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Discrete Series for p-adic $SO(2n)$ and Restrictions of Representations of $O(2n)$

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Abstract. In this paper we give a classification of discrete series for $SO(2n, F)$, F p-adic, similar to that of Mœglin–Tadić for the other classical groups. This is obtained by taking the Mœglin–Tadić classification for $O(2n, F)$ and studying how the representations restrict to $SO(2n, F)$. We then extend this to an analysis of how admissible representations of $O(2n, F)$ restrict.

1 Introduction

In [M-T] (also, cf. [Mœ2]), Mœglin and Tadić construct the discrete series for a number of families of classical groups. However, they only address discrete series for the split classical group $SO(2n, F)$, not the split ones. The basic reason for this is that the Weyl groups are different: the groups considered by Mœglin and Tadić all have Weyl groups of the form $W \cong \{$ permutations and sign changes on n letters $\}$, whereas the Weyl group for $SO(2n, F)$ requires the number of sign changes to be even. This introduces a number of complications, which we take a moment to discuss.

The complications go beyond simple changes in the combinatorics. For example, one datum that appears in the admissible triples used by Mœglin and Tadić in the classification of discrete series is the partial cuspidal support of an irreducible representation. For $SO(2n, F)$, there is not a corresponding notion of partial cuspidal support (or more precisely, the corresponding partial cuspidal support can consist of more than one representation; see Example 8.1). At a subtler level, for the groups they consider, the $Jord_\rho$ (where $Jord_\rho = \{(\rho', a) \in Jord \mid \rho' \cong \rho\}$) for different ρ are essentially independent of each other (cf. [M-T, Section 14.5] for a more detailed discussion). From the standpoint of [J1, J4], this has its roots in the observation (cf. [G1, G2]) that if $\rho_1, \dots, \rho_k$ are irreducible unitary supercuspidal representations of general linear groups and σ is an irreducible supercuspidal representation of an appropriate classical group, then $\mathrm{Ind}((\rho_1 \otimes \cdots \otimes \rho_1) \otimes \cdots \otimes (\rho_k \otimes \cdots \otimes \rho_k) \otimes \sigma)$ has 2^m components, where $m = |\{i \mid \mathrm{Ind}(\rho_i \otimes \sigma)$ is reducible $\}|$. For $SO(2n, F)$, the situation is different (cf. [G1]), *e.g.*, one can have $\mathrm{Ind}(\rho_1 \otimes \rho_2 \otimes \sigma_0)$ reducible even if both $\mathrm{Ind}(\rho_1 \otimes \sigma_0)$ and $\mathrm{Ind}(\rho_2 \otimes \sigma_0)$ are irreducible. At a more practical level, the μ^* structure of [T2], which figures prominently in the paper, did not have an $SO(2n, F)$ counterpart (though note the subsequent development of such in [J5]).

This leaves two obvious strategies for the classification of discrete series for the groups $SO(2n, F)$. One approach is to emulate the work of Mœglin and Tadić, making the requisite changes along the way. Another approach is to start with the

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Mœglin-Tadić classification of discrete series for the groups $O(2n,F)$ and study restrictions to $SO(2n,F)$. By a lemma of [G-K] (essentially Mackey theory, see Lemma 2.3 below), this is equivalent to studying when $\hat{c}\pi \cong \pi$, where $\hat{c}$ denotes the character of $O(2n,F)$ that is 1 on $SO(2n,F)$ and -1 on $O(2n,F)\setminus SO(2n,F)$. We adopt the latter approach. Note that this requires retaining the Basic Assumption (BA) of [M-T], which we do (though the former approach would certainly require something like this as well). Owing to its somewhat technical nature, we forgo a discussion of their Basic Assumption until Section 3, by which point the necessary background will have been introduced.

We use the results of [J4] (in particular, the extension of [J1] to $O(2n,F)$) to simplify matters. In particular, this reduces the problem of studying restrictions of general irreducible representations to studying restrictions for irreducible representations in $R((\rho,\alpha);\sigma)$, *i.e.*, with supercuspidal support on sets of the form $\{\nu^x\rho,\nu^{-x}\tilde{\rho}\}_{x\in\alpha+\mathbb{Z}}\cup\{\sigma\}$. More precisely, an irreducible π in $R((\rho,\alpha);\sigma)$ appears as a subquotient of some parabolically induced representation of the form $\mathrm{Ind}_P^G(\rho_1\otimes\cdots\otimes\rho_k\otimes\sigma)$ with each $\rho_i\in\{\nu^x\rho,\nu^{-x}\tilde{\rho}\}_{x\in\alpha+\mathbb{Z}}$ (see Section 2 for more). Here, ρ is an irreducible unitary supercuspidal representation of a general linear group, $\nu=|det|$ on a general linear group, σ is an irreducible supercuspidal representation of an even orthogonal group, and $0\le\alpha<1$ (if $\tilde{\rho}\cong\rho$, we take $0\le\alpha\le\frac{1}{2}$).

Now, let $\pi\in R((\rho,\alpha);\sigma)$ be an irreducible representation. By the Langlands classification (see Section 2), we may write $\pi=L(\nu^{x_1}\tau_1\otimes\cdots\otimes\nu^{x_k}\tau_k\otimes\tau)$, with τ_i an irreducible tempered representation of $GL(m_i,F)$, τ an irreducible tempered representation of $O(2m,F)$, and $x_1<\cdots<x_k<0$. Then $\hat{c}\pi\cong L(\nu^{x_1}\tau_1\otimes\cdots\otimes\nu^{x_k}\tau_k\otimes\hat{c}\tau)$ (see Lemma 2.4). In particular, $\hat{c}\pi\cong\pi$ if and only if either $\hat{c}\tau\cong\tau$ or $\tau=1$ (the trivial representation of $O(0,F)$, the trivial group). To address the question of when $\hat{c}\tau\cong\tau$, observe that by a result of Harish-Chandra (extended to $O(2n,F)$ in the appendix), $\tau\hookrightarrow\mathrm{Ind}(\delta_1\otimes\cdots\otimes\delta_\ell\otimes\delta)$, where δ_i is a discrete series representation of $GL(r_i,F)$ and δ is a discrete series representation of $O(2r,F)$. Note that since the inducing representation is unique up to conjugation, (the equivalence class of) δ is uniquely determined by τ. In particular, if $\hat{c}\delta\ncong\delta$, then $\hat{c}\tau\ncong\tau$. However, it is possible to have $\hat{c}\delta\cong\delta$ but still have $\hat{c}\tau\ncong\tau$. To better understand this, as well as motivating the definition of (1.1) below, we consider what is happening at the $SO(2n,F)$ level. Suppose $\hat{c}\delta\cong\delta$. We then have $c\delta_0\ncong\delta_0$, where δ_0 is a component of $\mathrm{Res}^O_{SO}\,\delta$, the restriction of δ to $SO(2r,F)$. Now, let τ_0 be a component of $\mathrm{Res}^O_{SO}\,\tau$. Then

$$\tau_0\hookrightarrow\mathrm{Ind}(\delta_1\otimes\cdots\otimes\delta_\ell\otimes\delta_0),$$

where δ_0 is the appropriate component of $\mathrm{Res}^O_{SO}\,\delta$. As long as $\rho\ncong\tilde{\rho}$ or ρ is a representation of $GL(d,F)$ with d even, the result of Harish-Chandra tells us δ_0 is uniquely determined by τ. (If these fail, one can have $\delta_1\otimes\cdots\otimes\delta_\ell\otimes\delta_0$ conjugate to $\delta_1\otimes\cdots\otimes\delta_\ell\otimes c\delta_0$ in $SO(2n,F)$, so δ_0 need not be uniquely determined.) Under these conditions, we then have $c\tau_0\cong\tau_0$ implies $c\delta_0\cong\delta_0$, which translates to $\hat{c}\tau\ncong\tau$ implies $\hat{c}\delta\ncong\delta$ via the lemma of [G-K]. We remark that this discussion also indicates why we need to allow the possibility of more than one representation in the partial cuspidal support for $SO(2n,F)$.

The analysis breaks into three cases. The simplest case is when $\hat{c}\sigma\ncong\sigma$. In this case, supercuspidal support considerations tell us an irreducible representation in $R((\rho,\alpha);\sigma)$ restricts irreducibly (cf. Theorem 4.1). The second case is when $\hat{c}\sigma \cong \sigma$ or $\sigma = 1$ and the following condition fails:

(1.1) $\rho \cong \tilde{\rho}$ and ρ is a representation of $GL(m,F)$ with m odd.

In this case, we use an approach from [J1] to show that an irreducible representation in $R((\rho,\alpha);\sigma)$ restricts reducibly (cf. Theorem 5.3). Note that in both of these cases, the results follow from general arguments and apply to discrete series; the Mœglin–Tadić classification is not used.

The third case is when $\hat{c}\sigma \cong \sigma$ or $\sigma = 1$, and (1.1) holds. In this case, we show that a nonsupercuspidal discrete series representation in $R((\rho,\alpha);\sigma)$ restricts irreducibly (cf. Theorem 6.5). We note that in this case, one must have $\alpha = 0$ to support discrete series. In this case, the Mœglin–Tadić classification is central to the argument. We then use this along with the fact that non-discrete series irreducible tempered representations embed in induced discrete series to study restrictions of irreducible tempered representations in $R((\rho,\alpha);\sigma)$. We note that in this case only $\alpha = 0$ and $\alpha = \frac{1}{2}$ support tempered representations. When $\alpha = 0$, they restrict irreducibly; when $\alpha = \frac{1}{2}$, they restrict reducibly (cf. Proposition 7.2). Finally, we use the Langlands classification to address the question for irreducible admissible representations. In this case, the restriction is irreducible unless $\alpha = 0$ and certain other conditions on the Langlands data are satisfied (cf. Proposition 7.3).

We take a moment to remark on the conditions arising in the third case ($\hat{c}\sigma \cong \sigma$ or $\sigma = 1$ and (1.1) holding). Let σ_0 be an irreducible representation that occurs in the restriction $\mathrm{Res}^{O}_{SO}\,\sigma$ (with $\sigma_0 = 1$–the trivial representation of the trivial group $SO(0,F)$–if $\sigma = 1$). Then the parabolically induced representations $\mathrm{Ind}(\nu^x\rho \otimes \sigma_0)$ and $\mathrm{Ind}(\nu^x\rho \otimes \sigma)$ of $SO(2n,F)$, $O(2n,F)$, resp., are reducible for the same values of $x \in \mathbb{R}$ except when the conditions for the third case are satisfied. When that happens, $\mathrm{Ind}(\rho \otimes \sigma_0)$ is irreducible, but $\mathrm{Ind}(\rho \otimes \sigma)$ reduces. It is essentially this difference that makes the third case subtler.

To unify the conditions $\hat{c}\sigma \cong \sigma$ and $\sigma = 1$, we formally define $\hat{c}1 = 1$ for the trivial representation of $O(0,F)$. Thus the three cases above become $\hat{c}\sigma \ncong \sigma$, $\hat{c}\sigma \cong \sigma$ with (1.1) failing, and $\hat{c}\sigma \cong \sigma$ with (1.1) holding. In the same spirit, we also use $1 \otimes e$ and $1 \otimes c$ (e and c denoting the usual representatives for $O(2n,F)/SO(2n,F)$; see Section 2) for the trivial representation of $SO(0,F)$, with different interpretations for parabolic induction (see Definition 2.1). These conventions simplify a number of statements in the paper.

It is not a difficult matter to combine the results about restrictions of representations in $R((\rho,\alpha);\sigma)$ to obtain results about restriction of general discrete series; we do this in Section 8. We note that we actually obtain a bit more for discrete series: we can define an action of $\hat{c}$ on the admissible triples that corresponds to the action of $\hat{c}$ on the corresponding discrete series.

We briefly describe the contents section by section. The next section reviews notation and background results. We also introduce the afore-mentioned convention to allow $\sigma = 1$ and $\sigma_0 = 1$ to be dealt with on an equal footing with other representations. In the third section, we discuss the construction of Mœglin–Tadić as well as a variation (of part of the construction) given in [T5, T6] (with some lemmas for later application to $SO(2n,F)$). Section 4 studies restrictions to $SO(2n,F)$ of irreducible representations in $R((\rho,\alpha);\sigma)$ for the case where $\hat{c}\sigma \ncong \sigma$; Section 5 when $\hat{c}\sigma \cong \sigma$ with (1.1) not holding. The more difficult case when $\hat{c}\sigma \cong \sigma$ and (1.1) holds is covered by Sections 6 and 7, with Section 6 addressing discrete series only and Section 7 building on the results of Section 6 to address irreducible tempered representations and irreducible admissible representations in general. In Section 8, we begin to put the pieces together. Theorem 8.4 is the main result on the restriction of discrete series, indicating when discrete series for $O(2n,F)$ reduce upon restriction to $SO(2n,F)$ in terms of their Mœglin–Tadić data. Building up from discrete series, in Section 9 we give corresponding results for the restriction of irreducible tempered representations and irreducible admissible representations. In Sections 10 and 11, we reformulate the results of Section 8 so that the definitions and statements are made without reference to objects for $O(2n,F)$. In Section 10, we define admissible triples for $SO(2n,F)$; in Section 11, we characterize the bijective correspondence between admissible triples and discrete series. We close with an appendix, which extends a result of Harish-Chandra for connected groups to $O(2n,F)$. In particular, it shows that if an irreducible tempered representation τ has $\tau \hookrightarrow \mathrm{Ind}(\delta_1)$ and $\tau \hookrightarrow \mathrm{Ind}(\delta_2)$ with δ_1, δ_2 discrete series of standard parabolic subgroups (cf. Section 2 for what we mean by standard parabolic subgroups for $O(2n,F)$), then δ_1 and δ_2 (and the corresponding Levi factors) are conjugate.

2 Notation and Preliminaries

Let F be a p-adic field with $charF = 0$. We make use of results from [G2] (both directly and indirectly) in this paper, hence need this assumption.

In parts of this paper, we work in the Grothendieck group (*i.e.*, with semisimplified representations) rather than with the actual composition series. To make the distinction, if π_1, π_2 are smooth finite-length representations, we write $\pi_1 = \pi_2$ if π_1 and π_2 have the same irreducible subquotients with same multiplicities. We write $\pi_1 \cong \pi_2$ if π_1 and π_2 are actually equivalent. We write $\pi = \pi_1 + \cdots + \pi_k$ if $m(\rho,\pi) = m(\rho,\pi_1) + \cdots + m(\rho,\pi_k)$ for every irreducible ρ, where $m(\rho,\pi)$ denotes the multiplicity of ρ in π. Similarly, we write $\pi \geq \pi_0$ if $m(\rho,\pi) \geq m(\rho,\pi_0)$ for every such ρ.

We use the notation of [B-Z] in parts of this paper. If $P = MU$ is a standard parabolic subgroup of G, then we let $i_{G,M}$ and $r_{M,G}$ denote the normalized induction and normalized Jacquet functors (or their semisimplifications), resp. We use Ind_{SO}^{O} (resp., Res_{SO}^{O}) for induction (resp., restriction) of representations from $SO(2n,F)$ to $O(2n,F)$ (resp., $O(2n,F)$ to $SO(2n,F)$).

The special orthogonal group $SO(2n,F)$, $n \geq 1$, is the group

$$SO(2n,F) = \{X \in SL(2n,F) \mid {}^{\tau}XX = I_{2n}\}.$$

Here ${}^{\tau}X$ denotes the matrix of X transposed with respect to the second diagonal. For $n = 1$, we get

$$SO(2,F) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \,\middle|\, \lambda \in F^{\times} \right\} \cong F^{\times}.$$

$SO(0,F)$ is defined to be the trivial group. The orthogonal group $O(2n,F)$, $n \geq 1$, is the group

$$O(2n,F) = \left\{ X \in GL(2n,F) \mid {}^{\tau}XX = I_{2n} \right\}.$$

We have $O(2n,F) = SO(2n,F) \rtimes C$, where C is the group $C = \{1, c\}$ with

$$c = \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix} \in O(2n,F),$$

which acts on $SO(2n,F)$ by conjugation. We take $O(0,F)$ to be the trivial group.

We now describe the standard parabolic subgroups of $SO(2n,F)$. First, fix the minimal parabolic subgroup $P_{\varnothing} \subset SO(2n,F)$ consisting of all upper triangular matrices in $SO(2n,F)$. Let $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$ denote the simple roots for $SO(2n,F)$. For a simple root α, let s_α denote the corresponding simple reflection. The standard parabolic subgroups have the form $P_\Phi = \langle P_\varnothing, s_\alpha \rangle_{\alpha \in \Phi}$, where Φ is a subset of Π. If $P = MU$ is a standard parabolic subgroup of $SO(2n,F)$, then $M = GL(m_1,F) \times \cdots \times GL(m_k,F) \times SO(2m,F)$ with $m_1 + \cdots + m_k + m = n$. More precisely, if $\alpha_{n-1}, \alpha_n \notin \Phi$, then $m = 0$ and $m_k = 1$; if exactly one of α_{n-1}, α_n is in Φ, then $m = 0$ and $m_k > 1$; and if $\alpha_{n-1}, \alpha_n \in \Phi$, then $m > 0$ and $m_k > 1$. Note that $c\alpha_i = \alpha_i$ for $i < n-1$ and $c\alpha_{n-1} = \alpha_n$. In particular, if Φ contains exactly one of α_{n-1}, α_n, then P_Φ and $P_{c\Phi} = c(P_\Phi)$ are standard parabolic subgroups that are conjugate in $O(2n,F)$.

For $O(2n,F)$, we use the standard parabolic subgroups used in [M-T, J4, B1], etc., (though this definition is not completely standard). In particular, fix the minimal parabolic subgroup $P_\varnothing \subset O(2n,F)$ consisting of all upper triangular matrices in $O(2n,F)$. Let $S = \{s_{\alpha_1}, \dots, s_{\alpha_{n-1}}, c\}$. The standard parabolic subgroups have the form $P_\Phi = \langle P_\varnothing, s \rangle_{s \in \Phi}$, where Φ is a subset of S. If $P = MU$ is a standard parabolic subgroup of $O(2n,F)$, then $M = GL(m_1,F) \times \cdots \times GL(m_k,F) \times O(2m,F)$ with $m_1 + \cdots + m_k + m = n$.

Suppose that $\rho_1, \dots, \rho_k$ are representations of $GL(m_1,F), \dots, GL(m_k,F)$ and σ is a representation of $O(2m,F)$. Let $G = O(2n,F)$, where $n = m_1 + \cdots + m_k + m$. Let $M = GL(m_1,F) \times \cdots \times GL(m_k,F) \times O(2m,F)$ be a standard Levi subgroup of G. Following [B-Z, T1], set

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma = i_{G,M}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma).$$

For $G = SO(2n,F)$, suppose that $\rho_1, \dots, \rho_k$ are representations of

$$GL(m_1,F), \dots, GL(m_k,F)$$

and σ_0 is a representation of $SO(2m,F)$, with $m > 0$. Let $G = SO(2n,F)$, where $n = m_1 + \cdots + m_k + m$. Again, we write

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma_0 = i_{G,M}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma_0).$$

To allow the trivial representations of $O(0,F), SO(0,F)$ to be dealt with on an equal footing with representations of $O(2n,F), SO(2n,F)$ for $n > 0$, rather than requiring special cases throughout the paper, we introduce a few conventions here. In terms of the actions of $c, \hat{c}$, we would like to have the trivial representation of $O(0,F)$ be fixed under the action of $\hat{c}$ while the trivial representation of $SO(0,F)$ is changed by the action of c. Thus, for the trivial representation of $O(0,F)$, we take $\hat{c}1 = 1$. For the trivial representation of $SO(0,F)$, we introduce the following convention.

Definition 2.1 We let both $1 \otimes e$ and $1 \otimes c$ denote the trivial representation of $SO(0,F)$, but with different interpretations when used with parabolic induction. In particular, suppose $P = MU$ is a standard parabolic subgroup with $\alpha_n \notin \Phi$. Then $M = GL(m_1,F) \times \cdots \times GL(m_k,F)$. For representations $\tau_1,\ldots,\tau_k$ of

$$GL(m_1,F),\ldots,GL(m_k,F),$$

we let $\tau_1 \otimes\cdots\otimes \tau_k \otimes (1\otimes e)$ denote a representation of M, while $\tau_1\otimes\cdots\otimes\tau_k\otimes(1\otimes c)$ denotes a representation of $c(M)$ (the Levi factor of the standard parabolic subgroup $c(P)$). Thus, we write

$$\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes e) = i_{G,M}(\tau_1 \times \cdots \times \tau_k),$$

and

$$\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes c) = c(i_{G,M}(\tau_1 \times \cdots \times \tau_k)),$$

noting that by Lemma 2.4, the latter is equivalent to $i_{G,c(M)}(\tau \otimes \cdots \otimes \tau_k)$. We remark that if $\alpha_{n-1}, \alpha_n \notin \Phi$, then M and $c(M)$ are the same, hence so are $\tau_1\times\cdots\times\tau_k\rtimes(1\otimes e)$ and $\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes c)$. In terms of the action of c, we take $c(1 \otimes e) = 1 \otimes c$ and $c(1 \otimes c) = 1 \otimes e$.

These are interpreted in the obvious way with respect to Ind_{SO}^{O} and Res_{SO}^{O}.

We note that while this simplifies matters in what follows, it has the consequence that rather than having only the trivial representation of $SO(0,F)$ in the discrete series, we have both $1 \otimes e$ and $1 \otimes c$.

We now discuss some structure theory from [Z, T2, B1]. First, let

$$R = \bigoplus_{n\geq 0} \mathcal{R}(GL(n,F)) \quad \text{and} \quad R[O] = \bigoplus_{n\geq 0} \mathcal{R}(O(2n,F)),$$

where $\mathcal{R}(G)$ denotes the Grothendieck group of the category of smooth finite-length representations of G. We define multiplication on R as follows: suppose ρ_1, ρ_2 are representations of $GL(n_1,F), GL(n_2,F)$, resp. We have that $M = GL(n_1,F) \times GL(n_2,F)$ is the Levi factor of a standard parabolic subgroup of $G = GL(n,F)$, where $n = n_1 + n_2$, and set $\tau_1 \times \tau_2 = i_{G,M}(\tau_1 \otimes \tau_2)$. This extends (after semisimplification) to give the multiplication $\times\colon R \times R \longrightarrow R$. To describe the comultiplication on R, let $M_{(i)}$ denote the standard Levi factor for $G = GL(n,F)$ having $M_{(i)} = GL(i,F) \times GL(n-i,F)$. For a representation τ of $GL(n,F)$, we define

$$m^*(\tau) = \sum_{i=0}^{n} r_{M_{(i)},G}\tau,$$

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the sum of semisimplified Jacquet modules (lying in $R \otimes R$). This extends to a map $m^*\colon R \to R \otimes R$. We note that with this multiplication and comultiplication (and antipode map given by the Zelevinsky involution, a special case of the general duality operator of [Aub, S-S]), R is a Hopf algebra. Similarly, if one extends $\rtimes$ from above to a map $\rtimes\colon R \otimes R[O] \to R[O]$, we have $R[O]$ as a module over R. Now, let $M_{(i)} = GL(i, F) \otimes O(2(n-i), F)$, a standard Levi factor for $G = O(2n, F)$. For a representation π of $O(2n, F)$, we define

$$\mu^*(\pi) = \sum_{i=0}^{n} r_{M_{(i)},G}\pi,$$

the sum of semisimplified Jacquet modules (lying in $R \otimes R[O]$). This extends to a map $\mu^*\colon R[O] \to R \otimes R[O]$. This gives $R[O]$ the structure of an M^*-module over R (cf. [B1, T2]):

Theorem 2.2 *Define $M^*\colon R \to R \otimes R$ by*

$$M^* = (m \otimes 1) \circ (\tilde{\ } \otimes m^*) \circ s \circ m^*,$$

where m denotes the multiplication $\times\colon R \otimes R \to R$, $\tilde{\ }$ denotes contragredient, and $s\colon R \otimes R \to R \otimes R$ the extension of the map defined on representations by $s\colon \tau_1 \otimes \tau_2 \mapsto \tau_2 \otimes \tau_1$. Then

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi),$$

where $\rtimes$ on the right-hand side is determined by $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \theta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \theta)$.

