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THE DYADIC TRACE AND ODD WEIGHT COMPUTATIONS FOR SIEGEL MODULAR CUSP FORMS

CRIS POOR AND DAVID S. YUEN

We define the concept of a special positive matrix. We use the dyadic trace to prove the result that $\dim S_4^k = 0$ for odd $k \leqslant 13$ and that $\dim S_4^{15} \leqslant 4$.

The computation of $\dim S_n^k$, the dimension of the space of Siegel modular cusp forms of degree n and weight k, may be facilitated by the use of the dyadic trace [**5**]. Recall the definition of the dyadic trace: for a positive definite $n \times n$ matrix T, we define the dyadic trace by $w(T) = \sup \sum_i \alpha_i$, where the supremum is taken over all dyadic representations $T = \sum_i \alpha_i \mu_i {}^t\mu_i$ with $\mu_i \in \mathbb{Z}^n \setminus \{0\}$ and positive $\alpha_i \in \mathbb{R}$.

The following result from [**5**] gives an explicit finite set of Fourier coefficients that uniquely determine cusp forms of a given weight. Let $f \in S_n^k$ have Fourier series $f(\Omega) = \sum_T a_T e(\langle T, \Omega \rangle)$, where the summation is over semi-integral positive definite matrices T (this means T has half integer entries but with integer diagonal entries); the notation is standard, see [**1**] or [**5**]. The result is that $f \equiv 0$ if and only if

$$(*) \qquad a_T = 0 \text{ whenever } w(T) \leqslant n \frac{2}{\sqrt{3}} \frac{k}{4\pi}.$$

The paper [**5**] discusses examples for even weights, and this paper addresses the case of odd weights k in S_4^k; namely, we prove the following theorem.

THEOREM. *$S_4^k = 0$ for odd $k \leqslant 13$ and $\dim S_4^{15} \leqslant 4$.*

PROOF: Define a positive definite symmetric $n \times n$ matrix T to be *special positive* if each element of its automorphism group $\mathrm{Aut}_{\mathbb{Z}}(T)$ has determinant 1. This is a class property. The Fourier coefficients of f satisfy

$$(**) \qquad a_{{}^t v T v} = \det(v)^k a_T$$

for all $v \in \mathrm{GL}_n(\mathbb{Z})$ [**2**, p.45]. Note that if $\mathrm{Aut}_{\mathbb{Z}}(T)$ has an element v with determinant -1, then k odd and (**) would imply that $a_T = 0$. Thus for k odd, the support of f consists entirely of special positive T.

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If T has a 1 on its diagonal, then the lattice corresponding to T has an element of norm 2, and so the reflection in this element's orthogonal hyperplane would stabilise the lattice. Thus such a T would have a reflection in $\mathrm{Aut}_{\mathbb{Z}}(T)$, and so such a T would not be special positive. Table 1 gives an initial list of representatives for all classes of special positive semi-integral T ordered by their dyadic traces. In particular, Table 1 contains all T with $w(T) < 6$. Table 1 was constructed using a computer program with Nipp's tables [**3**] as a database.

$w(T)$	$16 \det T$	$\# \mathrm{Aut}(T)$	T_{11}	T_{22}	T_{33}	T_{44}	$2T_{12}$	$2T_{13}$	$2T_{23}$	$2T_{14}$	$2T_{24}$	$2T_{34}$
5	105	8	2	2	2	2	2	1	0	0	1	2
5	121	24	2	2	2	2	2	1	0	1	1	2
5.5	145	4	2	2	2	2	2	1	0	−1	−1	1
5.5	153	4	2	2	2	2	1	1	0	1	1	2
6	161	4	2	2	2	2	2	1	0	0	1	0

Table 1.

Notice that there are no special positive matrices of dyadic trace less than 5. By use of (*), any cusp form $f \in S_4^k$ of odd weight k would vanish if $4(2/\sqrt{3})(k/4\pi) < 5$, which happens if $k < 13.61$. This implies that $S_4^k = 0$ for odd $k \leqslant 13$.

For $f \in S_4^{15}$, f is determined by the Fourier coefficients a_T for the special positive semi-integral classes $[T]$ with $w(T) \leqslant 4(2/\sqrt{3})(15/4\pi)$, which implies $w(T) \leqslant 5.52$. Table 1 shows that there are four such classes, which implies that $\dim S_4^{15} \leqslant 4$. This completes the proof of the theorem. ∎

The result for $k = 11$ is new. The results with $k = 13$ and $k = 9$ were previously proven in [**4**] using the techniques of theta series with pluri-harmonics. For $k = 17$, the dyadic trace bound turns out to imply $w(T) \leqslant 6.25$. The number of classes of special positive matrices with $w(T) \leqslant 6$ is 15. This implies $\dim S_4^{17} \leqslant 15$; but one might suspect the actual dimension is lower.

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Department of Mathematics
Fordham University
Bronx, NY 10458
e-mail: poor@fordham.edu

Math. and Computer Science Department
Lake Forest College
555 N. Sheridan Rd.
Lake Forest, IL 60045
e-mail: yuen@lfc.edu

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