\|Lu\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}.$$

Proof. The case $d = 1$ follows from Lemma 4.6. For case the $d \ge 2$, by Lemma 4.6 and using the proof of Lemma 3.7 in [17], we can obtain the desired result. □

Finally, we are ready to prove Theorem 4.9.

Proof of Theorem 4.9. To prove assertion (i), for $T = \infty$ and $u \in C_0^\infty$, this in turn is obtained from Lemmas 4.10 and 4.11 for the case in Assumption 4.1(γ) and Theorem 4.2 in [17], and an idea from Agmon; see also the proof of Theorem 3.2. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W_{p,\omega}^{1,2}$ solves $(L - V)v = \chi_{t<T}(L - V)u$. Assertion (ii) is established from assertion (i) by the method of continuity. By using assertion (iii) of Theorem 3.2 and assertion (iii) of Theorem 4.2, and adapting the proof of Theorem 3.1 in [17], we can prove assertion (iii) (see also the proof of Theorem 5.2 below).

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5. Hierarchically partially BMO coefficients

We consider in this section parabolic equations with more general coefficients introduced in [14]. The assumption is that for $ij > 1$, a^{ij} are measurable in $(t, x^1, \ldots, x^{\pi_{ij}})$ and BMO in the other coordinates, where

$$\pi_{ij} = \max(i, j) - 1.$$

In addition, we suppose a^{11} is measurable in t and BMO in the other coordinates. More precisely assumptions are related below.

We recall the definition of a_R^{11} in Section 3. For $R > 0$, we denote

$$a_R^* = \sup_{(t_0,x_0)\in\mathbb{R}^{d+1}} \sup_{r\le R} \sup_{(i,j)\neq(1,1)} \frac{1}{|Q_r(t_0,x_0)|} \int_{Q_r(t_0,x_0)} |a^{ij} - \bar{a}^{ij}| \, dx \, dt,$$

$$\bar{a}^{ij} = \bar{a}^{ij}(t, x^1, \ldots, x^{\pi_{ij}}) = \frac{1}{|B_r^{d-\pi_{ij}}(x_0^{\pi_{ij}+1}, \ldots, x_0^d)|} \times \int_{B_r^{d-\pi_{ij}}(x_0^{\pi_{ij}+1}, \ldots, x_0^d)} |a^{ij}(t, x^1, \ldots, x^{\pi_{ij}}, y^{\pi_{ij}}, \ldots, y^d)| \, dy^{\pi_{ij}+1} \cdots dy^d.$$

We impose the following assumptions on a^{ij}.

Assumption 5.1 (γ). There exists a positive constant R_0 such that $a_{R_0}^{11} + a_{R_0}^* \le \gamma$.

Clearly the assumptions above are weaker than those in Theorems 4.2 and 4.9 in terms of the regularity of a^{ij} for $i > 2$ or $j > 2$. Now, we first consider the divergence form parabolic equations.

Theorem 5.2. *Let V satisfy (3.2), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exists a constant $\gamma = \gamma(d, K, \delta, p, \omega) > 0$ such that under Assumption 5.1(γ) for any $T \in (-\infty, +\infty]$ the following assertions hold.*

(i) *For $u \in W^1_{p,\omega}(\mathbb{R}^{d+1}_T)$ $f = (f^1, \ldots, f^d) \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, $g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, and*

$$\mathcal{L}u - Vu = \operatorname{div} f + g.$$

Then there exist constants λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \gamma$ and ω, such that

$$\|\sqrt{V}u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g/\sqrt{V}\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)})$$

provided that $V \ge \lambda_0$.

(ii) *For any $V(t, x) \equiv \lambda > \lambda_0 = \lambda_0(p, K, \delta, \gamma, \omega)$ and $f, g \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, there exists a unique solution $u \in \mathcal{H}^1_{p,\omega}(\mathbb{R}^{d+1}_T)$ of the equation $\mathcal{L}u - \lambda u = \operatorname{div} f + g$ in $\mathbb{R}^{d+1}_T$ and satisfying*

$$\|u_t\|_{H^{-1}_{p,\omega}(\mathbb{R}^{d+1}_T)} + \lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \sqrt{\lambda}\|u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le N(\sqrt{\lambda}\|f\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|g\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)})$$

(iii) *In the case that $a^{11} = a^{11}(t)$, $a^{ij} = a^{ij}(t, x^1, \ldots, x^{\pi_{ij}})$, $ij > 1$, $b^j \equiv c \equiv 0$ and $V(t, x) \equiv \lambda$, we can take $\lambda_0 = 0$ in assertions* (i) *and* (ii).

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For simplicity, we only give a proof of Theorem 5.2 when $d \geq 3$, or $d > 3$ and π_{ij} is placed by $\widetilde{\pi}_{ij} = \min(\pi_{ij}, 2)$. The general case can be proved by an induction.

We first give the following result.

Lemma 5.3. *Let $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1})$, $a^i = b^i = 0$, $c = 0$. Then there exist constants $\mu_1 > 1, 0 < \gamma_1 < 1$ and N depending only on p, K, d, δ and ω such that, under Assumption 5.1(γ_2) with $\widetilde{\pi}_{ij}$ in place of π_{ij}, for any $T \in (-\infty, +\infty]$, for $u \in C_0^\infty$ vanishing outside $Q_{\mu_1^{-1}R_0}$ satisfying $\mathcal{L}u = \operatorname{div} f$, where $f \in L_{p,\omega}(\mathbb{R}_T^{d+1})$, we have*

$$\|u_x\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \leq N\Big(\|f\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} + \sum_{j=3}^{d} \|D_j u\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}\Big).$$

Proof. Similar to the proof Corollary 6.4 in [14], we can obtain the desired result. We omit the details. □

Following the lines of Sections 5 and 6 in [17], we have the next estimate of mean oscillations.

Lemma 5.4. *Let $1 < q < \infty$, $a^i = b^i = 0$, $c = 0$, $\sigma, \tau \in (1, \infty)$ satisfying $1/\sigma + 1/\tau = 1$. Assume $u \in C_0^\infty$ and $\mathcal{L}u = \operatorname{div} f$, where $f \in L_{q,loc}$. Then under Assumption 5.1(γ) with π_{ij} in place of π_{ij}, there exist an $\alpha = \alpha(d, \delta) \in (0, 1)$ and a positive constant N depending only d, σ, q and δ such that*

$$\begin{aligned}(|u_{x'} - (u_{x'})_{Q_r(t,x)}|^q)_{Q_r(t,x)} &\leq N k^{d+2} (|f|^q)_{Q_{kr}(t,x)} \\ &\quad + N(k^{-q\alpha} + k^{d+2}(\gamma_1)^{1/\sigma})(|Du|^{q\tau})^{1/\tau}_{Q_{kr}(t,x)}\end{aligned}$$

for any $k \geq 4$, $r \in (0, \infty)$, and $(t, x) \in \mathbb{R}^{d+1}$, provided that u vanishes outside Q_{R_0}.

Using Lemmas 5.3 and 5.4, we immediately obtain the following result.