As in [B-Z], we set $\nu = |det|$ for general linear groups. Let ρ be an irreducible representation of $GL(n, F)$. We say that ρ is essentially square-integrable (resp., essentially tempered) if there exists $e(\rho) \in \mathbb{R}$ such that $\nu^{-e(\rho)}\rho$ is square-integrable (resp., tempered). If ρ is an irreducible unitary supercuspidal representation of $GL(m, F)$, then $\nu^a\rho \times \nu^{a-1}\rho \times \cdots \times \nu^b\rho$ has a unique irreducible subrepresentation (resp., unique irreducible quotient), which we denote $\delta([\nu^a\rho, \nu^b\rho])$ (resp., $\zeta([\nu^a\rho, \nu^b\rho])$). The representation $\delta([\nu^a\rho, \nu^b\rho])$ is essentially square-integrable, and every irreducible essentially square-integrable representation of a general linear group has this form. Note that $\delta([\nu^a\rho, \nu^b\rho])$ and $\zeta([\nu^a\rho, \nu^b\rho])$ are dual under the duality operator of [Aub, S-S].

We now discuss the Langlands classification (subrepresentation version) and the Casselman criterion, first for $O(2n, F)$, then $SO(2n, F)$. We refer the reader to the appendix of [B-J2] and [M-T, Section 16] for a discussion of how the explicit descriptions below arise from the more general results in [S1, B-W, K, B-J1, C]. We note that $SO(2, F) \cong F^\times$ and $O(2, F) \cong F^\times \rtimes C$. Unitary characters of $F^\times$ then correspond to irreducible representations of $SO(2, F)$ that are square-integrable (in fact, compactly supported) mod center, hence may be viewed as discrete series (or even supercuspidal). Similar considerations apply to the following representations of $O(2, F)$: the trivial representation, $\hat{c}$, and $\mathrm{Ind}_{SO(2,F)}^{O(2,F)}\chi$ with χ a nontrivial unitary character of $F^\times \cong SO(2, F)$. We note that Mœglin and Tadíc [M-T] do not allow such an interpretation in their construction. (In general, if ρ is an irreducible unitary supercuspidal representation of $GL(r,F)$ and σ an irreducible supercuspidal representation of $O(2m,F)$, then the components of $\rho \rtimes \sigma$ are only tempered, not square-integrable. Treating these representations of $O(2,F)$ as non-square-integrable thus allows them to fit this general pattern.) We follow this convention throughout this paper.

We begin with the Langlands classification for $O(2n,F)$. Let $\tau_1,\dots,\tau_k$ be irreducible tempered representations of $GL(m_1,F),\dots,GL(m_k,F)$, resp., and τ an irreducible tempered representation of $O(2m,F)$ (possibly $m=0$ and $\tau=1$). If $x_1<\cdots<x_k<0$, then the induced representation $\nu^{x_1}\tau_1\times\cdots\times\nu^{x_k}\tau_k\rtimes\tau$ has a unique irreducible subrepresentation, which we denote $L(\nu^{x_1}\tau_1\otimes\cdots\otimes\nu^{x_k}\tau_k\otimes\tau)$. Further, every irreducible admissible representation of $O(2n,F)$ may be written uniquely in this form. The Langlands classification for $SO(2n,F)$ is similar: let $\tau_1,\dots,\tau_k$ be irreducible tempered representations of $GL(m_1,F),\dots,GL(m_k,F)$, resp., and τ_0 an irreducible tempered representation of $SO(2m,F)$ (possibly $m=0$ and $\tau_0=1\otimes e$ or $1\otimes c$). If $x_1<\cdots<x_k<0$, then the induced representation $\nu^{x_1}\tau_1\times\cdots\times\nu^{x_k}\tau_k\rtimes\tau$ has a unique irreducible subrepresentation, which we denote $L(\nu^{x_1}\tau_1\otimes\cdots\otimes\nu^{x_k}\tau_k\otimes\tau)$. Further, every irreducible admissible representation of $SO(2n,F)$ may be written uniquely in (exactly) one of these forms.

We now discuss the Casselman criterion for $O(2n,F)$, $n>1$. Suppose π is an irreducible representation of $O(2n,F)$. Suppose $\nu^{x_1}\rho_1\otimes\cdots\otimes\nu^{x_k}\rho_k\otimes\sigma\leq r_{M,G}\pi$ has ρ_i an irreducible unitary supercuspidal representation of $GL(m_i,F)$ for $i=1,\dots,k$, σ an irreducible supercuspidal representation of $O(2m,F)$, and $x_1,\dots,x_k\in\mathbb{R}$. The Casselman criterion tells us that if π is tempered, the following hold:

$$
\begin{aligned}
m_1x_1 &\geq 0\\
m_1x_1+m_2x_2 &\geq 0\\
&\vdots\\
m_1x_1+m_2x_2+\cdots+m_kx_k &\geq 0.
\end{aligned}
$$

Conversely, if these inequalities hold for any such $\nu^{x_1}\rho_1\otimes\cdots\otimes\rho_k\otimes\sigma$ (*i.e.*, ρ_i an irreducible unitary supercuspidal representation of $GL(m_i,F)$ and σ an irreducible supercuspidal representation of $O(2m,F)$) appearing in a Jacquet module of π, then π is tempered. The criterion for square-integrability is the same except that the inequalities are strict. To describe the Casselman criterion for $SO(2n,F)$, $n>1$, suppose π_0 is an irreducible representation of $SO(2n,F)$. Suppose $\nu^{x_1}\rho_1\otimes\cdots\otimes\nu^{x_k}\rho_k\otimes\sigma_0\leq r_{M,G}\pi_0$ has ρ_i an irreducible unitary supercuspidal representation of $GL(m_i,F)$ for $i=1,\dots,k$ and σ_0 an irreducible supercuspidal representation of $SO(2m,F)$ (also requiring that σ_0 be unitary if $m=1$; it is automatically unitarizable otherwise). The Casselman criterion tells us that if π is tempered, the following hold:

$$
\begin{aligned}
m_1x_1 &\geq 0\\
m_1x_1+m_2x_2 &\geq 0\\
&\vdots\\
m_1x_1+m_2x_2+\cdots+m_kx_k &\geq 0.
\end{aligned}
$$

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Conversely, if these inequalities hold for all such $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_k}\rho_k \otimes \sigma_0 \leq r_{M,G}\pi_0$, then π_0 is tempered. The criterion for square-integrability is the same except that the inequalities are strict.

The duality operator of [Aub, S-S] may be extended to the (non-connected) group $G = O(2n, F)$, $n \geq 1$ (cf. [J4]). We define the duality operator D_O by

$$D_O = \sum_{\Phi \subset S} (-1)^{|\Phi|} i_{G,M_\Phi} \circ r_{M_\Phi,G}.$$

Up to sign, D_O sends irreducible representations to irreducible representations. The duality operator for $O(2n, F)$ has the same basic properties as for connected groups (compare [J4, Theorem 6.1] with [Aub, Théorème 1.7]). Further, we have

$$D_{SO} \circ \mathrm{Res}^O_{SO} = \mathrm{Res}^O_{SO} \circ D_O \text{ and } D_O \circ \mathrm{Ind}^O_{SO} = \mathrm{Ind}^O_{SO} \circ D_{SO},$$

where D_{SO} denotes the duality operator for $SO(2n, F)$ (cf. [J4, Proposition 6.3]).

Recall that we let $\hat{c}\colon O(2n, F) \to \{\pm 1\}$ denote the nontrivial character of $O(2n, F)$, $n > 0$. The following is an immediate consequence of the results in [G-K, Section 2] (basically Mackey theory).

Lemma 2.3 *Suppose π, π_0 are irreducible representations of $O(2n, F)$, $SO(2n, F)$, resp., with $n > 0$ and $\pi_0 \leq \mathrm{Res}^O_{SO}\,\pi$. Then exactly one of the following holds:*

(i) *$\hat{c}\pi \cong \pi$, in which case $c\pi_0 \not\cong \pi_0$ and we have*

$$\mathrm{Ind}^O_{SO}\,\pi_0 \cong \mathrm{Ind}^O_{SO}\,c\pi_0 \cong \pi \textit{ and } \mathrm{Res}^O_{SO}\,\pi \cong \pi_0 \oplus c\pi_0.$$

(ii) *$\hat{c}\pi \not\cong \pi$, in which case $c\pi_0 \cong \pi_0$ and we have*

$$\mathrm{Ind}^O_{SO}\,\pi_0 \cong \pi \oplus \hat{c}\pi \textit{ and } \mathrm{Res}^O_{SO}\,\pi \cong \mathrm{Res}^O_{SO}\,\hat{c}\pi \cong \pi_0.$$

We now summarize some basic properties of the actions of $c, \hat{c}$.

Lemma 2.4 (i) *If M (resp., M_0) is a standard Levi factor for $G = O(2n, F)$ (resp., $G_0 = SO(2n, F)$) with $n > 1$ and θ (resp., θ_0) an admissible representation of M (resp., M_0), we have*

$$i_{G,M}\hat{c}\theta \cong \hat{c} \circ i_{G,M}\theta \textit{ and } i_{G_0,c(M_0)}c\theta_0 \cong c \circ i_{G_0,M_0}\theta_0.$$

In particular,

$$\hat{c}(\lambda_1 \times \cdots \times \lambda_k \rtimes \lambda) \cong \lambda_1 \times \cdots \times \lambda_k \rtimes \hat{c}\lambda,$$

and

$$c(\lambda_1 \times \cdots \times \lambda_k \rtimes \lambda_0) \cong \lambda_1 \times \cdots \times \lambda_k \rtimes c\lambda_0.$$

(ii) *If M (resp., M_0) is a standard Levi factor for $G = O(2n, F)$ (resp., $G_0 = SO(2n, F)$) with $n > 1$ and π (resp., π_0) an admissible representation of G (resp., G_0), we have*

$$r_{M,G}\hat{c}\pi \cong \hat{c} \circ r_{M,G}\pi \textit{ and } r_{c(M_0),G_0}c\pi_0 \cong c \circ r_{M_0,G_0}\pi.$$

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(iii) *The duality operators* D_O *and* D_{SO} *satisfy*

$$c \circ D_{SO} = D_{SO} \circ c \text{ and } \hat{c} \circ D_O = D_O \circ \hat{c}.$$

(iv)

$$\hat{c}L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \tau) = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \hat{c}\tau)$$

and

$$cL(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \tau_0) = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes c\tau_0).$$

Proof The only one of these claims that is not already in the literature is the first part of (iii), which we check:

$$\begin{aligned} D_{SO} \circ c(\pi_0) &= \sum_{I \subset \Pi} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G}(c\pi_0) \\ &= \sum_{I \subset \Pi} (-1)^{|I|} c \circ i_{G,M_{c(I)}} \circ r_{M_{c(I)},G}(\pi_0) \end{aligned}$$

by (i) and (ii) (noting $c^{-1} = c$). Since $c(\Pi) = \Pi$, as I runs through the subsets of Π, so does $c(I)$. It then follows that

$$D_{SO} \circ c(\pi_0) = c \circ \sum_{I \subset \Pi} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G}(\pi_0) = c \circ D_{SO}(\pi_0),$$

as needed.

As for the remaining results, (i) follows from [B-Z, Proposition 1.9(f)] and [B2, Corollary 4.1]. [J5, Lemma 3.2] gives (ii). The second part of (iii) is [J4, Corollary 6.4]. For (iv), see [B-J1, Proposition 4.5] and [B-J2, Lemma 4.6], which prove the same result for a slightly different form of the Langlands classification; the same considerations apply here. ■

We now discuss cuspidal reducibility for orthogonal groups and special orthogonal groups. Let ρ be an irreducible unitary supercuspidal representation of $GL(m, F)$ and σ_0 an irreducible supercuspidal representation of $SO(2r, F)$. If $\rho \otimes \sigma_0 \not\cong \tilde{\rho} \otimes c^m \sigma_0$, then $\nu^x \rho \rtimes \sigma_0$ is irreducible for all $x \in \mathbb{R}$; if $\rho \otimes \sigma_0 \cong \tilde{\rho} \otimes c^m \sigma_0$, there is a unique $x \geq 0$ such that $\nu^x \rho \rtimes \sigma_0$ is reducible. We call this value of x the cuspidal reducibility point for (ρ, σ) and, in a variation on the notation in [Mœ1], denote it by $\text{red}(\rho; \sigma_0)$ (if $\nu^x \rtimes \sigma_0$ is irreducible for all $x \in \mathbb{R}$, we write $\text{red}(\rho; \sigma_0) = \infty$). The uniqueness of x is a consequence of the results in [S2]; the Basic Assumption of [M-T] (which we also assume) implies that $x \in \frac{1}{2}\mathbb{Z}$. Characterizations of the particular value of x where reducibility occurs (assuming certain conjectures) are given in [Mœ1] and [Zh]; for the case of $\rho \otimes \sigma_0$ generic, see [Sh1, Sh2]. A corresponding result may be deduced for the (non-connected) orthogonal groups (cf. [B-J2, Corollary 4.4]): let σ be an irreducible supercuspidal representation of $O(2r, F)$ (allowing $r = 0$ and $\sigma = 1$). If $\rho \not\cong \tilde{\rho}$, then $\nu^x \rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$; if $\rho \cong \tilde{\rho}$, there is a unique $x \geq 0$ such that $\nu^x \rho \rtimes \sigma$ is reducible. Again, we denote this value of x by $\text{red}(\rho; \sigma)$. The proposition below (cf. [B-J2, Theorem 4.3]) relates the cuspidal reducibility points for orthogonal and special orthogonal groups.

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Proposition 2.5 *Suppose σ_0, σ are irreducible supercuspidal representations of $SO(2r, F)$, $O(2r, F)$, resp., with $\sigma_0 \leq \mathrm{Res}^{O}_{SO}\,\sigma$. Suppose ρ is an irreducible unitary supercuspidal representation of $GL(m, F)$. Then $\mathrm{red}(\rho;\sigma) = \mathrm{red}(\rho;\sigma_0)$ (possibly infinite) unless the following hold:* (i) *$\rho \cong \tilde{\rho}$ with m odd, and* (ii) *$\sigma_0 \ncong c\sigma_0$. In this case, $\mathrm{red}(\rho;\sigma_0) = \infty$, but $\mathrm{red}(\rho;\sigma) = 0$.*

Note that condition (ii) is equivalent to $\sigma \cong \hat{c}\sigma$ (including $\sigma = 1$); condition (i) is just (1.1).

Our analysis of discrete series uses the results of [J4, Section 7] (which extends the correspondence of [J1] to orthogonal groups). We take a moment to recall this correspondence, as well as discuss how it behaves under the action of $\hat{c}$.

Let ρ be an irreducible unitary supercuspidal representation of $GL(n, F)$, $\alpha \in \mathbb{R}$. Set

$$\mathcal{S}(\rho,\alpha) = \{\nu^{\beta}\rho, \nu^{-\beta}\tilde{\rho}\}_{\beta\in\alpha+\mathbb{Z}}.$$

If $\rho \cong \tilde{\rho}$, we may take $0 \leq \alpha \leq \frac{1}{2}$; otherwise $0 \leq \alpha < 1$. Suppose $\rho_1, \rho_2, \dots, \rho_m$ are irreducible, unitary, supercuspidal representations of $GL(n_1, F), \dots, GL(n_m, F)$, and $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$, with $0 \leq \alpha_i \leq \frac{1}{2}$ if $\rho_i \cong \tilde{\rho}_i$, $0 \leq \alpha_i < 1$ if not. Further, assume that $\mathcal{S}(\rho_1,\alpha_1), \dots, \mathcal{S}(\rho_m,\alpha_m)$ are disjoint. We let $R((\rho_1,\alpha_1), \dots, (\rho_m,\alpha_m))$ denote the Hopf subalgebra of R generated by representations with supercuspidal support in $\mathcal{S}(\rho_1,\alpha_1) \cup \mathcal{S}(\rho_2,\alpha_2) \cup \dots \cup \mathcal{S}(\rho_m,\alpha_m)$. We note that every irreducible representation of $O(2n, F)$ has supercuspidal support on a set of the form $\mathcal{S}(\rho_1,\alpha_1) \cup \mathcal{S}(\rho_2,\alpha_2) \cup \dots \cup \mathcal{S}(\rho_m,\alpha_m) \cup \{\sigma\}$, where σ is an irreducible supercuspidal representation of some $O(2r, F)$. Let $R((\rho_1,\alpha_1), \dots, (\rho_m,\alpha_m);\sigma) \subset R[O]$ be generated by representations with supercuspidal support in $\mathcal{S}(\rho_1,\alpha_1) \cup \mathcal{S}(\rho_2,\alpha_2) \cup \dots \cup \mathcal{S}(\rho_m,\alpha_m) \cup \{\sigma\}$, an M^*-module over $R((\rho_1,\alpha_1), \dots, (\rho_m,\alpha_m))$.

Note that $R((\rho,\alpha);\sigma)$ contains the supercuspidal representation σ; it supports non-supercuspidal discrete series if and only if $\alpha \equiv \mathrm{red}(\rho;\sigma)\, mod 1$ (requiring $\mathrm{red}(\rho;\sigma) < \infty$, so $\rho \cong \tilde{\rho}$–cf. [T4, Theorem 6.2] and [J2, Proposition 4.3.1]). It follows that $R((\rho,\alpha);\sigma)$ supports non-supercuspidal tempered representations if and only if $\alpha = 0$ or $\frac{1}{2}$.

We now recall some results from [J1, J4]. Suppose

$$\pi \in R\big((\rho_1,\alpha_1), (\rho_2,\alpha_2), \dots, (\rho_m,\alpha_m);\sigma\big)$$

is an irreducible representation. Then there exist irreducible representations $\tau_1, \tau_2, \dots, \tau_{m-1}$ of $GL(k_1, F), GL(k_2, F), \dots, GL(k_{m-1}, F)$, and an irreducible representation θ_m of $O(2k_m + r, F)$ such that

(i) $\pi \hookrightarrow \tau_1 \times \tau_2 \times \dots \times \tau_{m-1} \rtimes \theta_m$
(ii) $\tau_i \in R(\rho_i,\alpha_i)$ and $\theta_m \in R((\rho_m,\alpha_m);\sigma)$.

Further, θ_m is unique. Similarly, one could single out $(\rho_1,\alpha_1), \dots, (\rho_{m-1},\alpha_{m-1})$, resp., to produce $\theta_1, \dots, \theta_{m-1}$, resp., in $R((\rho_1,\alpha_1);\sigma), \dots, R((\rho_{m-1},\alpha_{m-1});\sigma)$, resp. Write $\psi_{(\rho_i,\alpha_i)}(\pi) = \theta_i$.

Definition 2.6 For a representation $\pi \in R((\rho_1,\alpha_1), (\rho_2,\alpha_2), \dots, (\rho_m,\alpha_m);\sigma)$, let $\mu^*_{(\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)}(\pi)$ denote the sum of every $\tau \otimes \theta \in \mu^*(\pi)$ such that

$$\tau \in R\big((\rho_1,\alpha_1), \dots, (\rho_k,\alpha_k)\big) \quad \text{and} \quad \theta \in R\big((\rho_{k+1},\alpha_{k+1}), \dots, (\rho_m,\alpha_m);\sigma\big).$$

Similarly, for a representation

$$\lambda \in R\big((\rho_1,\alpha_1),(\rho_2,\alpha_2),\dots,(\rho_m,\alpha_m)\big),$$

let $M^*_{(\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)}(\lambda)$ denote the sum of every $\tau\otimes\tau'$ in $M^*(\lambda)$ such that

$$\tau \in R\big((\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)\big) \quad \text{and} \quad \tau' \in R\big((\rho_{k+1},\alpha_{k+1}),\dots,(\rho_m,\alpha_m)\big).$$

We summarize the results we need in the following theorem. Additional properties may be found in [J1, Theorem 9.3, Proposition 9.8], and the refinements in [J1, Section 10].

Theorem 2.7 *Suppose (ρ_i,α_i) and σ are as above. Let $\mathrm{Irr}((\rho_1,\alpha_1),\dots,(\rho_m,\alpha_m);\sigma)$ denote the set of all irreducible representations of all $O(2n,F)$, $n\ge 0$, supported on $\mathcal{S}(\rho_1,\alpha_1)\cup\cdots\cup\mathcal{S}(\rho_m,\alpha_m)\cup\{\sigma\}$ and similarly for $\mathrm{Irr}((\rho_i,\alpha_i);\sigma),\dots,\mathrm{Irr}((\rho_m,\alpha_m);\sigma)$. Then the map*

$$\pi \longmapsto \psi_{(\rho_1,\alpha_1)}(\pi)\otimes\cdots\otimes\psi_{(\rho_m,\alpha_m)}(\pi)$$

implements a bijective correspondence

$$\mathrm{Irr}\big((\rho_1,\alpha_1),\dots,(\rho_m,\alpha_m);\sigma\big) \longleftrightarrow \mathrm{Irr}\big((\rho_1,\alpha_1);\sigma\big)\otimes\cdots\otimes\mathrm{Irr}((\rho_m,\alpha_m);\sigma\big).$$

We let Ψ denote the inverse map. We have the following:

(i) *With notation as in Definition 2.6,*

$$\mu^*_{(\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)}(\tau\rtimes\theta) = M^*_{(\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)}(\tau)\rtimes\mu^*_{(\rho_1,\alpha_1),\dots,(\rho_k,\alpha_k)}(\theta).$$

(ii) *Suppose we have irreducible representations $\tau\in R((\rho_1,\alpha_1),\dots,(\rho_i,\alpha_i))$ and $\theta\in R((\rho_{i+1},\alpha_{i+1}),\dots,(\rho_m,\alpha_m);\sigma)$. If $\tau\rtimes\sigma=\sum_j m_j\theta_j$ (a sum of irreducible representations with multiplicities), then*

$$\tau\rtimes\theta = \sum_j m_j\Psi\left(\psi_{(\rho_1,\alpha_1)}(\theta_j),\dots,\psi_{(\rho_i,\alpha_i)}(\theta_j),\psi_{(\rho_{i+1},\alpha_{i+1})}(\theta),\dots,\psi_{(\rho_m,\alpha_m)}(\theta)\right).$$

(iii) $\psi_{(\rho_i,\alpha_i)}(\hat{\pi}) = \widehat{\psi_{(\rho_i,\alpha_i)}(\pi)}$.