Proposition 5.5. *Let $1 < p < \infty$, $a^i = b^i = 0$, $c = 0$ and $\omega \in A_p(\mathbb{R}^{d+1})$. Let μ_1 and γ_1 be the constants in Theorem 5.2. Then exists a constant $\gamma_2 \in (0, \gamma_1]$ depending only on d, p, δ and ω such that under Assumption 5.1(γ_2) with $\widetilde{\pi}_{ij}$ in place of π_{ij}, for any $u \in C_0^\infty(Q_{\mu_1^{-1}R_0})$ and $f \in L_{p,\omega}(\mathbb{R}_T^{d+1})$ satisfying $\mathcal{L}u = \operatorname{div} f$, we have*

$$\|Du\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})} \leq N\|f\|_{L_{p,\omega}(\mathbb{R}_T^{d+1})}.$$

Now, we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Recall that for simplicity we replace π_{ij} in the assumption by $\widetilde{\pi}_{ij}$. The general case can be done by an induction.

To prove assertion (i), for $T = \infty$ and $u \in C_0^\infty$, this in turn is obtained from Theorem 5.2 and an idea from Agmon; see also the proof of Theorem 3.4. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W^1_{p,\omega}$ solves $(L - V)v = \chi_{t<T}(L - V)u$. Assertion (ii) is established from assertion (i) by the method of continuity. The proof of assertion (iii) is similar to that of assertion (iii) of Theorem 4.2, we omit the details here. As a consequence Theorem 5.2, we give the following result which will be used in Section 8.

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COROLLARY 5.6. *Let V satisfy (3.2), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exist constants λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \eta$ and ω, such that*

$$\|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le C\|h\|_{L_{p,\omega}(\mathbb{R}^{d+1})}$$

provided that $V \ge \lambda_0$, where $u = (\mathcal{L} - V)^{-1}h$.

Next, we consider the nondivergence form parabolic equations.

THEOREM 5.7. *Let V satisfy (3.1), $\omega \in A_p(\mathbb{R}^{d+1})$ with $1 < p < \infty$. Then there exists a constant $\gamma = \gamma(d, p, K, \delta, \omega) > 0$ such that under Assumption 5.1(γ) for any $T \in (-\infty, +\infty]$ the following assertions hold.*

(i) *For any $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$,*

$$\|Vu\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|\sqrt{V}u_x\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_{xx}\|_{L_{p,\omega}} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \le C\|(L - V)u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)}$$

provided that $V \ge \lambda_0$, where λ_0, N depending only on $p, K, d, \delta, C_0, \delta_0, \gamma$ and ω.

(ii) *For any $V(t, x) \equiv \lambda > \lambda_0 = \lambda_0(p, K, d, \delta, \gamma, \omega)$ and $f \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$, there exists a unique solution $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$ of the equation $Lu - \lambda u = f$ in $\mathbb{R}^{d+1}_T$.*

(iii) *In the case that $a^{11} = a^{11}(t)$, $a^{ij} = a^{ij}(t, x^1, \dots, x^{\pi_{ij}})$, $ij > 1$, $b^j \equiv c \equiv 0$ and $V(t, x) \equiv \lambda$, we can take $\lambda_0 = 0$ in assertions* (i) *and* (ii).

Next we only consider the situation that Assumption 5.1 holds. For the remaining case that the proof of Assumption 5.2 is similar, see also the proof of Theorem 6.9 in [17].

Now, we first consider the following equation

$$L_0u = -u_t + a^{ij}D_{ij}u,$$

where $a^{11} = a^{11}(t)$ and $a^{ij} = a^{ij}(t, x^1)$ for $ij > 1$.

LEMMA 5.8. *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{R}^{d+1})$, $T \in (-\infty, \infty]$. Then for any $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$ and $\lambda \ge 0$, we have*

$$\begin{aligned}\lambda\|u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} &+ \sqrt{\lambda}\|Du\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|D^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)} \\ &\le N\|L_0u\|_{L_{p,\omega}(\mathbb{R}^{d+1}_T)},\end{aligned} \tag{5.1}$$

where the constant N depends only on K, p, d, δ and ω. Moreover, for any $f \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$ and $\lambda > 0$ there is a unique $u \in W^{1,2}_{p,\omega}(\mathbb{R}^{d+1}_T)$ solving

$$L_0u - \lambda u = f \quad \text{in } \mathbb{R}^{d+1}_T.$$

PROOF. Similar to the proof of Theorem 6.11 in [14], we can obtain the desired result. We omit the details. □

In the sequel, we only consider the case $d = 3$, or $d > 3$ and π_{ij} is replaced by $\widetilde{\pi}_{ij}$ in Assumption 5.1. Like before, the general case follows by induction.

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LEMMA 5.9. *Let* $1 < p < \infty, \omega \in A_p(\mathbb{R}^{d+1}), b^j = 0, c = 0$. *Then there exist constants* γ_2, μ_2 *and* N, *depending only on* K, d, δ, p *and* $A_p(\omega)$ *such that under Assumption 5.1*(γ_2) *with* $\widetilde{\pi}$ *in place of* π, *for any* $u \in C_0^\infty(Q_{\mu_2^{-1}R_0})$ *we have*

$$\|D^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1})} \le N\|Lu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + N\sum_{i=3}^{d}\|D_i^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}. \tag{5.2}$$

PROOF. Set $f = Lu$. Note that u satisfies

$$-u_t + \sum_{i,j=1}^{2} a^{ij}D_{ij}u + \sum_{i=3}^{d} D_i^2u = f + \sum_{\max(i,j)>2}(\delta_{ij} - a^{ij})D_{ij}u.$$

The coefficients on the left-hand side above satisfy Assumption 4.1. Thus, by Theorem 3.5 in [17] and Lemma 4.6, for γ_2 sufficiently small and μ_2 sufficiently large depending only d, δ, p and ω,

$$\begin{aligned}&\|D^2u\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \|u_t\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\\ &\quad\le N\Big(\|Lu\|_{L_{p,\omega}(\mathbb{R}^{d+1})} + \sum_{\max(i,j)>2}\|D_{ij}u\|_{L_{p,\omega}(\mathbb{R}^{d+1})}\Big).\end{aligned}$$

By an inequality similar to (4.6), we get (5.2). Proposition 5.5 is proved. □

PROOF OF THEOREM 5.7. To prove assertion (i), for $T = \infty$ and $u \in C_0^\infty$, this in turn is obtained from Theorem 6.12 in [17] and Lemma 5.9, and an idea from Agmon; see also the proof of Theorem 3.2. For general $T \in (-\infty, \infty]$, we use the fact $u = v$ for $t < T$, where $v \in W^{1,2}_{p,\omega}$ solves $(L - V)v = \chi_{t<T}(L - V)u$. Assertion (ii) is established from assertion (i) by the method of continuity. Assertion (iii) is proved by Lemma 5.8.

6. Divergence equations on a half space

The object of this section is to establish the solvability of parabolic divergence equations on a half space.

THEOREM 6.1. *Let* $1 < p < \infty$, $\omega \in A_p(\mathbb{R}^{d+1}), \Omega = \mathbb{R}^d_+$ *and* $T \in (-\infty, \infty]$. *Then there is a constant* $\gamma = \gamma(d, K, \delta, p, \omega) > 0$ *such that under Assumption 4.1*(γ) *the following assertions hold.*

(i) *Assume* $u \in \mathcal{H}^1_{p,\omega}(\Omega_T), f, g \in L_{p,\omega}(\Omega_T)$. *There exists positive* λ_0 *and* N, *depending only on* d, δ, p, γ *and* ω, *such that*

$$\begin{aligned}&\|u_t\|_{H^{-1}_{p,\omega}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L_{p,\omega}(\Omega_T)} + \lambda\|u\|_{L_{p,\omega}(\Omega_T)}\\ &\quad\le N\sqrt{\lambda}\|g\|_{L_{p,\omega}(\Omega_T)} + N\|f\|_{L_{p,\omega}(\Omega_T)},\end{aligned} \tag{6.1}$$

provided that $\lambda \ge \lambda_0$ *and,*

$$\begin{cases} \mathcal{L}u - \lambda u = \operatorname{div} g + f & \text{in } \Omega_T,\\ u = 0, & \text{on } (-\infty, T) \times \partial\Omega. \end{cases} \tag{6.2}$$

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(ii) *For any $\lambda > \lambda_0$ and $f, g \in L_{p,\omega}(\Omega_T)$, there exists a unique $u \in \mathcal{H}^1_{p,\omega}(\Omega_T)$ of (6.2) satisfying (6.1).*

THEOREM 6.2. *The assertions of Theorem 6.1 hold true if (6.2) is replaced by*

$$\begin{cases} \mathcal{L}u - \lambda u = \operatorname{div} g + f & \text{in } \Omega_T, \\ a^{1j}D_j u + a^1 u = g_1, & \text{on } (-\infty, T) \times \partial\Omega. \end{cases} \tag{6.3}$$

We shall use the idea of odd/even extensions. For this purpose, we need the following lemma.

LEMMA 6.3. *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{R}^{d+1})$ and $-\infty \le S < T \le \infty$. Define $\bar\omega(t, x^1, x') = \omega(t, |x_1|, x')$. Then:*

(i) *$\bar\omega$ has the same properties in Lemma 2.1 as ω;*

(ii) *a function u belongs to $\mathcal{H}^1_{p,\bar\omega}((S,T) \times \mathbb{R}^d_+)$ if and only if its even extension $\widetilde{u}$ with respect to x^1 belongs to $\mathcal{H}^1_{p,\bar\omega}((S,T) \times \mathbb{R}^d_+)$; moreover, there exists $N = N(d) > 0$ such that*

$$N^{-1}\|u\|_{\mathcal{H}^{-1}_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le \|\widetilde{u}\|_{\mathcal{H}^{-1}_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le N\|u\|_{\mathcal{H}^{-1}_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)}, \tag{6.4}$$

$$N^{-1}\|u\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le \|\widetilde{u}\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le N\|u\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)}, \tag{6.5}$$

$$N^{-1}\|Du\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le \|D\widetilde{u}\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)} \le N\|Du\|_{L_{p,\bar\omega}((S,T)\times\mathbb{R}^d_+)}. \tag{6.6}$$

(iii) *a function u belongs to $\mathcal{H}^1_{p,\bar\omega}((S,T) \times \mathbb{R}^d_+)$ and vanishes on $(S,T) \times \partial\mathbb{R}^d_+$ if and only if its odd extension $\widetilde{u}$ with respect to x' belongs to $\mathcal{H}^1_{p,\bar\omega}((S,T) \times \mathbb{R}^d_+)$; moreover, we have (6.4)–(6.6).*

Next, we prove Theorems 6.1 and 6.2.

PROOF OF THEOREM 6.1. Define

$$\begin{aligned} &\widetilde{a}^{ij}(t,x) = \operatorname{sgn}(x^1)a^{ij}(t,|x^1|,x') \quad \text{for } i = 1,\ j \ge 2, \text{ or } j = 1, i \ge 2 \\ &\widetilde{a}^{ij}(t,x) = a^{ij}(t,|x^1|,x'), \quad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} &\widetilde{a}^1(t,x) = \operatorname{sgn}(x^1)a^1(t,|x^1|,x'), && \widetilde{a}^j(t,x) = a^j(t,|x^1|,x'), \quad j \ge 2, \\ &\widetilde{b}^1(t,x) = \operatorname{sgn}(x^1)a^1(t,|x^1|,x'), && \widetilde{b}^j(t,x) = a^j(t,|x^1|,x'), \quad j \ge 2, \\ &\widetilde{c}(t,x) = c(t,|x^1|,x'), && \widetilde{f}(t,x) = \operatorname{sgn}(x^1)f(t,|x^1|,x'), \\ &\widetilde{g}_1(t,x) = \operatorname{sgn}(x^1)g_1(t,|x^1|,x'), && \widetilde{g}_j(t,x) = g_j(t,|x^1|,x'), \quad j \ge 2. \end{aligned}$$

Clearly, if a^{ij}, a^i, b^i, c satisfy Assumption 4.1(γ), then the new coefficients $\widetilde{a}^{ij}, \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}$ satisfy Assumption 4.1(4γ). Moreover, $\widetilde{f}, \widetilde{g} \in L_{p,\bar\omega}(\mathbb{R}^{d+1}_T)$. Let $\widetilde{L}$ be the divergence form parabolic operator with coefficients $\widetilde{a}^{ij}, \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}$.

Due to Theorem 4.2 we can find $\gamma > 0$ and $\lambda_0 > 0$ such that there exists a unique solution $u \in \mathcal{H}^{-1}_{p,\bar\omega}((S,T) \times \mathbb{R}^d_+)$ of

$$\mathcal{L}u - \lambda u = \operatorname{div}\widetilde{g} + \widetilde{f} \quad \text{in } \mathbb{R}^{d+1}_T, \tag{6.7}$$

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provided that $\lambda \geq \lambda_0$. By the definition of the coefficients and the data, we have

$$\mathcal{L}u(t, -x^1, x') - \lambda u(t, -x^1, x') = -\operatorname{div} \widetilde{g} - \widetilde{f} \quad \text{in } \mathbb{R}^{d+1}_T.$$

Consequently, $-u(t, -x^1, x')$ is also a solution to (6.7). By the uniqueness of the solution, we obtain $u(t, x) = -u(t, -x^1, x')$. This implies that, as a function on $\mathbb{R}^d_+$, u has zero trace on the boundary and clearly u satisfies (6.2). The existence of the the solution is proved.

On the other hand, it is easy to see that if $u \in \mathcal{H}^1_{p,\bar{\omega}}((S, T) \times \mathbb{R}^d_+)$ is a solution to (7.2), then its odd extension with respect to x^1 is a solution to (6.7). So the uniqueness follows from Theorem 4.2. Using (4.1) and Theorem 6.1, we can prove (6.1). The theorem is proved. □

Proof of Theorem 6.2. We define $\widetilde{a}^{ij}, \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}$ and $\bar{\omega}$ as in the proof of Theorem 6.1. Let $\mathcal{L}$ be the divergence form parabolic operator with coefficients $\widetilde{a}^{ij}, \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}$. Different from above, we define

$$\widetilde{f}(t, x) = f(t, |x^1|, x'),$$
$$\widetilde{g}_1(t, x) = \operatorname{sgn} g_1(t, |x^1|, x'), \quad \widetilde{g}_j(t, x) = g_j(t, |x^1|, x'), \ j \geq 2.$$

Recall that $\widetilde{a}^{ij}$ satisfy Assumption 4.1(4γ). Clearly, $\widetilde{f}, \widetilde{g} \in L_{p,\omega}(\mathbb{R}^{d+1}_T)$. By Theorem 4.2, we can find $\gamma > 0$ and $\lambda_0 > 0$ such that there exists a unique solution $u \in \mathcal{H}^1_{p,\bar{\omega}}((S, T) \times \mathbb{R}^d_+)$ of (6.7) provided that $\lambda \geq \lambda_0$. By the definition of the coefficients and the data, we have

$$\mathcal{L}u(t, -x^1, x') - \lambda u(t, -x^1, x') = \operatorname{div} \widetilde{g} + \widetilde{f} \quad \text{in } \mathbb{R}^{d+1}_T.$$

Consequently, $u(t, -x^1, x')$ is also a solution to (6.7). By the uniqueness of the solution, we get $u(t, x) = u(t, -x^1, x')$.