(iv) *An irreducible representation $\pi\in R((\rho_1,\alpha_1),\dots,(\rho_m,\alpha_m);\sigma)$ is tempered (resp., square-integrable) if and only if $\psi_{(\rho_1,\alpha_1)}(\pi),\dots,\psi_{(\rho_m,\alpha_m)}(\pi)$ are all tempered (resp., square-integrable).*

(v) $\psi_{(\rho,\alpha)}(\hat{c}\pi)=\hat{c}\left(\psi_{(\rho,\alpha)}(\pi)\right)$ *(noting $\psi_{(\rho,\alpha)}(\hat{c}\pi)\in R((\rho,\alpha);\hat{c}\sigma)$).*

(vi) *Suppose that in the Langlands classification,*

$$\pi((\rho_i,\alpha_i);\sigma) = L(\nu^{x_1}\tau_1(\rho_i,\alpha_i)\otimes\cdots\otimes\nu^{x_k}\tau_k(\rho_i,\alpha_i)\otimes T((\rho_i,\alpha_i);\sigma))$$

for $i=1,\dots,m$. (NB: $\tau_j(\rho_i,\alpha_i)$ may be the trivial representation of $GL(0,F)$; $T(\rho_i,\alpha_i;\sigma)$ may just be σ.) Then

$$\begin{aligned}&\Psi\big((\pi(\rho_1,\alpha_1);\sigma),\dots,\pi((\rho_m,\alpha_m);\sigma))\big) = \\ &L\Big(\nu^{x_1}\big(\tau_1(\rho_1,\alpha_1)\times\cdots\times\tau_1(\rho_m,\alpha_m)\big)\otimes\cdots\otimes\nu^{x_k}\big(\tau_k(\rho_1,\alpha_1)\times\cdots\times\tau_k(\rho_m,\alpha_m)\big) \\ &\qquad\otimes\Psi\big(T\big((\rho_1,\alpha_1);\sigma\big),\dots,T\big((\rho_m,\alpha_m);\sigma\big)\big)\Big).\end{aligned}$$

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Proof The only claim not already in the literature is (v), which follows immediately from the characterization of $\psi_{(\rho,\alpha)}(\pi)$ above and Lemma 2.4. The remaining claims are in [J1] (cf. [J4, Theorem 7.2 and Remark 7.3]): see [J1, Proposition 7.4, Theorem 9.3 (4),(6),(7),(8)] and their refinements from Section 10 of [J1]. ∎

We take a moment to remark on a key obstacle to extending this result to $SO(2n, F)$, an issue that also arises when reformulating our results in terms of $SO(2n, F)$ only.

Remark 2.8 Suppose $\rho_1, \dots, \rho_\ell$ as above are pairwise inequivalent and self-contragredient. Let σ be as above with $\sigma_0 \le \mathrm{Res}^O_{SO}\,\sigma$ irreducible. Let us assume that $\rho_i \rtimes \sigma$ is reducible for each i. Set

$$I = \underbrace{\rho_1 \times \cdots \times \rho_1)}_{k_1} \times \cdots \times \underbrace{(\rho_\ell \times \cdots \times \rho_\ell)}_{k_\ell} \rtimes \sigma$$

and

$$I_0 = \underbrace{\rho_1 \times \cdots \times \rho_1)}_{k_1} \times \cdots \times \underbrace{(\rho_\ell \times \cdots \times \rho_\ell)}_{k_\ell} \rtimes \sigma_0.$$

By [G2], I has 2^ℓ components, pairwise inequivalent. Thus, choosing a component of $\rho_i \rtimes \sigma$ for each i is equivalent to choosing a component of I. More precisely, write $\rho_i \rtimes \sigma \cong T_1(\rho_i; \sigma) \oplus T_{-1}(\rho; \sigma)$. Then the choice of $\mathcal{T} \le I$ is equivalent to choosing $T_{\varepsilon_i}(\rho_i; \sigma)$, $1 \le i \le \ell$, where

$$\mathcal{T} \le \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes T_{\varepsilon_i}(\rho_i, \sigma)$$

and

$$\mathcal{T} \not\le \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes T_{-\varepsilon_i}(\rho_i, \sigma).$$

On the other hand, suppose $\hat{c}\sigma \cong \sigma$. If the ρ_i are enumerated so that $\rho_1, \dots, \rho_m$ do not satisfy (1.1) and $\rho_{m+1}, \dots, \rho_\ell$ do, then it follows from Proposition 2.5 that $\rho_i \rtimes \sigma_0$ is reducible for $1 \le i \le m$ and irreducible for $m + 1 \le i \le \ell$. By [G1, Theorems 6.8 and 6.11], I_0 has $2^{\ell-1}$ components. In particular, a choice of components of $\rho_i \rtimes \sigma_0$ for each i is not, in general, equivalent to a choice of components of I_0.

This issue arises when reformulating our results in terms of $SO(2n, F)$ data only (see the end of Section 3 as well as Remark 11.2 and the discussion immediately preceding it). It also is one of the key issues in preventing the preceding theorem from extending to $SO(2n, F)$; it represents a sort of interaction (in terms of reducibility) that prevents the different ρ_i from being treated separately. The identification of a component of I with components of $\rho_i \rtimes \sigma$ is essentially a special case of the correspondence in the theorem, one which is a starting point in proving the theorem.

3 The Mœglin–Tadić Construction

In this section, we review the construction of [M-T] for $O(2n, F)$. (This discussion also borrows freely from the review of the Mœglin–Tadić construction given in [Mu2].) There is an alternate characterization of part of the construction, given in [T5, T6], which we also discuss. We then close with some of lemmas for later use with $SO(2n, F)$.

Let π be an irreducible admissible representation of $O(2n, F)$. If π is not supercuspidal, we may write

$$\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_\ell}\rho_\ell \rtimes \sigma,$$

with $x_1, \ldots, x_\ell \in \mathbb{R}$, $\rho_1, \ldots, \rho_\ell$ irreducible, unitary, supercuspidal representations of general linear groups, and σ an irreducible supercuspidal representation of an orthogonal group (possibly $\sigma = 1$). Recall that Mœglin–Tadić do not treat representations of $O(2, F)$ as supercuspidal, so do not allow a representation of $O(2, F)$ to be the partial cuspidal support. Since $SO(2, F) \cong F^\times$, an irreducible representation of $O(2, F)$ can be embedded in a representation of the form $\mathrm{Ind}_{SO}^{O}\chi \cong \chi \rtimes 1$ with χ a character of $F^\times$. Thus we may take $\sigma = 1$ as the partial cuspidal support. The σ that appears is unique, and the partial cuspidal support of π is defined to be this σ.

Let δ be a discrete series representation for $O(2n, F)$. *Jord*(δ) is defined to be the set of pairs (ρ, a), where ρ is an irreducible unitary supercuspidal representation of a general linear group having $\rho \cong \tilde{\rho}$ and $a \in \mathbb{N}$, which satisfy the following:

(i) a is even if and only if the L-function $L(\rho, R_d, s)$ has a pole at $s = 0$. Here, for ρ a representation of $GL(d, F)$, $L(\rho, R_d, s)$ is the L-function defined by Shahidi (cf. [Sh1, Sh2]), where R_d the representation of $GL(d, \mathbb{C})$ on $\wedge^2\mathbb{C}^d$.

(ii) $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \delta$ is irreducible.

We note that the first condition ensures the parity of a matches the parity of $2\,\mathrm{red}(\rho; \sigma) + 1$. (Notice that the parity does not depend on σ, though the particular reducibility value does.) In the second condition, we note that

$$\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \sigma$$

is reducible for all $a \in \mathbb{N}$ having the correct parity and that satisfy $a \geq 2\,\mathrm{red}(\rho; \sigma) + 1$. Replacing σ with δ essentially produces irreducibility at the values of a that correspond to the segment ends for the generalized Steinberg representations of general linear groups that occur in the construction of δ (noting that some of these irreducibility points may correspond to segments that are degenerate, so do not actually appear in the construction). (Modulo these degenerate segments, the values of $\frac{a-1}{2}$ correspond to the nonnegative values of the a_i, b_i in [J3, Theorem 1.1]. This would allow for an alternate characterization of *Jord*(δ) in terms of Jacquet modules of π rather than induced representations built from π.)

We remark that, for convenience, we use representations in the following description of admissible triples when we actually want equivalence classes of representations; the reader should interpret the discussion below accordingly. (Working this way saves us from having to make a somewhat awkward but obvious definition of equivalence of triples.)

Let Trip denote the collection of all triples $(Jord, \sigma, \varepsilon)$ which satisfy the following:

(i) *Jord* is a finite set of pairs (ρ, a), where ρ is an irreducible unitary supercuspidal representation of a general linear group having $\tilde{\rho} \cong \rho$, and $a \in \mathbb{N}$ with a even if and only if $L(s, \rho, R_{d_\rho})$ has a pole at $s = 0$.

(ii) σ is an irreducible supercuspidal representation of an even orthogonal group.
(iii) $\varepsilon\colon S \to \{\pm 1\}$ is a function on a subset $S \subset Jord \cup (Jord \times Jord)$ that satisfies certain conditions, which we discuss in more detail momentarily.

Let us start by describing the domain S of ε. S contains all $(\rho, a) \in Jord$ except those having a odd and $(\rho, a') \in Jord(\sigma)$ for some $a' \in \mathbb{N}$; S contains $\big((\rho, a), (\rho', a')\big) \in Jord \times Jord$ when $\rho \cong \rho'$ and $a \neq a'$. Several compatibility conditions must also be satisfied:

(i) if $(\rho, a), (\rho, a') \in S$, we must have $\varepsilon((\rho, a), (\rho, a')) = \varepsilon(\rho, a)\varepsilon^{-1}(\rho, a')$;
(ii)

$$\varepsilon((\rho, a), (\rho, a'')) = \varepsilon((\rho, a), (\rho, a'))\varepsilon((\rho, a'), (\rho, a''))$$

for all (ρ, a), (ρ, a'), $(\rho, a'') \in Jord$ having a, a', a'' distinct; and
(iii) $\varepsilon((\rho, a), (\rho, a')) = \varepsilon((\rho, a'), (\rho, a))$ for all $\big((\rho, a), (\rho, a')\big) \in S$.

We follow the notation of [M-T] and, in light of (i) above, write $\varepsilon(\rho, a)\varepsilon^{-1}(\rho, a')$ for $\varepsilon\left((\rho, a), (\rho, a')\right)$ even when ε is undefined on (ρ, a) and (ρ, a') separately (*i.e.*, even when (ρ, a) and (ρ, a') are not in S).

We now discuss triples of alternated type. Suppose $(\rho, a) \in Jord$. We define (ρ, a_-) by taking $a_- = \max\{a' \in \mathbb{N} \mid (\rho, a') \in Jord \text{ and } a' < a\}$, noting that (ρ, a_-) may be undefined. Also, let us write $Jord_\rho = \{(\rho', a) \in Jord \mid \rho' \cong \rho\}$ and $Jord_\rho(\sigma) = \{(\rho', a) \in Jord(\sigma) \mid \rho' \cong \rho\}$. We call $(Jord, \sigma, \varepsilon) \in$ Trip a triple of alternated type if the following hold:

(i) $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = -1$ whenever (ρ, a_-) is defined, and
(ii) $|Jord_\rho| = |Jord'_\rho(\sigma)|$, where

$$Jord'_\rho(\sigma) = \begin{cases} Jord_\rho(\sigma) \cup \{(\rho, 0)\} & \text{if } a \text{ is even and } \varepsilon(\rho, min\ Jord_\rho) = 1, \\ Jord_\rho(\sigma) & \text{otherwise.} \end{cases}$$

We write Trip_{alt} for the subset of all alternated triples in Trip.

This brings us to admissible triples. First, suppose $(Jord, \sigma, \varepsilon) \in$ Trip has $(\rho, a) \in Jord$ with (ρ, a_-) defined and $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$. Set

$$Jord' = Jord \setminus \{(\rho, a), (\rho, a_-)\}$$

and let ε' be the restriction of ε to $S \cap [Jord' \cup (Jord' \times Jord')]$. One can check that $(Jord', \sigma, \varepsilon') \in$ Trip. We say that $(Jord', \sigma, \varepsilon')$ is subordinate to $(Jord, \sigma, \varepsilon)$. We say the triple $(Jord, \sigma, \varepsilon)$ is admissible if there is a sequence of triples $(Jord_i, \sigma, \varepsilon_i)$, $1 \leq i \leq k$, such that

/rm(i) $(Jord_1, \sigma, \varepsilon_1) = (Jord, \sigma, \varepsilon)$,
/rm(ii) $(Jord_{i+1}, \sigma, \varepsilon_{i+1})$ is subordinate to $(Jord_i, \sigma, \varepsilon_i)$ for all $1 \leq i \leq k-1$, and
/rm(iii) $(Jord_k, \sigma, \varepsilon_k)$ is of alternated type.

Let us call such a sequence of triples an admissible sequence (for $(Jord, \sigma, \varepsilon)$ or the discrete series representation associated to $(Jord, \sigma, \varepsilon)$ by Mœglin and Tadić). We write Trip_{adm} for the set of admissible triples.

Mœglin and Tadić establish a bijection between the set of all equivalence classes of discrete series for orthogonal groups (not including $O(2, F)$) and the set of all admissible triples. We now describe that correspondence. If δ is a discrete series representation for an orthogonal group, we write $(Jord(\delta), \sigma_\delta, \varepsilon_\delta)$ for the associated admissible triple. Here, $Jord(\delta)$ is as above and σ_δ is the partial cuspidal support of δ. It remains to describe ε_δ.

We first describe ε_δ on pairs. Suppose $(\rho, a) \in Jord_\delta$ with a_- defined. Then

(3.1)
$$\varepsilon_\delta(\rho, a)\varepsilon_\delta^{-1}(\rho, a_-) = 1$$
$$\Updownarrow$$
$$\text{there is an irreducible representation } \delta' \text{ such that } \delta \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta'.$$

This property is sufficient to define ε_δ on that part of S contained in $Jord(\delta) \times Jord(\delta)$ (use property (iii)(ii) from the definition of triple above). Now, suppose $(\rho, a) \in Jord(\delta)$ with a even. We then formally set $\varepsilon_\delta(\rho, 0) = 1$; (3.1) is then sufficient to determine $\varepsilon_\delta(\rho, a)$ for all such (ρ, a). If $(\rho, a) \in S$ with a odd (in which case there is no b with $(\rho, b) \in Jord(\sigma_\delta)$), we use normalized standard intertwining operators to define $\varepsilon_\delta(\rho, a)$ (cf. [Mœ2, Proposition 6.1]; the normalizations are taken from [Mœ1]). In particular, since $\delta([\nu^{-a+1/2}\rho, \nu^{a-1/2}\rho]) \rtimes \delta$ is irreducible, the normalized standard intertwining operator that sends

$$\nu^s\delta\Big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\Big) \rtimes \delta \longrightarrow \nu^{-s}\delta\Big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\Big) \rtimes \delta$$

is a scalar at $s = 0$. More precisely, if we let $\iota\colon g \longmapsto {}^{\tau}g^{-1}$ on general linear groups, the right-hand side is

$$i_{G,M}\Big(w_0\big(\nu^s\delta\big([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]\big) \otimes \delta\big)\Big) = \nu^s\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])) \circ \iota \rtimes \delta,$$

where M is the appropriate standard Levi factor, and if $M = GL(m, F) \times O(2n, F)$, w_0 corresponds to sign changes on the first m (diagonal) entries. Starting with a nonzero map $E\colon V_\rho \to V_\rho$ that intertwines ρ and $\rho \circ \iota$ (noting $\rho \circ \iota \cong \tilde{\rho}$) and has $E^2 = I$, we obtain a nonzero map for the equivalence

$$\delta\big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\big) \circ \iota \cong \delta\big(\big[\nu^{\frac{-a+1}{2}}\rho \circ \iota, \nu^{\frac{a-1}{2}}\rho \circ \iota\big]\big) \cong \delta\big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\big),$$

and then a map $\mathcal{E}$ giving the equivalence

$$\delta\big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\big) \rtimes \delta \cong \delta\big(\big[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\big]\big) \circ \iota \rtimes \delta.$$

Since $E^2 = I$, we have $\mathcal{E}^2 = I$. At $s = 0$, the normalized standard intertwining operator is a scalar multiple of $\mathcal{E}$; we let $\varepsilon_\delta(\rho, a)$ denote this scalar, necessarily ± 1. (We note that by [Mœ2], this is consistent with the characterization of $\varepsilon_\delta(\rho, a)\varepsilon_\delta^{-1}(\rho, a_-)$ above.) Mœglin-Tadić note that the function ε_δ is only partially defined in this case, which is due to the necessity of making a choice (of $\pm E$) in this process.

Before proceeding further, let us take a moment to recall the Basic Assumption under which the Mœglin and Tadić construction is done, and which we retain. Let ρ be an irreducible unitary supercuspidal representation of a general linear group having $\rho \cong \tilde{\rho}$ and σ an irreducible supercuspidal representation of some $O(2n, F)$. Then there is a unique nonnegative $x_\rho \in \mathbb{R}$ such that $\nu^{x_\rho}\rho \rtimes \sigma$ reduces (cf. [S2] and Proposition 2.5). The basic assumption is

$$x_\rho = \begin{cases} (a_{\rho,\max} + 1)/2 & \text{if } Jord_\rho(\sigma) \neq \varnothing, \\ \frac{1}{2} & \text{if } L(\rho, R_{d_\rho}, s) \text{ has a pole at } s = 0 \text{ and } Jord_\rho(\sigma) = \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{\rho,\max}$ is the largest value of a for which $(\rho, a) \in Jord$. The reader is referred to [M-T, Section 12] for more on this assumption.

In [T5,T6], Tadić gives another way of defining $\varepsilon_\delta(\rho, a)$ when $(\rho, a) \in S$ with a odd (cf. [T5, Sections 16.5 and 16.6]; with proofs and refinements in [T6]). In this case, the choice needed to fix ε on $Jord_\rho$ is a choice of components of $\rho \rtimes \sigma_\delta$ (which allows for a nice interpretation in terms of some available structures; cf. [J2, Section 3.3]). This definition will be useful when we work with admissible triples for $SO(2n, F)$ (cf. Section 11). We next review this definition.

First, we note that in his definition, Tadić uses the decomposition of [J1, J4] to restrict to the case where $\delta \in R((\rho, \alpha); \sigma)$ (with $\alpha = \text{red}(\rho; \sigma)$). We make this assumption while reviewing his construction. However, for $SO(2n, F)$, we do not have such a decomposition, so we need to work more generally. Thus, after reviewing Tadić's definition, we give some lemmas for later applications to $SO(2n, F)$.

Remark 3.1 With notation as in Section 2, let $\pi_i \in R(\rho_i, \alpha_i); \sigma)$, $1 \leq i \leq m$, be discrete series and $\pi \in R((\rho_1, \alpha_1), \dots, (\rho_m, \alpha_m); \sigma)$ the corresponding discrete series from Theorem 2.7. If π_i has Mœglin–Tadić data $(Jord_i, \sigma, \varepsilon_i)$, then π has data $(Jord, \sigma, \varepsilon)$, where

$$Jord = \bigcup_{i=1}^{m} Jord_i,$$

and $\varepsilon = \varepsilon_i$ on $S_\pi \cap [Jord_i \cup (Jord_i \times Jord_i)] = S_{\pi_i}$ (noting that $S_\pi = \cup_{i=1}^m S_{\pi_i}$). This is a result for the classification in [M-T] (cf. [M-T, Remark 14.5]) and holds by definition in the approach from [T5, T6].

We note that for a discrete series representation $\delta \in R((\rho, \alpha); \sigma)$, we have

$$Jord(\delta) = Jord_\rho(\delta) \cup \big(Jord(\sigma) \setminus Jord_\rho(\sigma)\big).$$

In particular, in the other direction, $\psi_{(\rho,\alpha)}(\delta_{(Jord,\sigma,\varepsilon)})$ has corresponding triple

$$\Big(Jord_\rho \cup \big(Jord(\sigma) \setminus Jord_\rho(\sigma)\big), \sigma, \varepsilon_\rho\Big),$$

where ε_ρ is the restriction of ε to

$$S \cap \Big(Jord_\rho \cup \big(Jord(\sigma) \setminus Jord_\rho(\sigma)\big)\Big) \cup \Big[\big(Jord_\rho \cup \big(Jord(\sigma) \setminus Jord_\rho(\sigma)\big)\big) \times \big(Jord_\rho \cup \big(Jord(\sigma) \setminus Jord_\rho(\sigma)\big)\big)\Big].$$

To start, we make a choice of components, writing $\rho \rtimes \sigma \cong \tau_1(\rho;\sigma) \oplus \tau_{-1}(\rho;\sigma)$. Then for $a \in \mathbb{N}$ and $\eta \in \{\pm 1\}$, let $\delta([\nu\rho, \nu^a\rho]; \tau_\eta(\rho;\sigma))$ denote the unique irreducible subrepresentation of $\delta([\nu\rho, \nu^a\rho]) \rtimes \tau_\eta(\rho;\sigma)$. If $a_{\max}$ is the largest value of a such that $(\rho, a) \in Jord$, we define $\varepsilon_\delta(\rho, a_{\max})$ as follows: $\varepsilon_\delta(\rho, a_{\max}) = \eta$ if and only if there is an irreducible θ such that

$$\delta \hookrightarrow \theta \rtimes \delta\big(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big]; \tau_\eta(\rho;\sigma)\big).$$

Observe that once $\varepsilon_\delta(\rho, a_{\max})$ is known, (3.1) is enough to determine ε_δ on S (recalling that even without $\varepsilon_\delta(\rho, a_{\max})$ known, (3.1) is enough to determine ε_δ on $S \cap (Jord \times Jord)$)

The following lemma (as well as Lemma 3.4) eliminates the need for the results of [J1, J4] in Tadić's definition. This will be of use when working with $SO(2n, F)$, where one does not have such results.

Lemma 3.2 *Let $\delta_\rho = \psi_{(\rho,0)}(\delta)$. Then there exists an irreducible θ such that*

$$\delta_\rho \hookrightarrow \theta \rtimes \delta\big(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big]; \tau_\eta(\rho;\sigma)\big)$$

if and only if there is an irreducible θ' such that

$$\delta \hookrightarrow \theta' \rtimes \delta\big(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho/bigr]; \tau_\eta(\rho;\sigma)\big)$$

Proof For ($\Rightarrow$), observe that it follows from [J1, J4] (cf. [J1, Section 7]) that there is an irreducible θ'' such that $\delta \hookrightarrow \theta'' \rtimes \psi_{(\rho,0)}(\delta)$. Therefore,

$$\delta \hookrightarrow \theta'' \rtimes \delta_\rho \hookrightarrow (\theta'' \times \theta) \rtimes \delta/bigl(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big]; \tau_\eta(\rho;\sigma)\big).$$

Since $\theta'' \times \theta$ is irreducible (by [Z]), the implication ($\Rightarrow$) follows.