Let $p' = p/(p-1)$. For any $h \in \mathcal{H}^1_{p',(\bar{\omega})^{-1/(p-1)}}((-\infty, T) \times \mathbb{R}^d_+)$, denote $\widetilde{h}$ to be its even extension with respect to x^1. Since u satisfies (6.7), then

$$\begin{aligned} &\int_{-\infty}^{T} \int_{\mathbb{R}^d} (-u_t \cdot \widetilde{h} - a^{ij} D_j u \cdot D_i \widetilde{h} - \widetilde{a}^i \cdot D_i \widetilde{h} + \widetilde{b}^i D_i u \cdot \widetilde{h} + (\widetilde{c} - \lambda) u \cdot \widetilde{h}) \, dx \, dt \\ &\quad = \int_{-\infty}^{T} \int_{\mathbb{R}^d} (-\widetilde{g}_i \cdot \widetilde{D}_i h + \widetilde{f} \cdot \widetilde{h}) \, dx \, dt. \end{aligned} \tag{6.8}$$

By the definition of $\widetilde{a}^{ij}, \widetilde{a}^i, \widetilde{b}^i, \widetilde{c}, \widetilde{g}$ and $\widetilde{f}$ as well as the evenness of u and $\widetilde{h}$, all terms inside the integrals in (6.8) are even with respect to x^1. Thus, (6.8) implies

$$\begin{aligned} &\int_{-\infty}^{T} \int_{\mathbb{R}^d} (-u_t \cdot h - a^{ij} D_j u \cdot D_i h - \widetilde{a}^i \cdot D_i h + \widetilde{b}^i D_i u \cdot h + (\widetilde{c} - \lambda) u \cdot h) \, dx \, dt \\ &\quad = \int_{-\infty}^{T} \int_{\mathbb{R}^d} (-g_i \cdot D_i h + f \cdot \widetilde{h}) \, dx \, dt. \end{aligned} \tag{6.9}$$

Since $h \in \mathcal{H}^1_{p,\bar{\omega}}((S, T) \times \mathbb{R}^d_+)$ is arbitrary, by the definition u solves (6.3). This proves the existence of the solution.

The uniqueness is obvious by using (6.9). Using (4.1) and Theorem 6.1, we can prove (6.1). The theorem is proved. □

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7. Nondivergence equations on a half space

The object of this section is to establish the solvability of nondivergence equations on a half space.

We first set

$$\bar{L}u = -u_t + a^{ij}D_{ij}u,$$

where the entries of coefficient matrices a^{ij} are measurable function of only $t \in \mathbb{R}$, i.e. $a^{ij} = a^{ij}(t)$ satisfying (1.3). With this operator $\bar{\mathcal{L}}$ we have the following theorem.

THEOREM 7.1. *Let* $1 < p < \infty, \Omega = \mathbb{R}^d_+$ *and* $T \in (-\infty, \infty]$. *Then there is a constant* $R_0 = R_0(d, K, \delta, p, \omega) > 0$ *such that under Assumption 3.1*(R_0) *the following assertions hold.*

(i) *Assume* $u \in W^{1,2}_p(\Omega_T)$. *There exist positive constants* λ_0 *and* N, *depending only on* d, δ, p, R_0, *such that*

$$\lambda\|u\|_{L_p(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L_p(\Omega_T)} + \|D^2u\|_{L_{p,\omega}(\Omega_T)} + \|u_t\|_{L_p(\Omega_T)} \le N\|\bar{L}u - \lambda u\|_{L_p(\Omega_T)}, \tag{7.1}$$

provided that $\lambda \ge \lambda_0$ *and*

$$u = 0, \quad \text{on } (-\infty, T) \times \partial\Omega. \tag{7.2}$$

(ii) *For any* $\lambda > \lambda_0$ *and* $f \in L_p(\Omega_T)$, *there exists a unique* $u \in W^{1,2}_p(\Omega_T)$ *satisfying* $\bar{\mathcal{L}}u - \lambda u = f$.

Theorem 7.1 is proved in [18].

As the applications of Theorem 7.1, we have the following results for $1 < p < \infty$.

LEMMA 7.2. *Let* $0 < r < R < \infty$ *and* $u \in W^{1,2}_{p,loc}(\Omega_T)$ *satisfy* (7.2). *Then*

$$\|u_t\|_{L_p(Q^+_r)} + \|D^2u\|_{L_p(Q^+_r)} \le N(\|\bar{L}u\|_{L_p(Q^+_r)} + \|Du\|_{L_p(Q^+_r)} + \|u\|_{L_p(Q^+_r)}),$$

where $N = N(K, d, \delta, r, R)$ *and* $Q^+_r = Q_r \bigcap(\mathbb{R}^d_+ \times (-\infty, T))$.

PROOF. See the proof of Lemma 5.2 in [26] or Lemma 7.1 in [18]. □

LEMMA 7.3. *Let* $0 < r < R < \infty$ *and* $u \in C^\infty_{loc}(\Omega_T)$ *satisfy* (7.2). *Assume that* $\bar{\mathcal{L}}u = 0$ *in* Q^+_R. *Then for any multi-index* γ, *we have*

$$\sup_{Q^+_r}|D^\gamma u| + \sup_{Q^+_r}|D^\gamma u_t| \le N(\|Du\|_{L_p(Q^+_r)} + \|u\|_{L_p(Q^+_r)}),$$

where $N = N(K, |\gamma|, d, \delta, r, R)$.

PROOF. See the proof of Lemma 5.8 in [26]. □

LEMMA 7.4. *Let* $u \in C^\infty_{loc}(\Omega_T)$ *satisfy* (7.2). *Assume that* $\bar{\mathcal{L}}u - \lambda u = 0$ *in* Q^+_2. *Then for any multi-index* γ, *we have*

$$\sup_{Q^+_1}|D^\gamma(D^2u)| + \sup_{Q^+_1}|D^\gamma u_t| \le N(\|u_t\|_{L_p(Q^+_2)} + \|D^2u\|_{L_p(Q^+_1)} + \sqrt{\lambda}\|u\|_{L_p(Q^+_1)}),$$

where $N = N(K, |\gamma|, p, d, \delta, r, R)$.

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PROOF. See the proof of Lemmas 5.9 in [26]. □

LEMMA 7.5. *Let $\lambda \geq 0, k \geq 2$ and $r \in (0, \infty)$. Let $u \in C^\infty_{loc}(\Omega_T)$ satisfy (7.2). Assume that $\bar{L}u - \lambda u = 0$ in Q^+_{kr}. Then*

$$\frac{1}{|Q^+_r|}\int_{Q^+_r} |D^2u(t,x) - (D^2u)_{Q^+_r}|^p \,dx\,dt \leq Nk^{-p}(|D^2u|^p + \lambda^p|Du|^p)_{Q_{kr}},$$

where $N = N(|\gamma|, K, p, d, \delta, r, R)$.

PROOF. See the proof of Lemma 5.10 in [26]. □

By using the results above, we can obtain the following result.

PROPOSITION 7.6. *Let $k \geq 4$ and $r \in (0, \infty)$. Let $u \in C^\infty_{loc}(\Omega_T)$ satisfy (7.2). Then*

$$\frac{1}{|Q^+_r|}\int_{Q^+_r} |D^2u(t,x) - (D^2u)_{Q^+_r}|^p \,dx\,dt \leq Nk^{-p}(|D^2u|^p + \lambda^p|Du|^p)_{Q_{kr}},$$

where $N = N(K, d, \delta, p)$.

By using Theorem 7.1, and adapting the standard process (compare with the proof of Theorem 5.1 in [15]), we can the main result in this section.

THEOREM 7.7. *Let $1 < p < \infty$, $\omega \in W(\mathbb{R}^{d+1})$, $\Omega = \mathbb{R}^d_+$ and $T \in (-\infty, \infty]$. Then there is a constant $R_0 = R_0(K, d, \delta, p, \omega) > 0$ such that under Assumption 3.1(R_0) the following assertions hold.*

(i) *Assume $u \in W^{1,2}_{p,\omega}(\Omega_T)$. There exist positive constants λ_0 and N, depending only on d, K, δ, p, R_0 and ω, such that*

$$\lambda\|u\|_{L_{p,\omega}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L_{p,\omega}(\Omega_T)} + \|D^2u\|_{L_{p,\omega}(\Omega_T)} + \|u_t\|_{L_{p,\omega}(\Omega_T)} \leq N\|Lu - \lambda u\|_{L_{p,\omega}(\Omega_T)},$$

provided that $\lambda \geq \lambda_0$ and

$$u = 0 \quad \text{on } (-\infty, T) \times \partial\Omega.$$

(ii) *For any $\lambda > \lambda_0$ and $f \in L_{p,\omega}(\Omega_T)$, there exists a unique $u \in W^{1,2}_{p,\omega}(\Omega_T)$ satisfying $\mathcal{L}u - \lambda u = f$.*

8. Equations on Morrey spaces

The object of this section is to establish the nondivergence and divergence equations' solvability results on Morrey spaces.

First of all we start with the definition of Morrey spaces. Let Ω be an open set in $\mathbb{R}^{d+1}$. Let $1 < p < \infty$ and $0 < \beta < d + 2$. We say that a locally integral function $f(t, x)$ belongs to the Morrey space $L^{p,\beta}(\Omega)$ if

$$\|f\|^p_{L^{p,\beta}(\Omega)} \equiv \sup_{(t,x)\in\Omega, r>0} \frac{1}{r^\lambda}\int_{Q_r(t,x)\cap\Omega} |f(t,y)|^p \,dy\,dt < \infty,$$

$$\|f\|_{W^{1,2}_{p,\beta}(\Omega)} = \|f\|_{L^{p,\beta}(\Omega)} + \|f_t\|_{L^{p,\beta}(\Omega)} + \|Df\|_{L^{p,\beta}(\Omega)} + \|D^2 f\|_{L^{p,\beta}(\Omega)} < \infty,$$

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and

$$\mathcal{H}^1_{p,\beta}(\Omega) = (1 - \triangle + \partial_t)^{1/2} W^{1,2}_{p,\beta}(\Omega), \quad H^{-1}_{p,\beta}(\Omega) = (1 - \triangle + \partial_t)^{1/2} L^{p,\beta}(\Omega).$$

In addition, we give some definitions that are slightly different from Section 2:

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad Q_r(t, x) = (t - r^2, t) \times B_r(x),$$
$$B'_r(x') = \{y \in \mathbb{R}^{d-1} : |x' - y'| < r\}, \quad Q'_r(t, x) = (t - r^2, t) \times B'_r(x'),$$
$$Q^+_r(t, x) = Q_r(t, x) \bigcap (\mathbb{R}^d_+ \times (-\infty, T)), \quad T \in (-\infty, \infty].$$

We next consider the nondivergence equation on Morrey spaces. Applying Theorem 3.2, we have the following result.

THEOREM 8.1. *Let $1 < p < \infty$, $0 < \beta < d + 2$, $\Omega = \mathbb{R}^d$ and $T \in (-\infty, \infty]$. Then there is a constant $R_0 = R_0(d, K, \beta, \delta, p) > 0$ such that under Assumption 3.1(R_0) the following assertions hold.*

(i) *Assume $u \in W^{1,2}_{p,\beta}(\Omega_T)$. There exist positive constants λ_0 and N, depending only on $d, \beta, K, \delta, p, R_0$, such that*

$$\lambda\|u\|_{L^{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \|D^2u\|_{L^{p,\beta}(\Omega_T)} + \|u_t\|_{L^{p,\beta}(\Omega_T)} \le N\|Lu - \lambda u\|_{L^{p,\beta}(\Omega_T)},$$

provided that $\lambda \ge \lambda_0$.

(ii) *For any $\lambda > \lambda_0$ and $f \in L^{p,\beta}(\Omega_T)$, there exists a unique $u \in W^{1,2}_{p,\beta}(\Omega_T)$ satisfying $\mathcal{L}u - \lambda u = f$.*

As the consequence of Theorem 7.7, we have the following result.

THEOREM 8.2. *Let $1 < p < \infty$, $0 < \beta < d + 2$, $\Omega = \mathbb{R}^d_+$ and $T \in (-\infty, \infty]$. Then there is a constant $R_0 = R_0(\beta, d, K, \delta, p) > 0$ such that under Assumption 3.1(R_0) the following assertions hold.*

(i) *Assume $u \in W^{1,2}_{p,\beta}(\Omega_T)$. There exist positive constants λ_0 and N, depending only on $d, \beta, K, \delta, p, R_0$, such that*

$$\lambda\|u\|_{L^{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \|D^2u\|_{L^{p,\beta}(\Omega_T)} + \|u_t\|_{L^{p,\beta}(\Omega_T)} \le N\|Lu - \lambda u\|_{L^{p,\beta}(\Omega_T)},$$

provided that $\lambda \ge \lambda_0$ and

$$u = 0 \quad \text{on } (-\infty, T) \times \partial\Omega.$$

(ii) *For any $\lambda > \lambda_0$ and $f \in L^{p,\beta}(\Omega_T)$, there exists a unique $u \in W^{1,2}_{p,\beta}(\Omega_T)$ satisfying $\mathcal{L}u - \lambda u = f$.*

By Theorems 8.1 and 8.2, we give one of the main results of this section.