For ($\Leftarrow$), we know $\delta_\rho \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_{\eta'}(\rho;\sigma))$ for some $\eta' \in \{\pm 1\}$ (in particular, $\eta' = \varepsilon_{\delta_\rho}(\rho, a_{\max})$); we need to show $\eta' = \eta$. Note that it follows from Frobenius reciprocity that

$$\theta \otimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_{\eta'}(\rho;\sigma)) \leq \mu^*(\delta_\rho).$$

Now, $\delta \hookrightarrow \theta' \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma))$. By [Z], we may write $\theta' \cong \theta \times \theta_\rho$ with $\theta_\rho \in R((\rho, 0))$ and $\theta \in R((\rho_1, \alpha_1), \dots, (\rho_m, \alpha_m))$ with $(\rho_i, \alpha_i) \neq (\rho, 0)$ for any i. Therefore,

$$\begin{gathered}\delta \hookrightarrow \theta \times \theta_\rho \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma)) \\ \Big\Downarrow \text{(cf. [J1, Lemma 5.5])} \\ \delta \hookrightarrow \theta \rtimes \delta'_\rho\end{gathered}$$

for some irreducible $\delta'_\rho \le \theta_\rho \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma))$. By $\psi_{(\rho,0)}$ considerations (*e.g.*, see Theorem 2.7(i)), we must have $\delta'_\rho \cong \delta_\rho$. Therefore,

$$\delta_\rho \le \theta_\rho \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma))$$
$$\Downarrow$$
$$\theta \otimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_{\eta'}(\rho;\sigma)) \le \mu^*\Big(\theta_\rho \rtimes \delta\big([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma)\big)\Big).$$

It now follows from a straightforward μ^* argument that $\eta = \eta'$ (use Theorem 2.2 and the fact that

$$\mu^*\Big(\delta\big([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma)\big)\Big) = 1 \otimes \delta\big([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho;\sigma)\big) + \sum_i \tau_i \otimes \theta_i,$$

where no θ_i contains $\nu^{\frac{a_{\max}-1}{2}}\rho$ in its supercuspidal support). ∎

We now give a variation of the preceding lemma that will also be used when working with $SO(2n, F)$. To start, we introduce a bit of notation. Suppose $\rho \ncong \rho'$ have $\rho' \rtimes \sigma$ and $\rho \rtimes \sigma$ reducible. By [G2], we may write

$$\rho \times \rho' \rtimes \sigma \cong \bigoplus_{i,j \in \{\pm 1\}} \tau_{i,j}(\rho, \rho'; \sigma),$$

where $\tau_{i,j}(\rho, \rho'; \sigma)$ is characterized by

$$\tau_{i,j}(\rho, \rho'; \sigma) \hookrightarrow \rho \rtimes \tau_j(\rho';\sigma) \text{ and } \rho' \rtimes \tau_i(\rho;\sigma).$$

Note 3.3 It follows from the characterization above and Lemma 5.1 below that

$$\hat{c}\tau_{i,j}(\rho, \rho'; \sigma) \cong \begin{cases} \tau_{-i,-j}(\rho, \rho'; \sigma) & \text{if } \rho, \rho' \text{ both satisfy (1.1)},\\ \tau_{-i,j}(\rho, \rho'; \sigma) & \text{if } \rho \text{ satisfies (1.1), but } \rho' \text{ does not},\\ \tau_{i,j}(\rho, \rho'; \sigma) & \text{if neither satisfies (1.1)}.\end{cases}$$

Lemma 3.4 *Suppose* $\rho \ncong \rho'$ *have* $\rho \rtimes \sigma$ *and* $\rho' \rtimes \sigma$ *reducible. For* $a, a' \in \mathbb{N}$ *odd and* $\eta, \eta' \in \{\pm 1\}$,

$$\delta\big([\nu\rho, \nu^{\frac{a-1}{2}}\rho]\big) \times \delta\big([\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\big) \rtimes \tau_{\eta,\eta'}(\rho, \rho'; \sigma)$$

has a unique irreducible subrepresentation that we denote

$$\delta\big([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{\eta,\eta'}(\rho, \rho'; \sigma)\big).$$

It is square-integrable. Further, in the notation of Section 2,

$$\psi_{(\rho,0)}\Big(\delta\big([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{\eta,\eta'}(\rho, \rho'; \sigma)\big)\Big) = \delta\big([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \tau_\eta(\rho;\sigma)\big),$$

and similarly for $\psi_{(\rho',0)}$.

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Proof Standard μ^* arguments tell us that

$$\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\otimes\tau_{\eta,\eta'}(\rho,\rho';\sigma)$$

occurs with multiplicity one in the following:

(i) $\mu^*\Big(\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\times\rho\times\rho'\rtimes\sigma\Big)$

(ii) $\mu^*\Big(\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\rtimes\tau_{\eta,\eta'}(\rho,\rho';\sigma)\Big)$

(iii) $\mu^*\Big(\delta\big(\big[\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\rtimes\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big];\tau_{\eta'}(\rho';\sigma)\big)\Big)$

(iv) $\mu^*\Big(\delta\big(\big[\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\rtimes\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big];\tau_{\eta}(\rho;\sigma)\big)\Big)$.

We also note that the induced representations appearing in (ii)–(iv) all embed in the induced representation appearing in (i). It then follows (from (ii) and Frobenius reciprocity) that

$$\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\rtimes\tau_{\eta,\eta'}(\rho,\rho';\sigma)$$

has a unique irreducible subrepresentation δ. Further (using (iii) and (iv)), δ must also appear as a subquotient of both

$$\delta\big(\big[\rho,\nu^{\frac{a-1}{2}}\rho\big]\big)\rtimes\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big];\tau_{\eta'}(\rho';\sigma)\big),$$

$$\delta\big(\big[\rho',\nu^{\frac{a'-1}{2}}\rho'\big]\big)\rtimes\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big];\tau_{\eta}(\rho;\sigma)\big).$$

In particular, it follows from $\psi_{(\rho,0)}$ and $\psi_{(\rho',0)}$ considerations (see Theorem 2.7(i)) that

$$\psi_{(\rho,0)}(\delta)=\delta\big(\big[\nu\rho,\nu^{\frac{a-1}{2}}\rho\big];\tau_{\eta}(\rho;\sigma)\big)$$

and

$$\psi_{(\rho',0)}(\delta)=\delta\big(\big[\nu\rho',\nu^{\frac{a'-1}{2}}\rho'\big];\tau_{\eta'}(\rho';\sigma)\big).$$

Now, Theorem 2.7(iv) tells us δ is square-integrable, finishing the proof. ■

Lemma 3.5 *With notation as in the previous lemma, let $\delta_{\rho,\rho'}\in R((\rho,0),(\rho',0);\sigma)$ be such that $\psi_{(\rho,0)}(\delta_{\rho,\rho'})=\psi_{(\rho,0)}(\delta)$ and $\psi_{(\rho',0)}(\delta_{\rho,\rho'})=\psi_{(\rho',0)}(\delta)$. Then there exist irreducible θ,θ' such that*

$$\delta_{\rho}\hookrightarrow\theta\rtimes\delta\big(\big[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho\big];\tau_{\eta}(\rho;\sigma)\big)$$

and

$$\delta_{\rho'}\hookrightarrow\theta'\rtimes\delta\big(\big[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho'\big];\tau_{\eta'}(\rho';\sigma)\big)$$

if and only if there is an irreducible θ'' such that

$$\delta \hookrightarrow \theta'' \rtimes \delta\big(\,[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,,\,[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,;\tau_{\eta,\eta'}(\rho,\rho';\sigma)\big)\,.$$

Proof The implication ($\Leftarrow$) is a straightforward consequence of the fact that

$$\delta\big(\,[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,,[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,;\tau_{\eta,\eta'}(\rho,\rho';\sigma)\big) \hookrightarrow \delta\big(\,[\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,\big) \rtimes \delta\big(\,[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,;\tau_{\eta'}(\rho';\sigma)\big)$$

and

$$\delta\big(\,[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,,[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,;\tau_{\eta,\eta'}(\rho,\rho';\sigma)\big) \hookrightarrow \delta\big(\,[\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,\big) \rtimes \delta\big(\,[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,;\tau_{\eta}(\rho;\sigma)\big)\,.$$

For the implication ($\Rightarrow$), we first argue that

$$\delta \hookrightarrow \theta'' \rtimes \delta\big(\,[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho]\,,\,[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,;\tau_{\xi,\xi'}(\rho,\rho';\sigma)\big)$$

for some $\xi,\xi' \in \{\pm 1\}$. To this end, we first show that δ has an admissible sequence $(Jord_i,\sigma,\varepsilon_i)$, $1 \le i \le k$, having $Jord_k = Jord_{k-1} \setminus \{(\rho,b),(\rho,a_{\max})\}$ for some b. It is not difficult to see that we may arrange $Jord_k = Jord_{k-1} \setminus \{(\rho,b),(\rho,a)\}$ for some a,b. It therefore suffices (by iteration) to show that if $|Jord_\rho| > 2$, then $(\rho,a_{\max})$ does not have to be part of the first pair removed from $Jord_\rho$. Suppose the first pair from $Jord_\rho$ is removed at the i-th stage. Let $a_{\max,-} = (a_{\max})_-$. If $\varepsilon_i(\rho,a_{\max})\varepsilon_i^{-1}(\rho,a_{\max,-}) \neq 1$, this is clear–the pair $(\rho,a_{\max,-}),(\rho,a_{\max})$ is not eligible to be removed. If $\varepsilon_i(\rho,a_{\max})\varepsilon_i^{-1}(\rho,a_{\max,-}) = 1$, we claim there is some $(\rho,a) \in Jord_\rho$ with $a \neq a_{\max}$ and $\varepsilon_i(\rho,a)\varepsilon_i^{-1}(\rho,a_-) = 1$. This follows from considering the resulting $Jord_{i+1} = Jord_i \setminus \{(\rho,a_{\max}),(\rho,a_{\max,-})\}$, which must then have $\varepsilon_{i+1}(\rho_a)\varepsilon_{i+1}^{-1}(\rho,a_-) = 1$ for some such a by admissibility. We may then remove this $\{(\rho,a),(\rho,a_-)\}$ at the i-th stage instead. A similar argument tells us we may arrange $Jord_{k-1} = Jord_{k-2} \setminus \{(\rho',b'),(\rho',a'_{\max})\}$ for some b'. It then follows that

$$\delta \hookrightarrow \delta_1 \times \cdots \times \delta_{k-3} \times \delta\big(\,[\nu^{\frac{-b'+1}{2}}\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho']\,\big) \times \delta\big(\,[\nu^{\frac{-b+1}{2}}\rho,\nu^{\frac{a_{\max}-1}{2}}\rho']\,\big) \rtimes \delta_{\mathrm{alt}}.$$

We give the argument below using "commuting arguments" ($\lambda_1 \times \lambda_2 \cong \lambda_2 \times \lambda_1$ when irreducible) and "inverting arguments" ($\lambda \rtimes \mu \cong \tilde{\lambda} \rtimes \mu$ when irreducible). We note that the irreducibility for the commuting arguments follows from [Z] (noting that $Jord_\rho$ and $Jord_{\rho'}$ do not contribute to δ_{alt} as $Jord_\rho(\sigma) = \varnothing$ and $Jord_{\rho'}(\sigma) = \varnothing$) while the irreducibility for the inverting arguments follows from [Mu1]. We obtain the following:

$$\delta \hookrightarrow \delta_1 \times \cdots \times \delta_{k-3} \times \delta([\nu^{\frac{-b'+1}{2}}\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']) \times \delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]) \rtimes \delta_{\mathrm{alt}}$$

$$\Downarrow$$

$$\delta \hookrightarrow \delta_1 \times \cdots \times \delta_{k-3} \times \delta([\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']) \times \delta([\nu^{\frac{-b'+1}{2}}\rho', \nu^{-1}\rho'])$$
$$\times \delta([\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]) \times \delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{-1}\rho]) \times \delta'_1 \times \cdots \times \delta'_m \rtimes \sigma$$

(commuting arguments)

$$\cong \delta_1 \times \cdots \times \delta_{k-3} \times \delta'_1 \times \cdots \times \delta'_m \times \delta([\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']) \times \delta([\rho, \nu^{\frac{a_{\max}-1}{2}}\rho])$$
$$\times \delta([\nu^{\frac{-b'+1}{2}}\rho', \nu^{-1}\rho']) \times \delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{-1}\rho]) \rtimes \sigma$$

(inverting and commuting arguments)

$$\cong \delta_1 \times \cdots \times \delta_{k-3} \times \delta'_1 \times \cdots \times \delta'_m \times \delta([\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho'])$$
$$\times \delta([\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]) \times \delta([\nu\rho', \nu^{\frac{b'-1}{2}}\rho']) \times \delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho]) \rtimes \sigma$$

(commuting arguments)

$$\cong \delta_1 \times \cdots \times \delta_{k-3} \times \delta'_1 \times \cdots \times \delta'_m \times \delta([\nu\rho', \nu^{\frac{b'-1}{2}}\rho'])$$
$$\times \delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho]) \times \delta([\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']) \times \delta([\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]) \rtimes \sigma$$

$$\Downarrow \text{(cf. [J1, Lemma 5.5])}$$

$$\delta \hookrightarrow \delta''_1 \times \cdots \times \delta''_m \rtimes \pi$$

for some irreducible $\pi \le \delta([\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']) \times \delta([\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]) \rtimes \sigma$ (writing the other representations that appear as $\delta''_1, \dots, \delta''_m$ for convenience). We need to show

$$\pi = \delta\big(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big], \big[\nu\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho'\big]; \tau_{\xi,\xi'}(\rho, \rho'; \sigma)\big)$$

for some $\xi, \xi' \in \{\pm 1\}$.

To this end, observe that (by [Mu1] and Theorem 2.7(ii),(vi)) the irreducible subquotients of

$$\delta\big(\big[\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho'\big]\big) \times \delta\big(\big[\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big]\big) \rtimes \sigma$$

are the following:

(i) $\delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']; \tau_{\xi,\xi'}(\rho, \rho'; \sigma))$ with $\xi, \xi' \in \{\pm 1\}$,

(ii) $L(\delta([\nu^{\frac{-a'_{\max}+1}{2}}\rho', \rho']) \otimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\xi(\rho; \sigma)))$ with $\xi \in \{\pm 1\}$,

(iii) $L(\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho, \rho]) \otimes \delta([\nu\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho']; \tau_{\xi'}(\rho'; \sigma)))$ with $\xi' \in \{\pm 1\}$, and

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(iv) the appropriate one of the following:

$$L(\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho,\rho])\otimes\delta([\nu^{\frac{-a'_{\max}+1}{2}}\rho',\rho'])\otimes\sigma) \text{ if } a_{\max}>a'_{\max},$$

$$L(\delta([\nu^{\frac{-a'_{\max}+1}{2}}\rho',\rho'])\otimes\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho,\rho])\otimes\sigma) \text{ if } a_{\max}<a'_{\max},$$

$$L(\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho,\rho])\times\delta([\nu^{\frac{-a'_{\max}+1}{2}}\rho',\rho'])\otimes\sigma) \text{ if } a_{\max}=a'_{\max}$$

(noting that only one of these will have the right form to satisfy the requirements for a Langlands subrepresentation).

We show that having π as one of the other subquotients would contradict the Casselman criterion for the square-integrability of δ. We note that any π other than

$$\delta\big(\big[\nu\rho,\nu^{\frac{a_{\max}-1}{2}}\rho\big],\big[\nu\rho',\nu^{\frac{a'_{\max}-1}{2}}\rho'\big];\tau_{\eta,\eta'}(\rho,\rho';\sigma)\big)$$

with $\xi,\xi'\in\{\pm1\}$ has

$$\pi\hookrightarrow\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\rho\big]\big)\rtimes\pi' \quad\text{or}\quad \pi\hookrightarrow\delta\big(\big[\nu^{\frac{-a'_{\max}+1}{2}}\rho',\rho'\big]\big)\rtimes\pi';$$

for purposes of an indirect argument, we assume the former. Then

$$\delta\hookrightarrow\delta''_1\times\cdots\times\delta''_m\times\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\rho\big]\big)\rtimes\pi'.$$

Now, observe that (cf. [Z]) if $c>s$ (c,d,s necessarily less than $a_{\max}$), then

$$\begin{aligned}\delta\big(\big[\nu^{\frac{-d+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho\big]\big) = \\ \delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho\big]\big) \\ +L\Big(\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho\big]\big),\delta\big(\big[\nu^{\frac{-d+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\big]\big)\Big),\end{aligned}$$

is a sum of irreducible representations; otherwise,

$$\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho])\times\delta([\nu^{\frac{-d+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])$$

is irreducible. In particular, any subquotient

$$\lambda\le\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu^{\frac{-d+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\big]\big)$$

has

$$\lambda\hookrightarrow\delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s'-1}{2}}\rho\big]\big)\times\delta\big(\big[\nu^{\frac{-d'+1}{2}}\rho,\nu^{\frac{c'-1}{2}}\rho\big]\big)$$

for some c',d',s' (a permutation of c,d,s). Therefore,

$$\delta\hookrightarrow\delta'_1\times\delta'_j\times\delta([\nu^{\frac{-d+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\times\delta([\nu^{\frac{-a_{\max}+1}{2}}\rho,\nu^{\frac{s-1}{2}}\rho])\times\delta'_{j+3}\times\cdots\times\delta_\ell\rtimes\pi'$$

$$\Downarrow \text{(by [J1, Lemma 5.5])}$$

$$\delta\hookrightarrow\delta'_1\times\delta'_j\times\lambda\times\cdots\delta'_{j+3}\times\delta_\ell\rtimes\pi'$$

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for some irreducible $\lambda \leq \delta([\nu^{\frac{-a_{\max}+1}{2}}\rho, \nu^{\frac{s-1}{2}}\rho]) \times \delta([\nu^{\frac{-d+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])$. It then follows that

$$\delta \hookrightarrow \delta_1' \times \delta_j' \times \delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho, \nu^{\frac{s'-1}{2}}\rho\big]\big) \times \delta\big(\big[\nu^{\frac{-d'+1}{2}}\rho, \nu^{\frac{c'-1}{2}}\rho\big]\big) \times \cdots \delta_{j+3}' \times \delta_\ell \rtimes \pi'$$

for some c', d', s' (a permutation of c, d, s). Iterating, we see that

$$\delta \hookrightarrow \delta\big(\big[\nu^{\frac{-a_{\max}+1}{2}}\rho, \nu^{\frac{t'-1}{2}}\rho\big]\big) \times \cdots,$$

contradicting the Casselman criterion for the square-integrability of δ. Thus we must have

$$\pi = \delta\big(\big[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho\big], \big[\nu\rho', \nu^{\frac{a'_{\max}-1}{2}}\rho'\big]; \tau_{\eta,\eta'}(\rho, \rho'; \sigma)\big)$$

with $\xi, \xi' \in \{\pm 1\}$, as claimed.

The remainder of the proof is similar to that of Lemma 3.2. ■

We close with one modification to Tadić's definition that will prove more convenient when shifting to $SO(2n, F)$. Let $\rho_1, \dots, \rho_\ell$ be inequivalent representations such that the following holds: $(\rho, a) \in S$ with a odd if and only if $\rho \cong \rho_i$ for some i. Instead of making ℓ choices of components, one for each $\rho_i \rtimes \sigma$, it follows from [G2] that it is equivalent to choose a single component $\mathcal{T}$ (out of the 2^ℓ possibilities)

$$\mathcal{T} \leq \rho_1 \times \cdots \times \rho_\ell \rtimes \sigma.$$

Then $T_\varepsilon(\rho_i; \sigma)$, $\varepsilon \in \{\pm 1\}$, is characterized by

$$\mathcal{T} \leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes T_1(\rho_i; \sigma)$$

but

$$\mathcal{T} \not\leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes T_{-1}(\rho_i; \sigma).$$

We remark that this modification is helpful in dealing with $SO(2n, F)$, where the reducibility is a bit subtler (see Remark 2.8 for a brief discussion of these subtleties).

4 The Case $\sigma \neq \hat{c}\sigma$

In this section, we consider $\mathrm{Res}^O_{SO}\,\pi$ for an irreducible $\pi \in R((\rho, \alpha); \sigma)$ when $\hat{c}\sigma \not\cong \sigma$. In this case, it is not difficult to show that $\mathrm{Res}^O_{SO}\,\pi$ is irreducible.

Theorem 4.1 *Suppose $\hat{c}\sigma \not\cong \sigma$. Let $\pi \in R((\rho, \alpha); \sigma)$ be irreducible. Then $\mathrm{Res}^O_{SO}\,\pi$ is irreducible.*

Proof The partial cuspidal support of π is σ; by Lemma 2.4(ii), the partial cuspidal support of $\hat{c}\pi$ is $\hat{c}\sigma$. Since $\hat{c}\sigma \not\cong \sigma$, it follows that $\hat{c}\pi \not\cong \pi$. The theorem now follows from Lemma 2.3. ■

Remark 4.2 We take a moment to compare the parameterizations for δ and $\hat{c}\delta$ under these circumstances. It is a fairly easy consequence of Lemma 2.4(i) that $Jord(\hat{c}\delta) = Jord(\delta)$. As noted in the preceding proof, $\sigma_{\hat{c}\delta} = \hat{c}\sigma_\delta$. It also follows from Lemma 2.4(i) that $\varepsilon_{\hat{c}\delta} = \varepsilon_\delta$ on that part of S contained in $Jord \times Jord$, and therefore also when ρ is a representation of $GL(m, F)$ with m even. If ρ is a representation of $GL(m, F)$ with m odd and $\varepsilon_\delta(\rho, a)$ defined for some a (so $\rho \rtimes \sigma$ is reducible), the definition from [T5, T6] requires a choice of one of the two components of $\rho \rtimes \sigma$. We make our choice consistent with the action of $\hat{c}$: if τ has been chosen for $\rho \rtimes \sigma$, we choose $\hat{c}\tau$ for $\rho \rtimes \hat{c}\sigma$. With this choice, $\hat{c}\delta$ has the corresponding triple $(Jord(\delta), \hat{c}\sigma_\delta, \varepsilon_\delta)$.

5 The Case $\sigma = \hat{c}\sigma$ with ρ not Satisfying (1.1)

In this section, we consider $\mathrm{Res}^O_{SO}\,\pi$ for an irreducible $\pi \in R((\rho, \alpha); \sigma)$ when $\hat{c}\sigma \cong \sigma$ but (1.1) not satisfied–*i.e.*, either ρ is a representation of $GL(m, F)$ with m even, or $\rho \not\cong \tilde{\rho}$ (noting that when $\rho \not\cong \tilde{\rho}$, there are no discrete series). In this case, we show that $\mathrm{Res}^O_{SO}\,\pi$ is reducible.

We start with a lemma, noting that part (iii) of the lemma requires (1.1) to be satisfied and is not used in this section (but is used later).