THEOREM 8.3. *Let $1 < p < \infty$, $0 < \beta < d + 2$, $T \in (-\infty, \infty]$ and Ω be a $C^{1,1}$ bounded domain with $C^{1,1}$ norm bounded by K. Then there is a constant $R_0 = R_0(d, K, \beta, \delta, p) > 0$ and $\lambda_0 = \lambda_0(d, K, \beta, \delta, p) > 0$ such that under Assumption 3.1(R_0) the following* *is true. For any* $f \in L^{p,\beta}(\Omega_T)$ *and* $\lambda > \lambda_0$*, there is a unique solution* $u \in W^{1,2}_{p,\beta}(\Omega_T)$ *to*

$$\begin{cases} Lu - \lambda u = f & \text{in } \Omega_T, \\ u = 0 & \text{on } (-\infty, T) \times \partial\Omega \end{cases}$$

and we have

$$\lambda\|u\|_{L^{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \|D^2u\|_{L^{p,\beta}(\Omega_T)} + \|u_t\|_{L^{p,\beta}(\Omega_T)} \le N\|f\|_{L^{p,\beta}(\Omega_T)},$$

where N *depends only on* d, p, β, δ, K *and* R_0.

Proof. By Theorem 8.1, we obtain the following interior estimate for any $0 < r < R < \infty$, $Q_r \subset Q_R \subset \Omega_T$ and $\lambda \ge \lambda_0$

$$\begin{aligned} &\lambda\|u\|_{L^{p,\beta}(Q_r)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(Q_r)} + \|D^2u\|_{L^{p,\beta}(Q_r)} + \|u_t\|_{L^{p,\beta}(Q_r)} \\ &\quad \le N(\|f\|_{L^{p,\beta}(Q_R)} + \|u\|_{L^{p,\beta}(Q_R)}). \end{aligned} \tag{8.1}$$

Similarly, Theorem 8.2 gives a boundary estimate: let $0 < r < R < \infty$, $f \in L^{p,\beta}(Q_R^+)$ and R_0 be the constant taken from Theorem 8.2. Then under Assumption 3.1(R_0), for any $\lambda \ge \lambda_0$ and $u \in W^{1,2}_{p,\beta}(Q_R^+)$, we have

$$\begin{aligned} &\lambda\|u\|_{L^{p,\beta}(Q_r^+)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(Q_r^+)} + \|D^2u\|_{L^{p,\beta}(Q_r^+)} + \|u_t\|_{L^{p,\beta}(Q_r^+)} \\ &\quad \le N(\|f\|_{L^{p,\beta}(Q_R^+)} + \|u\|_{L^{p,\beta}(Q_R^+)}), \end{aligned} \tag{8.2}$$

provided that $u = 0$ on Q_R' and

$$Lu - \lambda u = f \quad \text{in } Q_R^+.$$

It is well known that the ellipticity condition is preserved under a change of variables. Take $t_0 \in (-\infty, T)$, a point $x_0 \in \partial\Omega$ and a number $r_0 = r_0(\Omega)$, so that

$$\Omega \bigcap B_{r_0}(x_0) = \{x \in B_{r_0}(x_0) : x^1 > \phi(x')\}$$

in some coordinate system. We now locally flatten the boundary of $\partial\Omega$ by defining

$$y^1 = x^1 - \phi(x') := \Phi(x), \quad y^j = x^j := \Phi^j(x), \quad j \ge 2$$

under the assumptions of the theorem, Φ is a $C^{1,1}$ diffeomorphism in a neighborhood of x_0. It is easily seen that the coefficients of the new operator in the y-coordinates also satisfy Assumption 3.1 with a possibly different R_0. Thus, we can choose a sufficiently small R_0 such that from (8.2), for $X_0 = (t_0, x_0)$ and some $r_1 = r_1(\Omega) < r_0$,

$$\begin{aligned} &\lambda\|u\|_{L^{p,\beta}(\Omega_T \cap Q_{r_1}(X_0))} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T \cap Q_{r_1}(X_0))} \\ &\qquad + \|D^2u\|_{L^{p,\beta}(\Omega_T \cap Q_{r_1}(X_0))} + \|u_t\|_{L^{p,\beta}(\Omega_T \cap Q_{r_1}(X_0))} \\ &\quad \le N(\|f\|_{L^{p,\beta}(\Omega_T \cap Q_{r_0}(X_0))} + \|u\|_{L^{p,\beta}(\Omega_T \cap Q_{r_0}(X_0))}). \end{aligned} \tag{8.3}$$

By (8.1) and (8.3), and using a partition of the unity, one completes the proof for a sufficiently large λ_0. □

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Now we turn to the divergence case. From Theorems 4.2, 6.1 and 6.2, we have the following results.

THEOREM 8.4. *Let* $1 < p < \infty$, $0 < \beta < d + 2$, $\Omega = \mathbb{R}^d$ *and* $T \in (-\infty, \infty]$. *Then there is a constant* $\gamma = \gamma(d, K, \beta, \delta, p) > 0$ *such that under Assumption 4.1*(γ) *the following assertions hold.*

(i) *Assume* $u \in \mathcal{H}^1_{p,\beta}(\Omega_T)$, $f, g \in L^{p,\beta}(\Omega_T)$. *There exists positive* λ_0 *and* N, *depending only on* d, K, δ, p, γ *and* β, *such that*

$$\|u_t\|_{H^{-1}_{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \lambda\|u\|_{L^{p,\beta}(\Omega_T)} \le N\sqrt{\lambda}\|g\|_{L^{p,\beta}(\Omega_T)} + N\|f\|_{L^{p,\beta}(\Omega_T)}, \tag{8.4}$$

provided that $\lambda \ge \lambda_0$ *and*

$$\mathcal{L}u - \lambda u = \operatorname{div} g + f \quad \text{in } \Omega_T. \tag{8.5}$$

(ii) *For any* $\lambda > \lambda_0$ *and* $f, g \in L^{p,\beta}(\Omega_T)$, *there exists a unique* $u \in \mathcal{H}^1_{p,\beta}(\Omega_T)$ *of* (8.5) *satisfying* (8.4).

THEOREM 8.5. *Let* $1 < p < \infty$, $0 < \beta < d + 2$, $\Omega = \mathbb{R}^d_+$ *and* $T \in (-\infty, \infty]$. *Then there is a constant* $\gamma = \gamma(\beta, K, d, \delta, p) > 0$ *such that under Assumption 4.1*(γ) *the following assertions hold.*

(i) *Assume* $u \in \mathcal{H}^1_{p,\beta}(\Omega_T)$, $f, g \in L^{p,\beta}(\Omega_T)$. *There exists positive* λ_0 *and* N, *depending only on* $d, \delta, K, p, \gamma, \beta$ *and* θ, *such that*

$$\|u_t\|_{\mathcal{H}^{-1}_{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \lambda\|u\|_{L^{p,\beta}(\Omega_T)} \le N\sqrt{\lambda}\|g\|_{L^{p,\beta}(\Omega_T)} + N\|f\|_{L^{p,\beta}(\Omega_T)}, \tag{8.6}$$

provided that $\lambda \ge \lambda_0$ *and*

$$\begin{cases} \mathcal{L}u - \lambda u = \operatorname{div} g + f & \text{in } \Omega_T, \\ u = 0 & \text{on } (-\infty, T) \times \partial\Omega. \end{cases} \tag{8.7}$$

(ii) *For any* $\lambda > \lambda_0$ *and* $f, g \in L^{p,\beta}(\Omega_T)$, *there exists a unique* $u \in \mathcal{H}^1_{p,\beta}(\Omega_T)$ *of* (8.7) *satisfying* (8.6).