Lemma 5.1 *Suppose $\hat{c}\sigma \cong \sigma$. Let $\sigma_0 \leq \mathrm{Res}^O_{SO}\,\sigma$ be irreducible.*

(i) *Suppose $\rho \rtimes \sigma$ is irreducible. Then $\hat{c}(\rho \rtimes \sigma) \cong \rho \rtimes \sigma$.*

(ii) *Suppose $\rho \rtimes \sigma$ and $\rho \rtimes \sigma_0$ are both reducible. Write*

$$\rho \rtimes \sigma \cong \tau_1(\rho; \sigma) \oplus \tau_{-1}(\rho; \sigma).$$

Then $\hat{c}\tau_i(\rho; \sigma) \cong \tau_i(\rho; \sigma)$ for $i = \pm 1$.

(iii) *Suppose $\rho \rtimes \sigma$ is reducible but $\rho \rtimes \sigma_0$ is irreducible (noting that this requires (1.1) be satisfied; see Proposition 2.5). Write*

$$\rho \rtimes \sigma \cong \tau_1(\rho; \sigma) \oplus \tau_{-1}(\rho; \sigma).$$

Then $\hat{c}\tau_i(\rho; \sigma) \cong \tau_{-i}(\rho; \sigma)$ for $i = \pm 1$.

Proof We start with an observation that will allow us to address both the cases $\sigma = 1$ and $\sigma \neq 1$ together. Let $\sigma_0 \leq \mathrm{Res}^O_{SO}\,\sigma$ be irreducible. If ξ is a representation of a general linear group and $\sigma \neq 1$, then by Lemma 2.3 and induction in stages, we have

$$\xi \rtimes \sigma \cong \xi \rtimes \mathrm{Ind}^O_{SO}\,\sigma_0 \cong \mathrm{Ind}^O_{SO}(\xi \rtimes \sigma_0),$$

and this also holds when $\sigma = 1$.

We now consider (i). We have

$$\rho \rtimes \sigma \cong \mathrm{Ind}^O_{SO}(\rho \rtimes \sigma_0),$$

which implies $\rho \rtimes \sigma_0$ is irreducible and induces irreducibly to $O(2n, F)$. Therefore, by Lemma 2.3, $c(\rho \rtimes \sigma_0) \not\cong \rho \rtimes \sigma_0$ and $\hat{c}(\rho \rtimes \sigma) \cong \rho \rtimes \sigma$, as needed.

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We now consider (ii). We have

$$\tau_1(\rho;\sigma)\oplus\tau_{-1}(\rho;\sigma)\cong\rho\rtimes\sigma\cong \mathrm{Ind}_{SO}^{O}(\rho\rtimes\sigma_0)$$
$$\cong \mathrm{Ind}_{SO}^{O}\left(\tau_{1,0}(\rho;\sigma_0)\oplus\tau_{-1,0}(\rho;\sigma_0)\right),$$

where $\rho\rtimes\sigma_0=\tau_{1,0}(\rho;\sigma_0)\oplus\tau_{-1,0}(\rho;\sigma_0)$. Therefore, $\tau_{i,0}(\rho;\sigma_0)$, $i=\pm1$, induces irreducibly. Without loss of generality, we may write

$$\tau_i(\rho;\sigma)\cong \mathrm{Ind}_{SO}^{O}\,\tau_{i,0}(\rho;\sigma_0).$$

Further, by Lemma 2.3, it follows that for $i=\pm1$

$$c\tau_{i,0}(\rho;\sigma_0)\ncong\tau_{i,0}(\rho;\sigma_0),\qquad \hat{c}\tau_i(\rho;\sigma)\cong\tau_i(\rho;\sigma),$$

as needed.

We now address (iii). In this case, we have

$$\tau_1(\rho;\sigma)\oplus\tau_{-1}(\rho;\sigma)\cong\rho\rtimes\sigma\cong \mathrm{Ind}_{SO}^{O}(\rho\rtimes\sigma_0)\cong \mathrm{Ind}_{SO}^{O}\,\tau_0(\rho;\sigma_0),$$

with $\tau_0(\rho;\sigma_0)\cong\rho\rtimes\sigma_0$ (irreducible). Therefore, $\tau_0(\rho;\sigma_0)$ induces reducibly. It now follows from Lemma 2.3 that

$$c\tau_0(\rho;\sigma_0)\cong\tau_0(\rho;\sigma_0),\qquad \hat{c}\tau_i(\rho;\sigma)\ncong\tau_i(\rho;\sigma).$$

Further, since $\mathrm{Ind}_{SO}^{O}(\rho\rtimes\sigma_0)\cong\tau_i(\rho;\sigma)\oplus\hat{c}\tau_i(\rho;\sigma)$, we see that $\hat{c}\tau_i(\rho;\sigma)\cong\tau_{-i}(\rho;\sigma)$, as needed. ■

Lemma 5.2 *Suppose $\hat{c}\sigma\cong\sigma$ and ρ does not satisfy* (1.1), i.e., *either ρ is a representation of $GL(m,F)$ with m even, or $\rho\ncong\tilde{\rho}$. Suppose $\pi\in R((\rho,\alpha);\sigma)$ is an irreducible representation satisfying*

$$\pi\hookrightarrow\underbrace{\rho\times\cdots\times\rho}_{\ell}\rtimes\sigma$$

with $\ell>0$. Then $\hat{c}\pi\cong\pi$.

Proof First, suppose $\rho\rtimes\sigma$ is irreducible. It follows from [G2] that $\underbrace{\rho\times\cdots\times\rho}_{\ell}\rtimes\sigma$ is also irreducible. Thus $\pi=\underbrace{\rho\times\cdots\times\rho}_{\ell}\rtimes\sigma$. By Lemma 5.1(i),

$$\hat{c}(\rho\rtimes\sigma)\cong\rho\rtimes\sigma.$$

By Lemma 2.4(i),

$$\hat{c}\pi\cong\hat{c}(\underbrace{\rho\times\cdots\times\rho}_{\ell}\rtimes\sigma)\cong\underbrace{\rho\times\cdots\times\rho}_{\ell-1}\rtimes\hat{c}(\rho\rtimes\sigma)\cong\pi,$$

as needed.

Now, suppose $\rho \rtimes \sigma$ is reducible. It follows from [G2] that $\underbrace{\rho \times \cdots \times \rho}_{\ell} \rtimes \sigma$ has two components, so $\underbrace{\rho \times \cdots \times \rho}_{\ell-1} \rtimes \tau_i(\rho;\sigma)$ is irreducible for $i = \pm 1$. Thus, $\pi = \underbrace{\rho \times \cdots \times \rho}_{\ell-1} \rtimes \tau_i(\rho;\sigma)$ for some i. By Lemma 2.4(i) and 5.1(ii) (noting that Proposition 2.5 implies we are in Lemma 5.1(ii)),

$$\hat{c}\pi \cong \hat{c}\Big(\underbrace{\rho \times \cdots \times \rho}_{\ell-1} \rtimes \tau_i(\rho;\sigma)\Big) \cong \underbrace{\rho \times \cdots \times \rho}_{\ell-1} \rtimes \hat{c}\tau_i(\rho;\sigma) \cong \pi,$$

as needed. ■

Theorem 5.3 *Suppose $\hat{c}\sigma \cong \sigma$ and ρ does not satisfy* (1.1), i.e., *either ρ is a representation of $GL(m, F)$ with m even, or $\rho \not\cong \tilde{\rho}$. Let $\pi \in R((\rho, \alpha); \sigma)$ be irreducible. Then $\mathrm{Res}^O_{SO}\,\pi$ is reducible. In particular, we may write*

$$\mathrm{Res}^O_{SO}\,\pi \cong \pi_0 \oplus c\pi_0$$

with $c\pi_0 \not\cong \pi_0$.

Proof By Lemma 2.3, it suffices to show that $\hat{c}\pi \cong \pi$. The proof is by induction on the parabolic rank. The basis step follows from looking at the possible subquotients of $\nu^s\rho \rtimes \sigma$, $s \in \mathbb{R}$. If $\nu^s\rho \rtimes \sigma$ is irreducible, the result follows from Lemma 2.4(i). If $\nu^s\rho \rtimes \sigma$ is reducible and $s = 0$, the result follows from Lemma 5.1(ii) and Lemma 2.3. When $\nu^s\rho \rtimes \sigma$ is reducible and $s > 0$, then $\nu^s\rho \rtimes \sigma = L(\nu^{-s}\rho \otimes \sigma) + \delta(\nu^s\rho;\sigma)$, with $\delta(\nu^s\rho;\sigma)$ the square-integrable subrepresentation and $L(\nu^{-s}\rho \otimes \sigma)$ the Langlands quotient (written using subrepresentation data; see Section 2). By Lemma 2.4(iv),

$$\hat{c}L(\nu^{-s}\rho \otimes \sigma) \cong L(\nu^{-s}\rho \otimes \hat{c}\sigma) \cong L(\nu^{-s}\rho \otimes \sigma)$$

since $\hat{c}\sigma \cong \sigma$ by assumption. By duality (see Lemma 2.4(iii)), it follows that

$$\hat{c}\delta(\nu^s\rho;\sigma) \cong \delta(\nu^s\rho;\sigma).$$

The case $s \neq 0$ then follows.

For the inductive step, observe that by [J1, Corollary 4.2] (which also applies to $O(2n, F)$; cf. [J4]), (at least) one of the following holds:

(i) π is nontempered,
(ii) $D_O\pi$ is nontempered, or
(iii) $\pi \hookrightarrow \underbrace{\rho \times \cdots \times \rho}_{\ell} \rtimes \sigma$ for some ℓ.

We break the argument into three cases accordingly.

In case (i), write

$$\pi = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \tau).$$

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By Lemma 2.4(iv), we have

$$\hat{c}\pi = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \hat{c}\tau).$$

Thus $\hat{c}\pi \cong \pi$ if and only if $\hat{c}\tau \cong \tau$, which holds by inductive hypothesis.

For case (ii), recall that $D_O(\hat{c}\pi) \cong \hat{c}D_O\pi$ by Lemma 2.4(iii). It then follows that $\hat{c}\pi \cong \pi$ if and only if $\hat{c}D_O\pi \cong D_O\pi$, which follows from case (i).

Case (iii) follows from Lemma 5.2, finishing the proof. ∎

6 The Case $\sigma = \hat{c}\sigma$ with ρ Satisfying 1.1, Part I: Discrete Series

In this section, we consider discrete series representations in $R((\rho,\alpha);\sigma)$, when $\sigma \cong \hat{c}\sigma$ and ρ satisfies (1.1). We remark that $\text{red}(\rho;\sigma) = 0$ here, a consequence of Proposition 2.5, so we must have $\alpha = 0$ to support non-supercuspidal discrete series. In this case, Theorem 6.5 shows that $\hat{c}\delta \not\cong \delta$. More precisely, if $(Jord, \sigma, \varepsilon)$ is the admissible triple for δ, then $(Jord, \sigma, \hat{c}\varepsilon)$ is the admissible triple for $\hat{c}\delta$ (cf. Definition 6.2). Thus $\text{Res}^O_{SO}\,\delta$ is irreducible. We note that we have not worked in the generality of admissible representations, as we did in the previous cases, for a reason. An irreducible admissible representation $\pi \in R((\rho,\alpha);\sigma)$ may or may not have $\text{Res}^O_{SO}\,\pi$ reducible. We take up this issue in the next section.

Suppose $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ is a non-supercuspidal discrete series representation in $R((\rho,\alpha);\sigma)$. Recall that in this case, $Jord = Jord_\rho \cup (Jord(\sigma) \setminus Jord_\rho(\sigma))$, a disjoint union (cf. Remark 3.1). We note that the values of ε on $Jord_\rho$ are enough to determine ε on its domain. In particular, the values on $Jord_\rho$ determine the values on $Jord_\rho \times Jord_\rho$ by the compatability conditions required of triples; the values elsewhere are determined by ε_σ (*i.e.*, from the triple $(Jord(\sigma), \sigma, \varepsilon_\sigma)$ for σ). Write

$$Jord_\rho = \{(\rho, a_1), \dots, (\rho, a_k)\},$$

with $a_1 < \cdots < a_k$. We may then identify ε with the k-tuple $\varepsilon = (c_1, \dots, c_k)$, where $c_i = \varepsilon(\rho, a_i)$. We note that in order for $(Jord, \sigma, \rho)$ to be admissible, we must have k even (see [M-T, Section 14]). Further, in this situation, there are no nontrivial alternated triples (as $Jord_\rho(\sigma) = \varnothing$).

We note that the following lemma is not essential for the arguments below, but makes a number of claims clearer.

Lemma 6.1 *Suppose k is even and $\varepsilon = (c_1, \dots, c_k)$ with k even. Then ε is admissible,* i.e., *$(Jord, \sigma, \varepsilon)$ is admissible, if and only if $\langle \varepsilon, \varepsilon_{\text{alt},k} \rangle = 0$, where $\varepsilon_{\text{alt},k} = (\underbrace{1, -1, 1, -1, \dots, 1, -1}_{k})$ and $\langle \cdot, \cdot \rangle$ is the (restriction of the) usual inner product on $\mathbb{R}^k$.*

Proof The proof is by induction on k. For $k = 0$ there is nothing to prove; for $k = 2$, $(1, 1)$ and $(-1, -1)$ are clearly the admissible ϵ (cf. section 14 [M-T]). We now assume the lemma holds for $k - 2$ and check that it holds for k.

Suppose ε is admissible. Since $(Jord, \sigma, \varepsilon)$ is admissible and not alternated, there is some i with $\varepsilon(\rho, a_i)\varepsilon(\rho, a_{i-1})^{-1} = 1$ such that $Jord'' = Jord \setminus \{(\rho, a_i), (\rho, a_{i-1})\}$ and $\varepsilon'' = \varepsilon|_{Jord''}$ has $(Jord'', \sigma, \varepsilon'')$ admissible. Now,

$$\varepsilon'' = (c_1, \dots, c_{i-2}, c_{i+1}, \dots, c_k),$$

and $c_{i-1} = c_i$. By the inductive hypothesis,

$$0 = \langle \varepsilon'', \varepsilon_{\text{alt},k-2} \rangle = \sum_{j=1}^{i-1} (-1)^{j-1} c_j + \sum_{j=i+1}^{k} (-1)^{j-3} c_j$$
$$= \sum_{j=1}^{i-1} (-1)^{j-1} c_j + \sum_{j=i+1}^{k} (-1)^{j-1} c_j + [(-1)^{i-2} c_{i-1} + (-1)^{i-1} c_i]$$
$$= \langle \varepsilon, \varepsilon_{\text{alt},k} \rangle,$$

as needed.

Now, suppose $\langle \varepsilon, \varepsilon_{\text{alt},k} \rangle = 0$. Then there is some i such that $c_i = c_{i-1}$–if not, $\varepsilon = \pm\varepsilon_{\text{alt},k}$, and we would clearly have $\langle \varepsilon, \varepsilon_{\text{alt},k} \rangle \neq 0$. Let

$$Jord'' = Jord \setminus \{(\rho, a_i), (\rho, a_{i-1})\} \quad \text{and} \quad \varepsilon'' = (c_1, \dots, c_{i-2}, \dots, c_{i+1}, \dots, c_k)$$

(noting $\varepsilon'' = \varepsilon|_{Jord''}$). Reversing the calculation above shows $\langle \varepsilon'', \varepsilon_{\text{alt},k-2} \rangle = 0$, so by the inductive hypothesis, $(Jord'', \sigma, \varepsilon'')$ is admissible. Since $(Jord'', \sigma, \varepsilon'')$ is subordinate to $(Jord, \sigma, \varepsilon)$, it is also an admissible triple, as needed.

The lemma now follows by induction. ■

Definition 6.2 With $(Jord, \sigma, \epsilon)$ as above, write $\varepsilon = (c_1, \dots, c_k)$. Let

$$\hat{c}\varepsilon = (-c_1, -c_2, \dots, -c_k).$$

The following is fairly obvious even without the preceding lemma.

Corollary 6.3 *$(Jord, \sigma, \epsilon)$ is admissible if and only $(Jord, \sigma, \hat{c}\varepsilon)$ is admissible.*

Remark 6.4 For a fixed *Jord* as above, write $k = 2m$. Then the number of ε having $(Jord, \sigma, \varepsilon)$ admissible is $\binom{2m}{m}$. More generally, one can show inductively that the number of ε having $\langle \varepsilon, \varepsilon_{\text{alt},k} \rangle = 2j$ is $\binom{2m}{m+j}$.

Theorem 6.5 *Let $\delta_{(Jord,\sigma,\varepsilon)} \in R((\rho, 0); \sigma)$, ($\rho, \sigma$ as above) be a discrete series representation. Then $\hat{c}\delta_{(Jord,\sigma,\varepsilon)} = \delta_{(Jord,\sigma,\hat{c}\varepsilon)}$. In particular, $\operatorname{Res}^{O}_{SO} \delta_{(Jord,\sigma,\varepsilon)}$ is irreducible.*

Proof By Lemma 2.3, the irreducibility of $\operatorname{Res}^{O}_{SO} \delta_{(Jord,\sigma,\varepsilon)}$ follows once we show

$$\hat{c}\delta_{(Jord,\sigma,\varepsilon)} = \delta_{(Jord,\sigma,\hat{c}\varepsilon)}.$$

We prove this by induction on $|Jord_\rho|$. Since $|Jord_\rho| = 0$ just corresponds to σ, we start the induction with $|Jord_\rho| = 2$.

When $|Jord_\rho| = 2$, we have either $\varepsilon = (1, 1)$ or $(-1, -1)$. By the construction in [M-T], we have

$$\delta_{(Jord,\sigma,\varepsilon)} \hookrightarrow \delta([\nu^{\frac{-a_1+1}{2}} \rho, \nu^{\frac{a_2-1}{2}} \rho]) \rtimes \sigma.$$

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By duality (cf. Section 2) and [B-J2, Proposition 3.3],

$$\begin{aligned}
&D_O\delta_{(Jord,\sigma,\varepsilon)}\\
&\quad\le \zeta([\nu^{\frac{-a_2+1}{2}}\rho,\nu^{\frac{a_1-1}{2}}\rho])\rtimes\sigma\\
&\quad= L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}}\rho\otimes(\nu^{\frac{-a_1+1}{2}}\rho\times\nu^{\frac{-a_1+1}{2}}\rho)\\
&\qquad\quad\otimes\cdots\otimes(\nu^{-1}\rho\times\nu^{-1}\rho)\otimes\tau_1(\rho;\sigma))\\
&\qquad+L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}}\rho\otimes(\nu^{\frac{-a_1+1}{2}}\rho\times\nu^{\frac{-a_1+1}{2}}\rho)\\
&\qquad\quad\otimes\cdots\otimes(\nu^{-1}\rho\times\nu^{-1}\rho)\otimes\tau_{-1}(\rho;\sigma))\\
&\qquad+L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}-1}\rho\otimes\delta([\nu^{\frac{-a_1-1}{2}}\rho,\nu^{\frac{-a_1+1}{2}}\rho)\otimes\delta([\nu^{\frac{-a_1+1}{2}}\rho,\nu^{\frac{-a_1+3}{2}}\rho)\\
&\qquad\quad\otimes\cdots\otimes\delta([\nu^{-1}\rho,\rho])\otimes\sigma).
\end{aligned}$$

Since

$$\begin{aligned}
\pi_3 = L(&\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}-1}\rho\otimes\\
&\delta([\nu^{\frac{-a_1-1}{2}}\rho,\nu^{\frac{-a_1+1}{2}}\rho)\otimes\delta([\nu^{\frac{-a_1+1}{2}}\rho,\nu^{\frac{-a_1+3}{2}}\rho)\otimes\cdots\otimes\delta([\nu^{-1}\rho,\rho])\otimes\sigma)
\end{aligned}$$

is the unique irreducible quotient, Frobenius reciprocity implies

$$\mu^*(\pi_3)\ge\zeta([\nu^{\frac{-a_1+1}{2}}\rho,\nu^{\frac{a_2-1}{2}}\rho])\otimes\sigma.$$

Therefore, $\mu^*(D_O\pi_3)\ge\delta([\nu^{\frac{-a_2+1}{2}}\rho,\nu^{\frac{a_1-1}{2}}\rho])\otimes\sigma$. Since $a_1<a_2$, we see that $D_O\pi_3$ is not square-integrable (the Casselman criterion). Therefore, we have

$$\begin{aligned}
&D_O\left(\delta_{(Jord,\sigma,\varepsilon)}\right)=\\
&L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}}\rho\otimes(\nu^{\frac{-a_1+1}{2}}\rho\times\nu^{\frac{-a_1+1}{2}}\rho)\otimes\cdots\otimes(\nu^{-1}\rho\times\nu^{-1}\rho)\otimes\tau_i(\rho;\sigma))
\end{aligned}$$

for some $i =\in \{\pm1\}$. By Lemma 5.1(iii), $\hat{c}\tau_i(\rho;\sigma)=\tau_{-i}(\rho;\sigma)$. Therefore, by Lemma 2.4(iv), we have

$$\begin{aligned}
\hat{c}L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}}\rho\otimes(\nu^{\frac{-a_1+1}{2}}\rho\times\nu^{\frac{-a_1+1}{2}}\rho)\otimes\cdots\otimes(\nu^{-1}\rho\times\nu^{-1}\rho)\otimes\tau_1(\rho;\sigma))=\\
L(\nu^{\frac{-a_2+1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{-a_1-1}{2}}\rho\otimes(\nu^{\frac{-a_1+1}{2}}\rho\times\nu^{\frac{-a_1}{2}}\rho)\\
\otimes\cdots\otimes(\nu^{-1}\rho\times\nu^{-1}\rho)\otimes\tau_{-1}(\rho;\sigma)).
\end{aligned}$$

Since $\hat{c}\circ D_O=D_O\circ\hat{c}$ (cf. Lemma 2.4(iii)), we then have $\hat{c}\delta_{(Jord,\sigma,\varepsilon)}=\delta_{(Jord,\sigma,\hat{c}\varepsilon)}$, finishing the case $|Jord_\rho|=2$ and the basis step.

We now move to the inductive step, assuming the theorem holds for $|Jord_\rho|=k=2m$, $m>1$, and showing it holds for $|Jord_\rho|=2m+2$.

Let $(Jord, \sigma, \varepsilon)$ be an admissible triple with $\varepsilon = (c_1, \dots, c_{2m+2})$. Since there are no nontrivial alternating triples, we must have $c_i = c_{i+1}$ for some i. We consider two cases, depending on whether or not there is more than one such i.