THEOREM 8.6. *The assertions of Theorem 8.5 hold true if* (8.7) *is replaced by*

$$\begin{cases} \mathcal{L}u - \lambda u = \operatorname{div} g + f & \text{in } \Omega_T, \\ a^{1j}D_j u + a^1 u = g_1 & \text{on } (-\infty, T) \times \partial\Omega. \end{cases}$$

Now, we assume that the boundary $\partial\Omega$ of the domain Ω is locally the graph of a Lipschitz continuous function with small Lipschitz constant. More precisely, we make the following assumption containing a parameter $\rho \in (0, 1]$, which will be specified later.

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Assumption 8.7 (θ). There is a constant $R_1 \in (0, 1]$ such that, for any $x_0 \in \partial\Omega$ and $r \in (0, R_1]$, there exists a Lipschitz function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \bigcap B_r(x_0) = \{x \in B_r(x_0) : x^1 > \phi(x')\}$$

and

$$\sup_{x',y' \in B'_r(x'_0), x'=y'} \frac{|\phi(y') - \phi(x')|}{|y' - x'|} \leq \theta$$

in some coordinate system. Note that all C^1 domain satisfy this assumption for any $\theta > 0$.

We shall impose a little bit more regular assumption on a^{ij} near the boundary. For any $x \in \mathbb{R}^d$, denote

$$\mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|.$$

Assumption 8.8 (γ). There is a constant $R_1 \in (0, 1]$ such that, for any $x_0 \in \mathbb{R}^d$ with $\mathrm{dist}(x, \partial\Omega) \leq R_1$ and any $r \in (0, R_1]$, we have

$$\sup_{ij} r^{-2}|B_r(x_0)|^{-2} \int_{t-r^2}^{t} \int_{B_r(x_0)} |a^{ij}(s, x) - (a^{ij})_{B_r(x_0)}| \, dx \, ds \leq \gamma.$$

Theorem 8.9. *Let $1 < p < \infty$, $0 < \beta < d + 2$, $T \in (-\infty, \infty]$ and Ω be a bounded domain. Then there exist constants $R_0 = R_0(\beta, d, K, \delta, p)$, $\theta = \theta(\beta, d, K, \delta, p)$, $\gamma = \gamma(\beta, d, K, \delta, p)$ and $\lambda_0 = \lambda_0(\beta, d, K, \delta, p) > 0$ such that under Assumptions 4.1(γ_1), 8.7(θ) and 8.8(γ_2) the following is true. For any $f \in L^{p,\beta}(\Omega_T)$ and $\lambda > \lambda_0$, there is a unique solution $u \in H^1_{p,\beta}(\Omega_T)$ to*

$$\begin{cases} Lu - \lambda u = f & \text{in } \Omega_T, \\ u = 0 & \text{on } (-\infty, T) \times \partial\Omega, \end{cases}$$

and we have

$$\|u_t\|_{H^{-1}_{p,\beta}(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^{p,\beta}(\Omega_T)} + \lambda\|u\|_{L^{p,\beta}(\Omega_T)} \leq N\sqrt{\lambda}\|g\|_{L^{p,\beta}(\Omega_T)} + N\|f\|_{L^{p,\beta}(\Omega_T)},$$

where N depends only on $d, p, \beta, \delta, \lambda_0, K, \gamma_1, \gamma_2$ and θ.

Proof. By Theorems 8.4 and 8.5, and adapting the same arguments in the proof of Theorem 2.1 in [15] and Theorem 2.10 in [18], we can obtain the desired result. We omit the details here. □

9. Schrödinger-type operators

In this section, we will study the boundedness for parabolic-type operators such as $\nabla_x^2(L - V)^{-1}$, $V(L - V)^{-1}$, $V^{1/2}\nabla_x(L - V)^{-1}$, $\partial_t(L - V)^{-1}$, $V^{1/2}\nabla_x(\mathcal{L} - V)^{-1}$, $V^{1/2}(\mathcal{L} - V)^{-1}\nabla_x$, $V(\mathcal{L} - V)^{-1}$ and $\nabla_x(\mathcal{L} - V)^{-1}\nabla_x$ with positive potentials V satisfying (3.1) or (3.2).

Zhong [33], Shen [29] and Auscher and Ben [1] studied the $L_p(\mathbb{R}^d)$ boundedness for elliptic Schrödinger-type operators (that is, $\widetilde{L} = -\triangle + V$) with nonnegative potentials belonging to certain elliptic type reverse Hölder class $B_q(q > 1)$ (cf. (9.1) without t coordinate, see also [29]), and Okazawa [28] gave a $L_p(\mathbb{R}^d)$ estimate for Schrödinger-type operators with nonnegative potentials V, which satisfy the condition $|\nabla V| \le c_p V^{3/2}$. In addition, Kurata and Sugano [27] studied the $L_{p,\omega}(\mathbb{R}^d)$ and $L^{p,\beta}(\mathbb{R}^d)$ boundedness for uniformly elliptic operators $\widetilde{L} = (a^{ij}(x)u_{x_i}(x))_{x_j}$ with nonnegative potentials belonging to a certain elliptic-type reverse Hölder class $B_q(q > 1)$, where $a^{ij} \in C^\alpha$ with $\alpha \in (0, 1]$ and ω belong to a certain class of Muckenhoupt weights.

On the other hand, Gao and Jiang [21] considered the L^p-boundedness of the parabolic Schrödinger-type operator $\nabla_x^2(\partial_t - \triangle + V)^{-1}$ with certain potentials with space variable x. Recently, Carbonaro et al. [5] improved Gao and Jiang's result above by the potential V with the variables x, t, which is essentially the generalization to $\mathbb{R}^{n+1}$ of the condition of Gao and Jiang. More precisely, Carbonaro *et al.* in [5] proved $L^p(\mathbb{R}^{d+1})$-boundedness of operators $V(\partial_t - \triangle + V)^{-1}, \nabla_x^2(\partial_t - \triangle + V)^{-1}, \partial_t(\partial_t - \triangle + V)^{-1}$ if $0 \le V \in (PB)_p$ for $1 < p < \infty$. We say that a nonnegative locally L^p integral function $V(x, t)$ on $\mathbb{R}^{d+1}$ is said to belong to $(PB)_p(1 < p \le \infty)$ if there exists $C > 0$ such that the parabolic-type reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q V^p\,dx\,dt\right)^{1/p} \le C\left(\frac{1}{|Q|}\int_Q V\,dx\,dt\right) \tag{9.1}$$

holds for every parabolic cylinder Q in $\mathbb{R}^{d+1}$; see [5]. Clearly, $V(t, x) = \lambda e^{\sqrt{1+|x|^2+t^2}/\lambda^2} \notin (PB)_q$ for any $q > 1$ if $\lambda \ge 1$, but it satisfies (3.1) and (3.2).

Applying Theorem 5.2 and Lemma 5.3, we have the following result.

THEOREM 9.1. *Suppose that V satisfies (3.1), $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^{d+1})$. There are constants γ, λ_0 and N, depending only on $p, K, d, \delta, C_0, \delta_0$ and ω, such that under Assumption 5.1(γ) for $V \ge \lambda_0$,*

$$\begin{aligned}\|\nabla_x^2(L - V)^{-1} f\|_{L_{p,\omega}} + \|V(L - V)^{-1} f\|_{L_{p,\omega}} + \|V^{1/2}\nabla_x(L - V)^{-1} f\|_{L_{p,\omega}} \\ + |\partial_t(L - V)^{-1} f\|_{L_{p,\omega}} \le N\|f\|_{L_{p,\omega}}.\end{aligned}$$

THEOREM 9.2. *Suppose that V satisfies (3.2), $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^{d+1})$. Then there are constants γ, λ_0 and N, depending only on $p, K, d, \delta, C_0, \delta_0$ and ω, such that under Assumption 5.1(γ) for $V \ge \lambda_0$,*

$$\begin{aligned}\|\nabla_x(\mathcal{L} - V)^{-1}\nabla_x f\|_{L_{p,\omega}} + \|V^{1/2}\nabla_x(\mathcal{L} - V)^{-1} f\|_{L_{p,\omega}} \\ + \|V^{1/2}(\mathcal{L} - V)^{-1}\nabla_x f\|_{L_{p,\omega}} + \|V(\mathcal{L} - V)^{-1} f\|_{L_{p,\omega}} \le N\|f\|_{L_{p,\omega}}.\end{aligned}$$

As a consequence of Theorems 9.1 and 9.2, we have following results.

COROLLARY 9.3. *Suppose that V satisfies (3.1), $1 < p < \infty$ and $0 < \beta < d + 2$. There are constants γ, λ_0 and N, depending only on $p, K, d, \delta, C_0, \delta_0$ and β, such that under Assumption 5.1(γ) for $V \ge \lambda_0$,*

$$\begin{aligned}\|\nabla_x^2(L - V)^{-1} f\|_{L^{p,\beta}} + \|V(L - V)^{-1} f\|_{L^{p,\beta}} + \|V^{1/2}\nabla_x(L - V)^{-1} f\|_{L^{p,\beta}} \\ + \|\partial_t(L - V)^{-1} f\|_{L^{p,\beta}} \le N\|f\|_{L^{p,\beta}}.\end{aligned}$$

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COROLLARY 9.4. *Suppose that V satisfies (3.2),* $1 < p < \infty$ *and* $0 < \beta < d + 2$. *There are constants γ, λ_0 and N, depending only on $p, K, d, \delta, C_0, \delta_0$ and β, such that under Assumption 5.1(γ) for $V \geq \lambda_0$,*

$$\|\nabla_x(\mathcal{L} - V)^{-1}\nabla_x f\|_{L^{p,\beta}} + \|V^{1/2}\nabla_x(\mathcal{L} - V)^{-1} f\|_{L^{p,\beta}} + \|V^{1/2}(\mathcal{L} - V)^{-1}\nabla_x f\|_{L^{p,\beta}} + \|V(\mathcal{L} - V)^{-1} f\|_{L^{p,\beta}} \leq N\|f\|_{L^{p,\beta}}.$$

Finally, we study the boundedness of Schrödinger-type operators on variable L^p spaces. We consider a measure function $p : \mathrm{R}^{d+1} \to [1, \infty)$. Let $L^{p(\cdot)}(\mathrm{R}^{d+1})$ denotes the set of measurable functions f on R^{d+1} such that for some $\lambda > 0$,

$$\int_{\mathrm{R}^{d+1}} \Big(\frac{|f(t, x)|}{\lambda}\Big)^{p(t,x)} dx\,dt < \infty.$$

This set becomes a Banach function spaces when equipped with norm

$$\|f\|_{L^{p(\cdot)}} := \|f\|_{L^{p(\cdot)}(\mathrm{R}^{d+1})} = \inf\Big\{\lambda > 0 : \int_{\mathrm{R}^{d+1}} \Big(\frac{|f(t, x)|}{\lambda}\Big)^{p(t,x)} dx\,dt \leq 1\Big\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalize the standard L^p spaces: if $p(t, x) = p_0$ is constant, then $L^{p(\cdot)}(\mathrm{R}^{d+1})$ equals $L_{p_0}(\mathrm{R}^{d+1})$. They have many properties in common with the standard L^p spaces.

These spaces, and the corresponding variable Sobolev spaces, are of interest in their own right, and also for applications to partial differential equations and the calculus of variations. (See [8] and references therein.)

For conciseness, define $\mathcal{P}(\mathrm{R}^{d+1})$ be the set of measurable functions $p : \mathrm{R}^{d+1} \to [1, \infty)$ such that

$$p_- = \inf\{p(t, x) : (t, x) \in \mathrm{R}^{d+1}\} > 1, \; p_+ = \sup\{p(t, x) : (t, x) \in \mathrm{R}^{d+1}\} < \infty.$$

Let $\mathcal{B}(\mathrm{R}^{d+1})$ be the set of $p(\cdot) \in \mathcal{P}(\mathrm{R}^{d+1})$ such that M is bounded on $L^{p(\cdot)}$, where M denotes the parabolic Hardy–Littlewood maximal operator; see Section 2.

Combining Corollary 1.11 and Theorem 1.2 in [8], Theorems 9.1 and 9.2 together, we have following results.

COROLLARY 9.5. *Suppose that V satisfy (3.1), and $p(\cdot) \in \mathcal{B}(\mathrm{R}^{d+1})$. There are constants γ, λ_0 and N, depending only on $p(\cdot), K, d, \delta, C_0, \delta_0$, such that under Assumption 5.1(γ) for $V \geq \lambda_0$,*

$$\|\nabla_x^2(L - V)^{-1} f\|_{L^{p(\cdot)}} + \|V(L - V)^{-1} f\|_{L^{p(\cdot)}} + \|V^{1/2}\nabla_x(L - V)^{-1} f\|_{L^{p(\cdot)}} + \|\partial_t(L - V)^{-1} f\|_{L^{p(\cdot)}} \leq N\|f\|_{L^{p(\cdot)}}.$$

COROLLARY 9.6. *Suppose that V satisfy (3.2), and $p(\cdot) \in \mathcal{B}(\mathrm{R}^{d+1})$. There are constants γ, λ_0 and N, depending only on $p(\cdot), K, d, \delta, C_0, \delta_0$, such that under Assumption 5.1(γ) for $V \geq \lambda_0$,*

$$\|\nabla_x(\mathcal{L} - V)^{-1}\nabla_x f\|_{L^{p(\cdot)}} + \|V^{1/2}\nabla_x(\mathcal{L} - V)^{-1} f\|_{L^{p(\cdot)}} + \|V^{1/2}(\mathcal{L} - V)^{-1}\nabla_x f\|_{L^{p(\cdot)}} + \|V(\mathcal{L} - V)^{-1} f\|_{L^{p(\cdot)}} \leq N\|f\|_{L^{p(\cdot)}}.$$

We remark that, in fact, Corollary 1.11 and Theorem 1.2 in [8] are proved in the elliptic case, but it is easy to see that Corollary 1.11 and Theorem 1.2 in [8] are also true in the parabolic case.

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