Case 1: $c_i = c_{i+1}$ and $c_j = c_{j+1}$ for some $i \neq j$
Let $\varepsilon_1 = \varepsilon = (c_1, \dots, c_{2m+2})$ and set

$$\varepsilon_2 = (c_1, \dots, c_{i-1}, -c_i, -c_{i+1}, c_{i+2}, \dots, c_{2m+2})$$

and

$$\varepsilon_3 = (c_1, \dots, c_{j-1}, -c_j, -c_{j+1}, c_{j+2}, \dots, c_{2m+2})$$

(*i.e.*, changing the signs of the $i, i+1$ and $j, j+1$ entries, respectively). By Lemma 6.1, $(Jord, \sigma, \varepsilon_\ell)$, $\ell = 1, 2, 3$, are all admissible; let $\delta_\ell = \delta_{(Jord,\sigma,\varepsilon_\ell)}$. Now, set

$$\varepsilon^* = (c_1, \dots, c_{i-1}, c_{i+2}, \dots, c_{2m+2}) \quad \text{and} \quad \varepsilon^{**} = (c_1, \dots, c_{j-1}, c_{j+2}, \dots, c_{2m+2})$$

(*i.e.*, deleting the $i, i+1$ and $j, j+1$ entries, respectively). By [M-T, Lemma 5.1], we have

$$\delta_1, \delta_2 \hookrightarrow \delta([\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho]) \rtimes \delta_{(Jord^*,\sigma,\varepsilon^*)}$$

and

$$\delta_1, \delta_3 \hookrightarrow \delta([\nu^{\frac{-a_j+1}{2}}\rho, \nu^{\frac{a_{j+1}-1}{2}}\rho]) \rtimes \delta_{(Jord^{**},\sigma,\varepsilon^{**})},$$

where $Jord^* = Jord \setminus \{(\rho, a_i), (\rho, a_{i+1})\}$, $Jord^{**} = Jord \setminus \{(\rho, a_j), (\rho, a_{j+1})\}$ and $\varepsilon^*, \varepsilon^{**}$ are the restrictions above (noting that the resulting triples are admissible). By the inductive hypothesis,

$$\hat{c}\delta_{(Jord^*,\sigma,\varepsilon^*)} = \delta_{(Jord^*,\sigma,\hat{c}\varepsilon^*)} \quad \text{and} \quad \hat{c}\delta_{(Jord^{**},\sigma,\varepsilon^{**})} = \delta_{(Jord^{**},\sigma,\hat{c}\varepsilon^{**})}.$$

Therefore, (cf. Lemma 2.4)

$$\begin{aligned}
\hat{c}\delta_1, \hat{c}\delta_2 &\hookrightarrow \hat{c}\left(\delta\Big(\Big[\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho\Big]\Big) \rtimes \delta_{(Jord^*,\sigma,\varepsilon^*)}\right) \\
&\cong \delta\Big(\Big[\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho\Big]\Big) \rtimes \hat{c}\delta_{(Jord^*,\sigma,\varepsilon^*)} \\
&\cong \delta\Big(\Big[\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho\Big]\Big) \rtimes \delta_{(Jord^*,\sigma,\hat{c}\varepsilon^*)}.
\end{aligned}$$

Similarly,

$$\hat{c}\delta_1, \hat{c}\delta_3 \hookrightarrow \delta\Big(\Big[\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho\Big]\Big) \rtimes \delta_{(Jord^{**},\sigma,\hat{c}\varepsilon^{**})}.$$

[M-T, Lemma 5.1] applied directly to $\delta_{(Jord,\sigma,\hat{c}\varepsilon)}$ tells us

$$\delta_{(Jord,\sigma,\hat{c}\varepsilon_1)}, \delta_{(Jord,\sigma,\hat{c}\varepsilon_2)} \hookrightarrow \delta\Big(\Big[\nu^{\frac{-a_i+1}{2}}\rho, \nu^{\frac{a_{i+1}-1}{2}}\rho\Big]\Big) \rtimes \delta_{(Jord^*,\sigma,\hat{c}\varepsilon^*)}$$

and

$$\delta_{(Jord,\sigma,\hat{c}\varepsilon_1)}, \delta_{(Jord,\sigma,\hat{c}\varepsilon_3)} \hookrightarrow \delta\Big(\Big[\nu^{\frac{-a_j+1}{2}}\rho, \nu^{\frac{a_{j+1}-1}{2}}\rho\Big]\Big) \rtimes \delta_{(Jord^{**},\sigma,\hat{c}\varepsilon^{**})}.$$

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It then follows that

$$\{\hat{c}\delta_1, \hat{c}\delta_2\} = \{\delta_{(Jord,\sigma,\hat{c}\varepsilon_1)}, \delta_{(Jord,\sigma,\hat{c}\varepsilon_2)}\} \quad \text{and} \quad \{\hat{c}\delta_1, \hat{c}\delta_3\} = \{\delta_{(Jord,\sigma,\hat{c}\varepsilon_1)}, \delta_{(Jord,\sigma,\hat{c}\varepsilon_3)}\}.$$

Therefore, $\hat{c}\delta_1 = \hat{c}\delta_{(Jord,\sigma,\varepsilon_1)} = \delta_{(Jord,\sigma,\hat{c}\varepsilon_1)}$, as needed for Case 1. Further, and this is needed in Case 2 below, we can also conclude $\hat{c}\delta_i = \hat{c}\delta_{(Jord,\sigma,\varepsilon_i)} = \delta_{(Jord,\sigma,\hat{c}\varepsilon_i)}$ for $i = 2, 3$ as well.

Case 2: $c_i = c_{i+1}$ and $c_j \neq c_{j+1}$ for all $j \neq i$

We remark that in this case, we cannot compare two different induced representations containing $\delta_{(Jord,\sigma,\varepsilon)}$ as we did in Case 1. Instead, we essentially realize this as the δ_2 from Case 1.

First, we claim that $i \neq 1, 2m+1$. If $i = 1$, the only triple subordinate to $(Jord, \sigma, \varepsilon)$ would be $(Jord', \sigma, \varepsilon')$, where $\varepsilon' = (c_3, c_4, \dots, c_{2m+2})$ and $Jord' = Jord \setminus \{(\rho, 2a_1+1), (\rho, 2a_2+1)\}$. Since $c_j \neq c_{j+1}$ for all $j \neq 1$ and $m > 1$, we have $(Jord', \sigma, \varepsilon')$ a nontrivial alternating triple, a contradiction. A similar argument shows $i \neq 2m+1$. Therefore, $\varepsilon = (c_1, \dots, c_{i-1}, c_i, c_{i+1}, c_{i+2}, \dots, c_{2m+2})$ with $(c_1, \dots, c_{i-1})$ and $(c_{i+2}, \dots, c_{2m+2})$ alternating. Further, by the Case 2 assumption, we must have $c_{i-1} = c_{i+2} = -c_i = -c_{i+1}$.

Set $\varepsilon_2 = \varepsilon$ and $\varepsilon_1 = (c_1, \dots, c_{i-1}, -c_i, -c_{i+1}, c_{i+2}, \dots, c_{2m+2})$. Then $(Jord, \sigma, \varepsilon_1)$ is an admissible triple which falls under Case 1. Further, $(Jord, \sigma, \varepsilon_2)$ is exactly the admissible triple for δ_2 in Case 1. Therefore, by the results of Case 1, we have

$$\hat{c}\delta_i = \hat{c}\delta_{(Jord,\sigma,\varepsilon_i)} = \delta_{(Jord,\sigma,\hat{c}\varepsilon_i)}$$

for $i = 1, 2$, as needed.

The theorem now follows from induction. ■

7 The Case $\sigma = \hat{c}\sigma$ with ρ Satisfying (1.1), Part II: Admissible Representations

In this section, we consider the question of when an irreducible $\pi \in R((\rho, \alpha); \sigma)$ has $\mathrm{Res}^{O}_{SO}\,\pi$ irreducible, or equivalently, when $\hat{c}\pi \cong \pi$. In the previous section, we showed that if π is a non-supercuspidal discrete series, then $\hat{c}\pi \ncong \pi$. In the general case, the answer is not so simple. For $\pi \in R((\rho, \alpha); \sigma)$, the answer depends on α. We use the result for discrete series and the classification of irreducible tempered representations in terms of discrete series to deal with the case of π tempered. We then use the result for tempered representations and the Langlands classification to deal with π irreducible admissible.

Note 7.1 When $\rho \rtimes \sigma$ is reducible, which includes the situation under consideration, $R((\rho, \alpha); \sigma)$ contains non-supercuspidal discrete series (*i.e.*, other than σ) only for $\alpha = 0$, and non-supercuspidal tempered representations only for $\alpha = 0, \frac{1}{2}$.

Proposition 7.2 *Suppose $\sigma \cong \hat{c}\sigma$ and ρ satisfies* (1.1), i.e., *$\rho \cong \tilde{\rho}$ and ρ is a representation of $GL(m, F)$ with m odd.*

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(i) *If* $\tau \in R((\rho, \frac{1}{2}); \sigma)$ *is a non-supercuspidal, irreducible, tempered representation, then* $\hat{c}\tau \cong \tau$. *In particular,* $\mathrm{Res}_{SO}^{O} \tau$ *is reducible.*
(ii) *If* $\tau \in R((\rho, 0); \sigma)$ *is a non-supercuspidal, irreducible, tempered representation, then* $\hat{c}\tau \not\cong \tau$. *In particular,* $\mathrm{Res}_{SO}^{O} \tau$ *is irreducible.*

Proof We start with (i). Suppose $a \in \mathbb{N}$. By [B-J2, Proposition 3.3], we have $\zeta([\nu^{-a+\frac{1}{2}}\rho, \nu^{a-\frac{1}{2}}\rho]) \rtimes \sigma$ is irreducible. It then follows from duality (cf. Section 2) that $\delta([\nu^{-a+\frac{1}{2}}\rho, \nu^{a-\frac{1}{2}}\rho]) \rtimes \sigma$ is irreducible. Now, by the classification of irreducible tempered representations, we have

$$\tau \hookrightarrow \delta_1 \times \cdots \times \delta_\ell \rtimes \sigma$$

for some discrete series $\delta_1, \ldots, \delta_\ell$ (noting we must have σ as the $O(2n, F)$ representation as $R((\rho, \frac{1}{2}); \sigma)$ admits no non-supercuspidal discrete series). To have $\tau \in R((\rho, \frac{1}{2}); \sigma)$, we must have $\delta_i = \delta([\nu^{-a_i+\frac{1}{2}}\rho, \nu^{a_i-\frac{1}{2}}\rho])$, $a_i \in \mathbb{Z}$, for $i = 1, \ldots, \ell$. Thus $\delta_i \rtimes \sigma$ is irreducible for all i. By [G2], we then have $\delta_1 \times \cdots \times \delta_\ell \rtimes \sigma$ irreducible, so

$$\tau = \delta_1 \times \cdots \times \delta_\ell \rtimes \sigma.$$

Since $\hat{c}\sigma \cong \sigma$, we have (as in the proof of Lemma 5.1)

$$\delta_1 \times \cdots \times \delta_\ell \rtimes \sigma \cong \mathrm{Ind}_{SO}^{O}(\delta_1 \times \cdots \times \delta_\ell \rtimes \sigma_0),$$

with $\sigma_0 \leq \mathrm{Res}_{SO}^{O} \sigma$ irreducible. Therefore, $\delta_1 \times \cdots \times \delta_\ell \rtimes \sigma_0$ (necessarily irreducible) induces irreducibly. Part (i) now follows from Lemma 2.3.

We now turn to (ii). Note that if τ is square-integrable, the result follows from Theorem 6.5. If not, we have $\tau \hookrightarrow \delta_1 \times \cdots \times \delta_\ell \rtimes \delta$ with $\delta_1, \ldots, \delta_\ell$ discrete series of general linear groups and δ a discrete series in $R((\rho, 0); \sigma)$.

If $\delta \neq \sigma$, then $\hat{c}\delta \not\cong \delta$ (cf. Theorem 6.5). Now, by Lemma 2.4(i), $\hat{c}\tau \hookrightarrow \delta_1 \times \cdots \times \delta_\ell \rtimes \hat{c}\delta$. The uniqueness up to conjugacy of the inducing data in the classification of irreducible tempered representations (see the appendix) then implies that $\hat{c}\tau \not\cong \tau$ (since $\hat{c}\delta \not\cong \delta$).

This reduces us to the case $\delta = \sigma$. We start with the case $\ell = 1$. First, we note that for $a \in \mathbb{Z}$ with $a \geq 0$, we have (see [B-J2, Proposition 3.3])

$$\begin{aligned}\zeta([\nu^{-a}\rho, \nu^{a}\rho]) \rtimes \sigma \cong\ & L((\nu^{-a}\rho \times \nu^{-a}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \tau_1(\rho; \sigma)) \\ & \oplus L((\nu^{-a}\rho \times \nu^{-a}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \tau_{-1}(\rho; \sigma)).\end{aligned}$$

By Lemma 2.4(iv) and Lemma 5.1(iii), we have

$$\begin{aligned}&\hat{c}L((\nu^{-a}\rho \times \nu^{-a}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \tau_i(\rho; \sigma)) \\ &\quad \cong L((\nu^{-a}\rho \times \nu^{-a}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \hat{c}\tau_i(\rho; \sigma)) \\ &\quad \cong L((\nu^{-a}\rho \times \nu^{-a}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \tau_{-i}(\rho; \sigma)).\end{aligned}$$

Therefore, if we write

$$\delta([\nu^{-a}\rho,\nu^{a}\rho]) \rtimes \sigma \cong T_1([\nu^{-a}\rho,\nu^{a}\rho];\sigma) \oplus T_{-1}([\nu^{-a}\rho,\nu^{a}\rho];\sigma),$$

duality (cf. section 2) tells us

$$\hat{c}T_i([\nu^{-a}\rho,\nu^{a}\rho];\sigma) \cong T_{-i}([\nu^{-a}\rho,\nu^{a}\rho];\sigma).$$

The case $\ell = 1$ follows.

Now, suppose $\ell > 1$. Then

$$\begin{gathered} \tau \hookrightarrow \delta_1 \times \cdots \times \delta_{\ell-1} \rtimes (\delta_\ell \rtimes \sigma) \\ \Downarrow \\ \tau \hookrightarrow \delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_i(\delta_\ell;\sigma) \end{gathered}$$

for some i, where we write $\delta_\ell \rtimes \sigma \cong T_1(\delta_\ell;\sigma) \oplus T_{-1}(\delta_\ell;\sigma)$. It follows from the fact that $\delta_1 \times \cdots \times \delta_{\ell-1} \times \delta_\ell \rtimes \sigma$ decomposes with multiplicity one (cf. [G2]) that $\delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_1(\delta_\ell;\sigma)$ and $\delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_{-1}(\delta_\ell;\sigma) \cong \hat{c}(\delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_1(\delta_\ell;\sigma))$ have no components in common. Since

$$\tau \hookrightarrow \delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_i(\delta_\ell;\sigma) \quad \text{and} \quad \hat{c}\tau \hookrightarrow \hat{c}(\delta_1 \times \cdots \times \delta_{\ell-1} \rtimes T_i(\delta_\ell;\sigma)),$$

we see that $\hat{c}\tau \not\cong \tau$, as needed. ■

Proposition 7.3 *Suppose $\sigma \cong \hat{c}\sigma$ and ρ satisfies* (1.1). *Let $\pi \in R((\rho,\alpha);\sigma)$ be a non-supercuspidal, irreducible, admissible representation.*

(i) *If $\alpha \neq 0$, then $\hat{c}\pi \cong \pi$. In particular, $\mathrm{Res}^{O}_{SO}\,\pi$ is reducible.*
(ii) *If $\alpha = 0$, write $\pi = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_\ell}\tau_\ell \otimes \tau)$. Then $\hat{c}\pi \cong \pi$ if and only if $\tau = \sigma$. In particular, $\mathrm{Res}^{O}_{SO}\,\pi$ is reducible if and only if $\tau = \sigma$.*

Proof By Lemma 2.4(iv),

$$\hat{c}L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_\ell}\tau_\ell \otimes \tau) \cong L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_\ell}\tau_\ell \otimes \hat{c}\tau).$$

If $\alpha \notin \{0,\frac{1}{2}\}$, then $\tau = \sigma$ (since there are no non-supercuspidal tempered representations except when $\alpha \in \{0,\frac{1}{2}\}$) and (i) is immediate. If $\alpha \in \{0,\frac{1}{2}\}$, (ii) follows from Proposition 7.2. ■

8 Discrete Series for $SO(2n,F)$ via Restriction

In this section, we classify the non-supercuspidal discrete series for $SO(2n,F)$ (cf. Theorem 8.4). In particular, we combine the results from the previous sections to

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give a characterization in terms of restrictions from $O(2n,F)$. Note that in Sections 10 and 11, these results are reformulated in terms of admissible triples for $SO(2n,F)$.

We start with the notion of partial cuspidal support for representations of special orthogonal groups, which plays an important role in Theorem 8.4 and the results of Sections 10 and 11. Let π_0 be an irreducible admissible representation of $SO(2n,F)$. Suppose π_0 is not supercuspidal. If there is a standard Levi M having $r_{M,G}\pi_0 \geq \nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_\ell}\rho_\ell \otimes \sigma_0$ with $\rho_1,\ldots,\rho_\ell,\sigma_0$ supercuspidal, we say that σ_0 is in the partial cuspidal support of π_0. In particular, if there is a standard Levi factor M having $r_{M,G}\pi \geq \nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_\ell}\rho_\ell$ (resp., $r_{M,G}\pi \geq c(\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_\ell}\rho_\ell)$) with $\rho_1,\ldots,\rho_\ell$ supercuspidal, we say $1\otimes e$ (resp., $1 \otimes c$) is in the partial cuspidal support of π_0. Let π be an irreducible representation of $O(2n,F)$ such that $\mathrm{Res}^O_{SO}\pi \geq \pi_0$. If σ is the partial cuspidal support of π, then any σ_0 in the partial cuspidal support of π_0 must satisfy $\sigma_0 \leq \mathrm{Res}^O_{SO}\sigma$–an easy consequence of the observation $r_{M^0,G^0}\pi_0 \leq r_{M^0,G^0} \circ \mathrm{Res}^O_{SO}\pi = \mathrm{Res}^O_{SO} \circ r_{M,G}\pi$. In particular, the only possibilities are that there is a unique such σ_0 in the partial cuspidal support, or the partial cuspidal support is $\{\sigma_0, c\sigma_0\}$ with $c\sigma_0 \not\cong \sigma_0$. The following example shows that the latter can occur.

Example 8.1 Suppose $\sigma \cong \hat{c}\sigma$ and (ρ,σ) satisfies (1.1). Write $\mathrm{Res}^O_{SO}\sigma \cong \sigma_0 \oplus c\sigma_0$, where $c\sigma_0 \not\cong \sigma_0$. For $a \in \mathbb{N}$, let $\delta_\eta = \delta([\nu\rho,\nu^a\rho];\tau_\eta(\rho;\sigma))$, $\eta \in \{\pm 1\}$ be the discrete series representations defined in Section 3. These are dual in the sense of [Aub, S-S] (cf. [J4] for $O(2n,F)$) to the generalized degenerate principal series subquotients $L(\nu^{-a}\rho \otimes \nu^{-a+1}\rho \otimes \cdots \otimes \nu^{-1}\rho \otimes \tau_\eta(\rho;\sigma))$ (cf. [B-J2, Proposition 3.3]). It follows that $\hat{c}\delta_\eta \cong \delta_{-\eta}$ (cf. Lemma 2.4(iv) and Lemma 5.1(iii)) and their Jacquet modules have $r_{M,G}\delta_\eta = \nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu\rho \otimes \rho \otimes \sigma$ for the appropriate M. (These are similar to the generalized Steinberg representations of [Mu3], but fall short of his definition as the first such representations in the families–$\tau_\eta(\rho;\sigma)$–are not discrete series.) Then $\mathrm{Res}^O_{SO}\delta_1 = \mathrm{Res}^O_{SO}\delta_{-1} = \delta_0$ a discrete series representation for $SO(2n,F)$. Therefore,

$$\begin{aligned} r_{M^0,G^0}\delta_0 &\cong \mathrm{Res}^O_{SO} \circ r_{M,G}\delta_\eta \cong \mathrm{Res}^O_{SO}(\nu^a\rho \otimes \cdots \otimes \rho \otimes \sigma) \\ &\cong (\nu^a\rho \otimes \cdots \otimes \rho \otimes \sigma_0) \oplus (\nu^a\rho \otimes \cdots \otimes \rho \otimes c\sigma_0). \end{aligned}$$

In particular, δ_0 is a discrete series for $SO(2n,F)$ that has $\{\sigma_0, c\sigma_0\}$ for its partial cuspidal support.

Lemma 8.2 *Let $\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_k}\rho_k \rtimes \sigma_0$ be an irreducible representation of $SO(2n,F)$, where $\rho_1,\ldots,\rho_k$ (not necessarily distinct) are irreducible, unitary, supercuspidal representations of $GL(m_1,F),\ldots,GL(m_k,F)$ (resp., $x_1,\ldots,x_k \in \mathbb{R}$), and σ_0 an irreducible supercuspidal representation of $SO(2m,F)$.*

(i) *If $\sigma_0 \cong c\sigma_0$, then π has partial cuspidal support σ_0.*
(ii) *If $\sigma_0 \not\cong c\sigma_0$ and $m_1,\ldots,m_k$ are all even, then π has partial cuspidal support σ_0.*
(iii) *If $\sigma_0 \not\cong c\sigma_0$ and some m_i is odd, then π has partial cuspidal support $\{\sigma_0, c\sigma_0\}$.*

Proof Part (i) is clear. Part (ii) follows from [J5] applied to $\nu^{x_1}\rho_1 \times \cdots \times \nu^{x_k}\rho_k \rtimes \sigma_0$. (Roughly speaking, since $m_1,\ldots,m_k$ are all even, there are no leftover sign changes available to change σ_0 to $c\sigma_0$.)

For (iii), let i be the largest value for which m_i is odd. Then noting that $\nu^{x_i}\rho_i \times \nu^{x_j}\rho_j \cong \nu^{x_j}\rho_j \times \nu^{x_i}\rho_i$ (by irreducibility), we have

$$\begin{gathered}
\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_k}\rho_k \rtimes \sigma_0 \\
\Downarrow \\
\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_{i-1}}\rho_{i-1} \times \nu^{x_{i+1}}\rho_{i+1} \times \cdots \times \nu^{x_k}\rho_k \times \nu^{x_i}\rho_i \rtimes \sigma_0 \\
\Downarrow \text{ (cf. [J1, Lemma 5.5])} \\
\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_{i-1}}\rho_{i-1} \times \nu^{x_{i+1}}\rho_{i+1} \times \cdots \times \nu^{x_k}\rho_k \rtimes \theta
\end{gathered}$$

for some irreducible $\theta \leq \nu^{x_i}\rho_i \rtimes \sigma_0$. By Proposition 2.5, $\theta = \nu^{x_i}\rho_i \rtimes \sigma_0$. Since $\nu^{x_i}\rho_i \rtimes \sigma_0$ contains both σ_0 and $c\sigma_0$ in its supercuspidal support, the result follows. ■

Definition 8.3 Let $(Jord, \sigma, \varepsilon)$ be an admissible triple.

(i) If $\hat{c}\sigma \not\cong \sigma$, we define $\hat{c}(Jord, \sigma, \varepsilon) = (Jord, \hat{c}\sigma, \varepsilon)$.
(ii) If $\hat{c}\sigma \cong \sigma$, we define $\hat{c}(Jord, \sigma, \varepsilon) = (Jord, \sigma, \hat{c}\varepsilon)$, where $\hat{c}\varepsilon$ is given by

$$\hat{c}\varepsilon(\rho, a) = \begin{cases} -\varepsilon(\rho, a) & \text{if } \rho \text{ satisfies (1.1)}, \\ \varepsilon(\rho, a) & \text{if not}, \end{cases}$$

and ε remains unchanged on pairs.

Theorem 8.4 *Let $(Jord, \sigma, \varepsilon) \in \mathrm{Trip}_{adm}$. Then $\hat{c}\delta_{(Jord,\sigma,\varepsilon)} = \delta_{\hat{c}(Jord,\sigma,\varepsilon)}$. Further, if $\sigma_0 \leq \mathrm{Res}^O_{SO}\, \sigma$ is irreducible, we have the following:*

(i) *Suppose $\hat{c}\sigma \not\cong \sigma$. If the discrete series are parameterized as in Remark 4.2, then $\hat{c}(Jord, \sigma, \varepsilon) \not\cong (Jord, \sigma, \varepsilon)$ and*

$$\mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)} \cong \mathrm{Res}^O_{SO}\, \delta_{(Jord,\hat{c}\sigma,\varepsilon)}$$

is a discrete series representation for $SO(2n, F)$ having partial cuspidal support σ_0. Every discrete series representation of an even special orthogonal group having partial cuspidal support σ_0 may be written uniquely this way, up to the choice of σ or $\hat{c}\sigma$.

(ii) *Suppose $\hat{c}\sigma \cong \sigma$ and there is no $(\rho, a) \in Jord$ with ρ satisfying (1.1). Then $\hat{c}(Jord, \sigma, \varepsilon) = (Jord, \sigma, \varepsilon)$ and*

$$\mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)} \cong \delta_{\sigma_0} \oplus \delta_{c\sigma_0},$$

with $c\delta_{\sigma_0} \cong \delta_{c\sigma_0}$, a direct sum of inequivalent discrete series having partial cuspidal support σ_0 and $c\sigma_0$, resp. Every discrete series representation of an even special orthogonal group having partial cuspidal support σ_0 or $c\sigma_0$ may be written uniquely this way.

(iii) *Suppose $\hat{c}\sigma \cong \sigma$ and there is some $(\rho, a) \in Jord$ with ρ satisfying (1.1). Then $\hat{c}(Jord, \sigma, \varepsilon) \neq (Jord, \sigma, \varepsilon)$ and*

$$\mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)} \cong \mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\hat{c}\varepsilon)}$$

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is a discrete series representation for $SO(2n, F)$ *having partial cuspidal support* $\{\sigma_0, c\sigma_0\}$. *Every discrete series representation of an even special orthogonal group having partial cuspidal support* $\{\sigma_0, c\sigma_0\}$ *may be written uniquely this way, up to the choice of* ε *or* $\hat{c}\varepsilon$.

Proof To see that $\hat{c}\delta_{(Jord,\sigma,\varepsilon)} = \delta_{\hat{c}(Jord,\sigma,\varepsilon)}$, observe that by Theorem 2.7(v),

$$\hat{c}\psi_{(\rho,\alpha)}(\delta_{(Jord,\sigma,\varepsilon)}) = \psi_{(\rho,\alpha)}(\hat{c}\delta_{(Jord,\sigma,\varepsilon)}).$$

By Remark 4.2 and the fact that $Jord(\sigma) = Jord(\hat{c}\sigma)$ (if $\hat{c}\sigma \not\cong \sigma$), Theorem 5.3 (if $\hat{c}\sigma \cong \sigma$ and (1.1) not satisfied), Theorem 6.5 (if $\hat{c}\sigma \cong \sigma$ and (1.1) satisfied), and Remark 3.1,

$$\hat{c}\psi_{(\rho,\alpha)}\delta_{(Jord,\sigma,\varepsilon)} = \begin{cases} \psi_{(\rho,\alpha)}(\delta_{(Jord,\hat{c}\sigma,\varepsilon)}) & \text{if } \sigma \not\cong \hat{c}\sigma, \\ \psi_{(\rho,\alpha)}(\delta_{(Jord,\sigma,\varepsilon)}) & \text{if } \sigma \cong \hat{c}\sigma \text{ and } \rho \text{ does not satisfy (1.1)}, \\ \psi_{(\rho,\alpha)}(\delta_{(Jord,\sigma,\hat{c}\varepsilon)}) & \text{if } \sigma \cong \hat{c}\sigma \text{ and } \rho \text{ satisfies (1.1)}. \end{cases}$$

It now follows from Remark 3.1 that $\hat{c}\delta_{(Jord,\sigma,\varepsilon)} = \delta_{\hat{c}(Jord,\sigma,\varepsilon)}$.

In (i)–(iii), the equality or inequality of $\hat{c}(Jord, \sigma, \varepsilon)$ and $(Jord, \sigma, \varepsilon)$ follows immediately from the discussion above. The reducibility of $\mathrm{Res}^{O}_{SO}\,\delta_{(Jord,\sigma,\varepsilon)}$ is then an immediate consequence of Lemma 2.3; square-integrability of the components is automatic. The supercuspidal support claims are covered by Lemma 8.2. The fact that any discrete series of $SO(2n, F)$ with the given properties may be written uniquely as such a restriction is a straightforward consequence of Lemma 2.3. ■

9 Restrictions of Irreducible Admissible Representations

In this section, we address the general question of when an irreducible admissible representation of $O(2n, F)$ has $\mathrm{Res}^{O}_{SO}\,\pi$ reducible. Proposition 9.1 addresses the case π tempered, using the classification of tempered representations to reduce to a corresponding question about discrete series (covered by Theorem 8.4). Proposition 9.2 uses the Langlands classification to address the admissible case, which easily reduces to a corresponding question about tempered representations (covered by Proposition 9.1)

Proposition 9.1 *Let* τ *be an irreducible tempered representation of* $O(2n, F)$. *Write*

$$\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \delta,$$

with $\delta_1, \ldots, \delta_k$ *discrete series for general linear groups and* δ *a discrete series representation for an orthogonal group (possibly* $\delta = 1$*).*

(i) *If* $\hat{c}\delta \not\cong \delta$, *then* $\hat{c}\tau \not\cong \tau$. *In particular,* $\mathrm{Res}^{O}_{SO}\,\tau$ *is irreducible.*
(ii) *If* $\hat{c}\delta \cong \delta$, *then* $\hat{c}\tau \cong \tau$ *if and only if no* δ_i *has the form* $\delta([\nu^{-a}\rho, \nu^{a}\rho])$ *with* $a \in \mathbb{N} \cup \{0\}$ *and* ρ *satisfying* (1.1). *In particular,* $\mathrm{Res}^{O}_{SO}\,\tau$ *is reducible if and only if no* δ_i *has this form.*

Note that the question of whether $\hat{c}\delta \cong \delta$ *can be addressed using Theorem* 8.4.

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Proof Part (i) follows from an argument like that used in the proof of Proposition 7.2. By Lemma 2.4(i),

$$\hat{c}\tau \hookrightarrow \hat{c}(\delta_1 \times \cdots \times \delta_k \rtimes \delta) \cong \delta_1 \times \cdots \times \delta_k \rtimes \hat{c}\delta.$$

Since $\hat{c}\delta \ncong \delta$, it follows from the appendix that $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ and $\delta_1 \times \cdots \times \delta_k \rtimes \hat{c}\delta$ have no irreducible subquotients in common, so $\hat{c}\tau \ncong \tau$.

For (ii), observe that $\hat{c}\delta \cong \delta$ implies $\hat{c}\sigma \cong \sigma$ (cf. Theorem 8.4). Now, first suppose some $\delta_i \cong \delta([\nu^{-a}\rho, \nu^a\rho])$, with $a \in \mathbb{N} \cup \{0\}$ and ρ satisfying (1.1). Then by Proposition 7.2(ii), $\psi_{(\rho,0)}(\tau)$ (cf. Section 2) has $\hat{c}\psi_{(\rho,0)}(\tau) \ncong \psi_{(\rho,0)}(\tau)$. Therefore, by Theorem 2.7(v) $\hat{c}\tau \ncong \tau$. Now, suppose no δ_i has this form. Then for any (ρ,α) having $\psi_{(\rho,\alpha)}(\tau)$ nontrivial (*i.e.*, not σ), we claim $\hat{c}\psi_{(\rho,\alpha)}(\tau) \cong \psi_{(\rho,\alpha)}(\tau)$. If ρ does not satisfy (1.1), the claim follows from Theorem 5.3; if ρ satisfies (1.1), it follows from Proposition 7.2(i). The result now follows from Theorem 2.7(v). ■

Proposition 9.2 *Let π be an irreducible admissible representation of $O(2n,F)$. Write*

$$\pi = L(\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \tau)$$

(the Langlands classification; cf. Section 2). Then $\hat{c}\pi \cong \pi$, and in particular, $\mathrm{Res}^{O}_{SO}\pi$ *is reducible if and only if $\hat{c}\tau \cong \tau$. Note that the question of whether $\hat{c}\tau \cong \tau$ may be addressed using Proposition* 9.1.

Proof This follows immediately from Lemma 2.4(iv). ■

10 Admissible Triples for $SO(2n,F)$

In this section, we define admissible triples for $SO(2n,F)$. Theorem 10.7 establishes an explicit bijective correspondence between admissible triples for $SO(2n,F)$, modulo an equivalence relation $\sim$, and discrete series for $SO(2n,F)$. The correspondence in Theorem 10.7 is described via restrictions of discrete series for $O(2n,F)$. In the next section, we characterize the correspondence along the lines of [M-T], without reference to representations of $O(2n,F)$.

We take a moment to give a general discussion motivating our definition.

First, suppose we have $\sigma \ncong \hat{c}\sigma$. By Theorem 8.4(i), the discrete series for special orthogonal groups having partial cuspidal support σ_0 may be parameterized by the admissible triples $(Jord, \sigma, \varepsilon)$ (or $(Jord, \hat{c}\sigma, \varepsilon)$). Thus, in this case, our goal is essentially to replace σ (or $\hat{c}\sigma$) by σ_0 in the definition of admissible triple; *i.e.*, we want to retain the same combinatorial conditions on $Jord$ and ε but reformulate them in terms of σ_0.

Now, suppose we have $\sigma \cong \hat{c}\sigma$ but (1.1) not satisfied. By Theorem 8.4(ii), the discrete series having partial cuspidal support σ_0 (resp., $c\sigma_0$) may be parameterized by the admissible triples having partial cuspidal support σ. (Recall that for $\sigma_0 = 1 \otimes e$ or $\sigma_0 = 1 \otimes c$, this should be interpreted as in Section 8.) Thus, again our goal is essentially to keep the same definition of admissible triple, but with σ_0 (resp., $c\sigma_0$) replacing σ.

The case $\sigma \cong \hat{c}\sigma$ and (1.1) introduces a new issue. By Theorem 8.4(iii), the discrete series having partial cuspidal support $\{\sigma_0, c\sigma_0\}$ may be parameterized by the admissible triples $(Jord, \sigma, \varepsilon)$ modulo $\sim$, where $(Jord, \sigma, \varepsilon) \sim (Jord', \sigma', \varepsilon')$ if $Jord' = Jord$, $\sigma' = \sigma$, and $\varepsilon' = \varepsilon$ or $\hat{c}\varepsilon$. Here, we again want to replace σ by σ_0 in the definition of admissible triple, but also need to introduce a quotient by $\sim$ in formulating the bijective correspondence.

To work without reference to representations of $O(2n, F)$, we must first reformulate the definition of $Jord(\delta)$ to obtain a corresponding definition for $Jord(\delta_0)$, where $\delta_0 \leq \mathrm{Res}^{O}_{SO}\,\delta$ are discrete series representations. The following lemma indicates the changes needed.

Lemma 10.1 *Suppose $\delta_0 \leq \mathrm{Res}^{O}_{SO}\,\delta$ are discrete series. Then $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible if and only if the following hold:*

(i) $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0$ *is irreducible.*

(ii) *If* $\delta_0 \not\cong c\delta_0$, *then* $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0 \not\cong c(\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0)$.

Proof First, suppose $\delta_0 \not\cong c\delta_0$. Then $\delta \cong \mathrm{Ind}^{O}_{SO}\,\delta_0$, so we have

$$\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta \cong \mathrm{Ind}^{O}_{SO}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big).$$

Now, suppose $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible. It is then immediate that $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0$ must be irreducible, so (i) holds. Further, it must also induce irreducibly to $O(2n, F)$, from which we see that

$$c\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \not\cong \delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0,$$

so that (ii) holds. In the other direction, suppose (i) and (ii) hold. Then

$$\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0$$

is an irreducible representation having

$$c\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \not\cong \delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0.$$

Therefore, it induces irreducibly to $O(2n, F)$, so

$$\mathrm{Ind}^{O}_{SO}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \cong \delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta$$

is irreducible, as needed.

Now, suppose $\delta_0 \cong c\delta_0$. We note that in this case, condition (ii) does not arise. We also observe that $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \hat{c}\delta \cong \hat{c}(\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta)$ (cf. Lemma 2.4(i)), so $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible if and only if

$$\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \hat{c}\delta$$

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is. First, suppose that $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0$ is irreducible. Since

$$\mathrm{Ind}_{SO}^{O}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big)$$

has at most two components and is equivalent to

$$\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta\Big) \oplus \Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \hat{c}\delta\Big),$$

we see that $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible. In the other direction, suppose $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible. Since $\hat{c}\delta \not\cong \delta$, it follows from the appendix that

$$\begin{aligned}\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta &\not\cong \delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \hat{c}\delta \\ &\cong \hat{c}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta\Big).\end{aligned}$$

Therefore, by Lemma 2.3, $\mathrm{Res}_{SO}^{O}(\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta)$ is irreducible. Therefore,

$$\begin{aligned}&\mathrm{Res}_{SO}^{O} \circ \mathrm{Ind}_{SO}^{O}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \\ &\quad\cong \mathrm{Res}_{SO}^{O}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta\Big) \oplus \Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \hat{c}\delta)\Big)\end{aligned}$$

has two components. Now (cf. [B-J2, Lemma 4.1]),

$$\begin{aligned}&\mathrm{Res}_{SO}^{O} \circ \mathrm{Ind}_{SO}^{O}\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \cong \\ &\quad\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big) \oplus c\Big(\delta\big(\big[\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho\big]\big) \rtimes \delta_0\Big).\end{aligned}$$

Thus $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0$ must be irreducible, as needed. ■

Let δ_0 be a discrete series representation for $SO(2n, F)$. $Jord(\delta_0)$ is defined to be the set of pairs (ρ, a) having $\rho \cong \tilde{\rho}$ and $a \in \mathbb{N}$ that satisfy the following:

(i) a is even if and only if the L-function $L(\rho, R_d, s)$ has a pole at $s = 0$. (Again, for ρ a representation of $GL(d, F)$, $L(\rho, R_d, s)$ is the L-function defined by Shahidi, where R_d is the representation of $GL(d, \mathbb{C})$ on $\wedge^2\mathbb{C}^d$).

(ii) $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0$ is irreducible.

(iii) If $\delta_0 \not\cong c\delta_0$, then $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0 \not\cong c(\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta_0)$.

Note that for $\Sigma_0 = \{\sigma_0, c\sigma_0\}$, we define $Jord(\Sigma_0) = Jord(\sigma_0) = Jord(c\sigma_0)$ (noting that the last two sets are clearly equal). The following is an immediate consequence of Lemma 10.1.

Corollary 10.2 $Jord(\delta_0) = Jord(\delta)$.

We are now ready to define Trip_{SO}. Again, for convenience, we use representations in the following description of admissible triples when we actually want equivalence classes of representations; the reader should interpret the discussion below accordingly. Trip_{SO} is the collection of all triples $(Jord, \Sigma_0, \varepsilon)$ that satisfy the following:

(i) $Jord$ is a finite (possibly empty) set of pairs (ρ, a), where ρ is an irreducible, unitary, supercuspidal representation of a general linear group having $\tilde{\rho} \cong \rho$, and $a \in \mathbb{N}$ with a even if and only if $L(s, \rho, R_{d_\rho})$ has a pole at $s = 0$.

(ii) If $Jord$ does not satisfy (1.1) (*i.e.*, there is no $(\rho, a) \in Jord$ with ρ satisfying (1.1)), then $\Sigma_0 = \{\sigma_0\}$, where σ_0 is an irreducible, unitary, supercuspidal representation of some $SO(2n, F)$, $n \neq 1$. If $Jord$ satisfies (1.1), then $\Sigma_0 = \{\sigma_0, c\sigma_0\}$ (noting that if $c\sigma_0 \cong \sigma_0$, we again have $\Sigma_0 = \{\sigma_0\}$). When Σ_0 consists of a single element σ_0, we will normally write $(Jord, \sigma_0, \varepsilon)$ rather than $(Jord, \{\sigma_0\}, \varepsilon)$ for the triple.

(iii) $\varepsilon\colon S \to \{\pm 1\}$ is a function on a subset $S \subset Jord \cup (Jord \times Jord)$ (defined below) and satisfying conditions (a)–(c) below.

Let us start by describing the domain S of ε. S contains all $(\rho, a) \in Jord$ except those having a odd and $(\rho, a') \in Jord(\Sigma_0)$ for some $a' \in \mathbb{N}$; S contains $\big((\rho, a), (\rho', a')\big) \in Jord \times Jord$ when $\rho \cong \rho'$ and $a \neq a'$. Several compatibility conditions must also be satisfied:

(a) if $(\rho, a), (\rho, a') \in S$, we must have $\varepsilon\left((\rho, a), (\rho, a')\right) = \varepsilon(\rho, a)\varepsilon^{-1}(\rho, a')$;

(b)
$$\varepsilon\left((\rho, a), (\rho, a'')\right) = \varepsilon\left((\rho, a), (\rho, a')\right)\varepsilon\left((\rho, a'), (\rho, a'')\right)$$
for all $(\rho, a), (\rho, a'), (\rho, a'') \in Jord$ having a, a', a'' distinct; and

(c) $\varepsilon\left((\rho, a), (\rho, a')\right) = \varepsilon\left((\rho, a'), (\rho, a)\right)$ for all $\left((\rho, a), (\rho, a')\right) \in S$.

We follow the notation of [M-T] and, in light of (i) above, write $\varepsilon(\rho, a)\varepsilon^{-1}(\rho, a')$ for $\varepsilon\left((\rho, a), (\rho, a')\right)$ even when ε is undefined on (ρ, a) and (ρ, a') separately (*i.e.*, even when (ρ, a) and (ρ, a') are not in S).

Let $\sigma_0 \in \Sigma_0$. In what follows, we let σ be a component of $\mathrm{Ind}_{SO}^{O}\sigma_0$ (making a choice, if necessary). The possible σ do not depend on the choice of $\sigma_0 \in \Sigma_0$.

Lemma 10.3 *$(Jord, \sigma, \varepsilon) \in \mathrm{Trip}_O$ if and only if $(Jord, \Sigma_0, \varepsilon) \in \mathrm{Trip}_{SO}$.*

Proof This follows immediately from the definitions and the fact that $Jord(\sigma) = Jord(\Sigma_0)$ (cf. Corollary 10.2). ■

We now discuss triples of alternated type. Suppose $(\rho, a) \in Jord$. We again define (ρ, a_-) by taking $a_- = \max\{a' \in \mathbb{N} \mid (\rho, a') \in Jord \text{ and } a' < a\}$ (noting that (ρ, a_-) may be undefined). Also, let us write $Jord_\rho = \{(\rho', a) \in Jord \mid \rho' \cong \rho\}$, $Jord_\rho(\Sigma_0) = \{(\rho', a) \in Jord(\Sigma_0) \mid \rho' \cong \rho\}$, and

$$Jord'_\rho(\Sigma_0) = \begin{cases} Jord_\rho(\Sigma_0) \cup \{(\rho, 0)\} & \text{if } a \text{ is even and } \varepsilon(\rho, min\ Jord_\rho) = 1, \\ Jord_\rho(\Sigma_0) & \text{otherwise.} \end{cases}$$

We call $(Jord, \Sigma_0, \varepsilon) \in \mathrm{Trip}_{SO}$ a triple of alternated type if the following hold: (i) $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = -1$ whenever (ρ, a_-) is defined, and (ii) $|Jord_\rho| = |Jord'_\rho(\Sigma_0)|$. We write $\mathrm{Trip}_{SO,\mathrm{alt}}$ for the subset of all alternated triples in Trip_{SO}.

Lemma 10.4 $(Jord, \sigma, \varepsilon) \in \mathrm{Trip}_{O,\mathrm{alt}}$ *if and only if* $(Jord, \Sigma_0, \varepsilon) \in \mathrm{Trip}_{SO,\mathrm{alt}}$.

Proof It follows immediately from Corollary 10.2 that $Jord'(\sigma) = Jord'(\Sigma_0)$. The lemma now follows from the definitions. ■

This brings us to admissible triples. Here we observe a difference with the case of $O(2n, F)$, owing to the dependence of Σ_0 on $Jord$ (in particular, on whether (1.1) holds). First, suppose $(Jord, \Sigma_0, \varepsilon) \in \mathrm{Trip}$ has $(\rho, a) \in Jord$ with (ρ, a_-) defined and $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$. Set $Jord' = Jord \setminus \{(\rho, a), (\rho, a_-)\}$ and let ε' be the restriction of ε to $S \cap [Jord' \cup (Jord' \times Jord')]$. If $Jord$ satisfies (1.1) but $Jord'$ does not, let $\Sigma_0' = \{\sigma_0\}$ (*i.e.*, choose an element of Σ_0); otherwise, let $\Sigma_0' = \Sigma_0$. One can check that $(Jord', \sigma, \varepsilon') \in \mathrm{Trip}_{SO}$. We say that $(Jord', \Sigma_0', \varepsilon')$ is subordinated to $(Jord, \Sigma_0, \varepsilon)$. We say the triple $(Jord, \Sigma_0, \varepsilon)$ is admissible if there is a sequence of triples $(Jord_i, \Sigma_0^{(i)}, \varepsilon_i)$, $1 \le i \le k$, such that (i) $(Jord_1, \Sigma_0^{(1)}, \varepsilon_1) = (Jord, \Sigma_0, \varepsilon)$, (ii) $(Jord_{i+1}, \Sigma_0^{(i+1)}, \varepsilon_{i+1})$ is subordinated to $(Jord_i, \Sigma_0^{(i)}, \varepsilon_i)$ for all $1 \le i \le k-1$, and (iii) $(Jord_k, \Sigma_0^{(k)}, \varepsilon_k)$ is of alternated type. Note that the choice of $\sigma_0 \in \Sigma_0$ that may need to be made does not affect admissibility since $(Jord, \sigma_0, \varepsilon) \in \mathrm{Trip}_{SO}$ (resp., $\mathrm{Trip}_{SO,\mathrm{alt}}$) if and only if $(Jord, c\sigma_0, \varepsilon) \in \mathrm{Trip}_{SO}$ (resp., $\mathrm{Trip}_{SO,\mathrm{alt}}$). If a choice is required at the j-th step, one can replace $(Jord_i, \sigma_0, \varepsilon_i)$ for $i \ge j$ with $(Jord_i, c\sigma_0, \varepsilon_i)$ and still satisfy the conditions for admissibility (in fact, one can show that the choice made does not matter). Let us call such a sequence of triples an admissible sequence. We write $\mathrm{Trip}_{SO,adm}$ for the set of admissible triples.

Lemma 10.5 $(Jord, \sigma, \varepsilon) \in \mathrm{Trip}_{O,adm}$ *if and only if* $(Jord, \Sigma_0, \varepsilon) \in \mathrm{Trip}_{SO,adm}$.

Proof It is a routine matter to check that if the sequence $(Jord_i, \sigma, \varepsilon_i)$ satisfies the conditions needed to have $(Jord, \sigma, \varepsilon)$ admissible, then the corresponding sequence $(Jord_i, \Sigma_0^{(i)}, \varepsilon_i)$ satisfies the conditions for $(Jord, \Sigma_0, \varepsilon)$ to be admissible (noting that $(Jord_k, \sigma, \varepsilon_k) \in \mathrm{Trip}_{O,\mathrm{alt}}$ if and only if $(Jord_k, \Sigma_0^{(k)}, \varepsilon_k) \in \mathrm{Trip}_{SO,\mathrm{alt}}$ by Lemma 10.4). ■

We now turn to the task of classifying discrete series for $SO(2n, F)$, $n \neq 1$, using admissible triples to parameterize them. More precisely, we use $\mathrm{Trip}_{SO,adm}/\sim$, where the equivalence relation $\sim$ is defined below:

Definition 10.6 We define the equivalence relation $\sim$ on Trip_{SO} by setting $(Jord, \Sigma_0, \varepsilon) \sim (Jord', \Sigma_0', \varepsilon')$ if the following all hold:

(i) $Jord = Jord'$,

(ii) $\Sigma_0 = \Sigma_0'$, and

(iii) $\varepsilon' = \begin{cases} \varepsilon & \text{or } \hat{c}\varepsilon \text{ if } \Sigma_0 = \{\sigma_0, c\sigma_0\} \text{ (with } \sigma_0 \ncong c\sigma_0\text{) and } Jord \text{ satisfies (1.1)} \\ \varepsilon & \text{otherwise.} \end{cases}$

Here, $\hat{c}\varepsilon$ is defined as in Definition 8.3(ii).

We remark that the equivalence class containing $(Jord, \Sigma_0, \varepsilon)$ has two elements if and only if $|\Sigma_0| = 2$ and there is some $\rho \in Jord$ which satisfies (1.1); otherwise, the equivalence class contains only one element.

Theorem 10.7 *The discrete series representations for $SO(2n, F)$, $n \neq 1$, are in bijective correspondence with $\mathrm{Trip}_{SO,adm}/\sim$. More precisely, we have the following:*

(i) *If the triple $(Jord, \sigma_0, \varepsilon)$ has $c\sigma_0 \cong \sigma_0$, then we take*

$$\delta_{(Jord,\sigma_0,\varepsilon)} = \mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)}$$

(noting that the restriction is irreducible by Theorem 8.4(i)*).*

(ii) *If the triple $(Jord, \sigma_0, \varepsilon)$ has $c\sigma_0 \ncong \sigma_0$, then we take*

$$\delta_{(Jord,\sigma_0,\varepsilon)} \leq \mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)},$$

choosing the component whose partial cuspidal support is σ_0 (noting the other component has partial cuspidal support $c\sigma_0$; cf. Theorem 8.4(ii)*).*

(iii) *If the triple $(Jord, \{\sigma_0, c\sigma_0\}, \varepsilon)$ has $c\sigma_0 \ncong \sigma_0$, then we take*

$$\delta_{(Jord,\{\sigma_0,c\sigma_0\},\varepsilon)} = \mathrm{Res}^O_{SO}\, \delta_{(Jord,\sigma,\varepsilon)}$$

(noting that the restriction is irreducible by Theorem 8.4(iii)*).*

Proof This follows from Theorem 8.4 and Lemma 10.5. ■

11 Discrete Series for $SO(2n, F)$ via Admissible Triples

Let δ_0 be a discrete series representation of $SO(2n, F)$, $n \neq 1$, and $(Jord_{\delta_0}, \Sigma_{\delta_0}, \varepsilon_{\delta_0}) \in \mathrm{Trip}_{SO,adm}$ an admissible triple associated with δ_0 by Theorem 10.7 (noting that if $Jord$ satisfies (1.1), replacing ε_{δ_0} by $\hat{c}\varepsilon_{\delta_0}$ will produce a $\sim$-equivalent associated triple). Our aim in this section is to describe the data $(Jord_{\delta_0}, \Sigma_{\delta_0}, \varepsilon_{\delta_0})$ in terms of δ_0, along the lines of [M-T, T5, T6].

Let δ be a discrete series representation of $O(2n, F)$ such that $\delta_0 \leq \mathrm{Res}^O_{SO}\, \delta$. Write $(Jord_\delta, \sigma_\delta, \varepsilon_\delta)$ for its associated admissible triple. Applying Theorem 10.7, the [M-T] definition (cf. Section 3 of this paper), and Corollary 10.2 successively, we get

$$Jord_{\delta_0} = Jord_\delta = Jord(\delta) = Jord(\delta_0).$$

In addition, it follows from Theorem 10.7 that Σ_{δ_0} is the partial cuspidal support of δ_0. Thus, all that remains is to describe ε in terms of δ_0. To this end, we start with a lemma.

Lemma 11.1 *Let τ be an irreducible admissible representation of $GL(m, F)$, and let π, π_0 be irreducible admissible representations of $O(2n, F)$, $SO(2n, F)$, resp., with $\pi_0 \leq \mathrm{Res}^O_{SO}\, \pi$. Then there is an irreducible representation π' of $O(2(n-m), F)$ having $\pi \hookrightarrow \tau \rtimes \pi'$ if and only if there is an irreducible representation π'_0 of $SO(2(n-m), F)$ having $\pi_0 \hookrightarrow \tau \rtimes \pi'_0$. Note that this includes the possibility that $\pi' = 1$ and $\pi'_0 = 1 \otimes e$ or $1 \otimes c$.*

Proof We prove the case $m < n$; $m = n$ is essentially the same.

($\Rightarrow$): Choose π_0' such that $\pi' \leq \mathrm{Ind}_{SO}^{O} \pi_0'$. Then

$$\begin{gathered}\pi \hookrightarrow \tau \rtimes \pi' \hookrightarrow \tau \rtimes \mathrm{Ind}_{SO}^{O} \pi_0' \cong \mathrm{Ind}_{SO}^{O}(\tau \rtimes \pi_0') \\ \Downarrow \\ \pi_0 \hookrightarrow \mathrm{Res}_{SO}^{O} \circ \mathrm{Ind}_{SO}^{O}(\tau \rtimes \pi_0') \cong (\tau \rtimes \pi_0') \oplus c(\tau \rtimes \pi_0')\end{gathered}$$

(cf. [B-J2, Lemma 4.1]). Therefore, (cf. Lemma 2.4(i))

$$\begin{gathered}\pi_0 \hookrightarrow (\tau \rtimes \pi_0') \oplus (\tau \rtimes c\pi_0') \\ \Downarrow \\ \pi_0 \hookrightarrow \tau \rtimes \pi_0' \text{ or } \pi_0 \hookrightarrow \tau \rtimes c\pi_0',\end{gathered}$$

and the result follows.

($\Leftarrow$): Here, we have

$$\pi \hookrightarrow \mathrm{Ind}_{SO}^{O} \pi_0 \hookrightarrow \mathrm{Ind}_{SO}^{O} \tau \rtimes \pi_0' \cong \tau \rtimes \mathrm{Ind}_{SO}^{O} \pi_0'.$$

It follows that there is some irreducible $\pi' \leq \mathrm{Ind}_{SO}^{O} \pi_0'$ such that $\pi \hookrightarrow \tau \rtimes \pi'$, as needed. ■

We note that by Theorem 10.7, we may take $\varepsilon_{\delta_0} = \varepsilon_\delta$ (though again, $\hat{c}\varepsilon_\delta$ also works if (1.1) is satisfied). Since $\hat{c}\varepsilon_\delta$ and ε_δ differ only on those $(\rho, a) \in \mathit{Jord}$ satisfying (1.1), we see that $\varepsilon_{\delta_0}|_{S\cap(\mathit{Jord}\times \mathit{Jord})}$ does not depend on which is used. It then follows immediately from the lemma above and (3.1) that

(11.1)
$$\begin{gathered}\varepsilon_{\delta_0}(\rho, a)\varepsilon_{\delta_0}^{-1}(\rho, a_-) = 1 \\ \Updownarrow \\ \text{there is an irreducible representation } \delta_0' \text{ such that } \delta_0 \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta_0'\end{gathered}$$

(noting we could have $\delta_0' = 1 \otimes e$ or $1 \otimes c$). By condition (iii)(b) in the definition of triple (cf. Section 10), this is sufficient to define $\varepsilon_{\delta_0}(\rho, a)\varepsilon_{\delta_0}^{-1}(\rho, b)$ for all pairs $((\rho, a), (\rho, b)) \in S$.

For $(\rho, a) \in S$ with a even, we cannot have (1.1) satisfied. Therefore, $\varepsilon_{\delta_0}(\rho, a)$ is again independent of whether ε_δ or $\hat{c}\varepsilon_\delta$ is used. We again want $\varepsilon_{\delta_0}(\rho, a) = \varepsilon_\delta(\rho, a)$. This may be effected by formally setting $\varepsilon_{\delta_0}(\rho, 0) = 1$ and using equation (11.1) to define $\varepsilon_{\delta_0}(\rho, a)$ for $(\rho, a) \in S$.

For $(\rho, a) \in S$ with a odd, we can have (ρ, a) satisfying (1.1). In this case, we give a characterization similar to that of [T5, T6]. Let $\rho_1, \dots, \rho_\ell$ be inequivalent representations such that the following holds: $(\rho, a) \in S$ with a odd if and only if $\rho \cong \rho_i$ for some i. To start, we make a choice of one component

$$\mathcal{S} \leq \rho_1 \times \cdots \times \rho_\ell \rtimes \sigma_0 \cong \rho_1 \times \cdots \times \rho_\ell \rtimes c\sigma_0.$$

We remark that when $c\sigma_0 \not\cong \sigma_0$, the equivalence follows from the general observation that if ρ is a representation of $GL(m, F)$ with m odd, then $\rho \otimes c\sigma_0$ is a Weyl conjugate of $\rho \otimes \sigma_0$.

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In the case $\sigma_0 \cong c\sigma_0$, this induced representation has 2^ℓ components (cf. [G1, Theorems 5.16, 5.19, 5.20 (mislabeled), and 6.5]). In the case where $\sigma_0 \not\cong c\sigma_0$ and no ρ_i satisfies (1.1), there are also 2^ℓ components (cf. [G1, Theorems 5.9, 5.19, and 6.5]). In these cases, each $\rho_i \rtimes \sigma_0$ is reducible; choosing a component of $\rho_1 \times \cdots \times \rho_\ell \rtimes \sigma_0$ is equivalent to choosing components of $\rho_i \rtimes \sigma_0$ for $i = 1, \ldots, \ell$. In particular, we have $\rho_i \rtimes \sigma_0 \cong \varsigma_1(\rho; \sigma_0) \oplus \varsigma_{-1}(\rho; \sigma_0)$, where the components are characterized by

$$\mathcal{S} \leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes \varsigma_1(\rho; \sigma_0)$$

and

$$\mathcal{S} \not\leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_\ell \rtimes \varsigma_{-1}(\rho; \sigma_0).$$

In the case where $\sigma_0 \not\cong c\sigma_0$ and some ρ_i satisfies (1.1), this induced representation has $2^{\ell-1}$ components (cf. [G1, Theorems 5.8, 5.9, 5.16, 5.19, 6.5, 6.8, and 6.11]). In this case, for each ρ_i satisfying (1.1), $\rho_i \rtimes \sigma_0$ is irreducible, so we do not have the option of making the choices separately. We also note that if $\ell = 1$, there is no choice to be made; $\rho_1 \rtimes \sigma_0$ is irreducible. (In the corresponding situation for $O(2n, F)$, there is a choice of components for $\rho_1 \rtimes \sigma$ to be made. Roughly speaking, this choice is used to distinguish between members of a pair of discrete series, both of which have the same restriction to $SO(2n, F)$; cf. Theorem 8.4(iii).)

Remark 11.2 For $O(2n, F)$, we fixed a choice of components $\mathcal{T} \leq \rho_1 \times \cdots \times \rho_\ell \rtimes \sigma$ (see the end of Section 3). If we want the parameterizations of discrete series to behave well with respect to Res^O_{SO} and Ind^O_{SO}, we should choose $\mathcal{S} \leq \mathrm{Res}^O_{SO} \mathcal{T}$. In what follows, we assume this holds. In particular, we may choose $\mathcal{S}$ arbitrarily and then make an appropriate choice of $\mathcal{T}$ (*i.e.*, $\mathcal{T} \leq \mathrm{Ind}^O_{SO} \mathcal{S}$).

Lemma 11.3 *Suppose δ_0, δ are discrete series with $\delta_0 \leq \mathrm{Res}^O_{SO} \delta$ and $\delta \not\cong 1$. For θ irreducible, we have*

$$\mathrm{Res}^O_{SO}(\theta \rtimes \delta) \cong \theta \rtimes \mathrm{Res}^O_{SO} \delta \cong \begin{cases} \theta \rtimes \delta_0, & \textit{if } c\delta_0 \cong \delta_0, \\ (\theta \rtimes \delta_0) \oplus (\theta \rtimes c\delta_0), & \textit{if } c\delta_0 \not\cong \delta_0. \end{cases}$$

Proof The first claim follows from [J4, Proposition 3.5] (an extension of [B-Z, Lemma 2.12] to non-connected groups). The second claim now follows from Lemma 2.3. ■

We now break the analysis into three cases, along the same lines as Theorems 8.4 and 10.7.

Case 1. $c\sigma_0 \cong \sigma_0$: Observe that in this case, if we have chosen $\mathcal{T}$ as in Remark 11.2, then

$$\mathrm{Res}^O_{SO} \tau_\eta(\rho; \sigma) = \varsigma_\eta(\rho; \sigma_0)$$

(by Theorem 4.1). For $a \in \mathbb{N}$ with a odd and $\eta \in \{\pm 1\}$, we claim that $\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \varsigma_\eta(\rho; \sigma_0)$ has a unique irreducible subrepresentation, which is square-integrable. In particular, any irreducible subrepresentation δ_0 satisfies

$$\delta_0 \hookrightarrow \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \tau_\eta(\rho;\sigma_0)$$
$$\Downarrow$$
$$\mathrm{Ind}_{SO}^{O}\,\delta_0 \hookrightarrow \mathrm{Ind}_{SO}^{O}\left(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \varsigma_\eta(\rho;\sigma_0)\right)$$
$$\Downarrow \text{ (cf. Theorem 4.1 and Lemma 2.3)}$$
$$\delta \oplus \hat{c}\delta \hookrightarrow \left(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \tau_\eta(\rho;\sigma)\right) \oplus \left(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \tau_\eta(\rho;\hat{c}\sigma)\right)$$
$$\Downarrow$$
$$\delta \hookrightarrow \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \tau_\eta(\rho;\sigma)) \text{ or } \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \tau_\eta(\rho;\hat{c}\sigma)).$$

It then follows from Theorem 8.4 that

$$\delta_0 \cong \mathrm{Res}_{SO}^{O}\,\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\sigma)) \cong \mathrm{Res}_{SO}^{O}\,\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\hat{c}\sigma)).$$

The claim then follows. We denote this subrepresentation by

$$\delta_0 = \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\varsigma_\eta(\rho;\sigma_0)).$$

Lemma 11.4 *Let δ and δ_0 be discrete series with $\delta_0 \leq \mathrm{Res}_{SO}^{O}\,\delta$. For irreducible θ,*

$$\delta \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\sigma)) \text{ if and only if } \delta_0 \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\varsigma_\eta(\rho;\sigma_0)).$$

Proof First, by Theorem 8.4, $\mathrm{Res}_{SO}^{O}\,\delta \cong \delta_0$. For ($\Rightarrow$), Lemma 11.3 gives

$$\begin{aligned}\delta_0 \cong \mathrm{Res}_{SO}^{O}\,\delta &\hookrightarrow \mathrm{Res}_{SO}^{O}\left(\theta \rtimes \delta\big(\left[\nu\rho, \nu^{\frac{a-1}{2}}\rho\right];\tau_\eta(\rho;\sigma)\big)\right)\\ &\cong \theta \rtimes \delta\big(\left[\nu\rho, \nu^{\frac{a-1}{2}}\rho\right];\varsigma_\eta(\rho;\sigma_0)\big),\end{aligned}$$

as needed. On the other hand, for ($\Leftarrow$) we have

$$\delta_0 \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\varsigma_\eta(\rho;\sigma_0))$$
$$\Downarrow$$
$$\begin{aligned}\delta \oplus \hat{c}\delta \cong \mathrm{Ind}_{SO}^{O}\,\delta_0 \hookrightarrow\ &\big(\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\sigma))\big) \oplus\\ &\big(\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\hat{c}\sigma))\big)\end{aligned}$$
$$\Downarrow$$
$$\delta \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho];\tau_\eta(\rho;\sigma))$$

by partial cuspidal support considerations, as needed. ∎

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In light of Lemma 3.2 and the preceding lemma, we may define $\varepsilon_{\delta_0}(\rho, a_{\max}) = \eta$ if and only if there is an irreducible θ such that $\delta_0 \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho; \sigma_0))$.

Case 2. $c\sigma_0 \not\cong \sigma_0$ and (1.1) not satisfied: Observe that in this case, if we have chosen $\mathcal{T}$ as in Remark 11.2, then

$$\mathrm{Res}^{O}_{SO}\, \tau_\eta(\rho; \sigma) = \varsigma_\eta(\rho; \sigma_0) \oplus c\varsigma_\eta(\rho; \sigma_0)$$

(by Lemmas 5.1(ii) and 2.3(i)). An argument similar to that in Case 1 tells us that for $a \in \mathbb{N}$ with a odd and $\eta \in \{\pm 1\}$, we may define $\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \tau_\eta(\rho; \sigma_0))$ to be the unique irreducible subrepresentation of $\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \varsigma_\eta(\rho; \sigma_0)$, which is square-integrable. We note that (cf. Theorem 8.4)

$$\begin{aligned}\mathrm{Ind}^{O}_{SO}\, \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \varsigma_\eta(\rho; \sigma_0)) &\cong \mathrm{Ind}^{O}_{SO}\, \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \varsigma_\eta(\rho; c\sigma_0)) \\ &\cong \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \tau_\eta(\rho; \sigma)).\end{aligned}$$

Lemma 11.5 *Let δ and δ_0 be discrete series with $\delta_0 \leq \mathrm{Res}^{O}_{SO}\, \delta$. For irreducible θ,*

$$\delta \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \tau_\eta(\rho; \sigma)) \textit{ if and only if } \delta_0 \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]; \varsigma_\eta(\rho; \sigma_0)).$$

Proof The proof is similar to that of Lemma 11.4. ■

We remark that the proof uses the assumption that (1.1) is not satisfied (for all of *Jord*, not just ρ) in order to apply partial cuspidal support considerations.

Again, in light of Lemmas 3.2 and 11.5, we may define $\varepsilon_{\delta_0}(\rho, a_{\max}) = \eta$ if and only if there is an irreducible θ such that $\delta_0 \hookrightarrow \theta \rtimes \delta([\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]; \tau_\eta(\rho; \sigma_0))$.

Case 3. $c\sigma_0 \not\cong \sigma_0$ and (1.1) satisfied: In Section 3, we noted that a choice of $\mathcal{T}$ was equivalent to choosing components $\tau_1(\rho_i; \sigma) \leq \rho_i \rtimes \sigma$ for $i = 1, \dots, \ell$. Since $\rho_i \rtimes \sigma_0 \cong \rho_i \rtimes c\sigma_0$ is irreducible for $i = 1, \dots, \ell$, we cannot work quite the same way for $SO(2n, F)$. Instead, we work with $\rho_1, \dots, \rho_\ell$ in pairs. In particular, for $1 \leq i < j \leq k$, $\rho_i \times \rho_j \rtimes \sigma_0 \cong \rho_i \times \rho_j \rtimes c\sigma_0$ has two components (cf. [G1]); we denote these by $\varsigma_\varepsilon(\rho_i, \rho_j; \Sigma_0)$, $\varepsilon \in \{\pm 1\}$. They are characterized by

$$\mathcal{S} \leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_{j-1} \times \rho_{j+1} \times \cdots \times \rho_\ell \rtimes \varsigma_1(\rho_i, \rho_j; \Sigma_0)$$

and

$$\mathcal{S} \not\leq \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_{j-1} \times \rho_{j+1} \times \cdots \times \rho_\ell \rtimes \varsigma_{-1}(\rho_i, \rho_j; \Sigma_0).$$

(We remark that it would be awkward to try to choose components of $\rho_i \times \rho_j \rtimes \sigma_0$ directly, as this would require making $\ell - 1$ such choices and having the remaining $\frac{(\ell-1)(\ell-2)}{2}$ choices imposed by a compatibility constraint.) Suppose $\rho, \rho' \in \{\rho_1, \dots, \rho_\ell\}$, $\rho \not\cong \rho'$, with ρ satisfying (1.1). If we choose $\mathcal{T}$ as in Remark 11.2, it follows from Note 3.3 that

$$\mathrm{Res}^{O}_{SO}\, \tau_{\eta,\eta'}(\rho, \rho'; \sigma) = \begin{cases} \varsigma_{\eta\eta'}(\rho, \rho'; \Sigma_0) & \text{if } \rho' \text{ also satisfies (1.1)} \\ \varsigma_{\eta'}(\rho, \rho'; \Sigma_0) & \text{if } \rho' \text{ does not satisfy (1.1).} \end{cases}$$

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For $a, a' \in \mathbb{N}$ with a, a' odd and $\eta \in \{\pm 1\}$, we define

$$\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \varsigma_\eta(\rho, \rho'; \Sigma_0)\big)$$

to be the unique irreducible subrepresentation of

$$\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,\big) \times \delta\big(\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,\big) \rtimes \varsigma_\eta(\rho, \rho'; \Sigma_0),$$

which is square-integrable. We note that (cf. Theorem 8.4 and Lemma 3.4)

$$\begin{aligned}\delta\big(\,&[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \varsigma_\eta(\rho, \rho'; \Sigma_0)\big)\\ &\cong \mathrm{Res}^O_{SO}\,\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \tau_{\eta,1}(\rho, \rho'; \sigma)\big)\\ &\cong \mathrm{Res}^O_{SO}\,\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \tau_{-\eta,-1}(\rho, \rho'; \sigma)\big)\end{aligned}$$

if ρ' satisfies (1.1) and

$$\begin{aligned}\delta\big(\,&[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \varsigma_\eta(\rho, \rho'; \Sigma_0)\big)\\ &\cong \mathrm{Res}^O_{SO}\,\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \tau_{1,\eta}(\rho, \rho'; \sigma)\big)\\ &\cong \mathrm{Res}^O_{SO}\,\delta\big(\,[\nu\rho, \nu^{\frac{a-1}{2}}\rho]\,,\,[\nu\rho', \nu^{\frac{a'-1}{2}}\rho']\,; \tau_{-1,\eta}(\rho, \rho'; \sigma)\big)\end{aligned}$$

if not.

Lemma 11.6 *Let δ and δ_0 be discrete series with $\delta_0 \leq \mathrm{Res}^O_{SO}\,\delta$. Let $\rho, \rho' \in \{\rho_1, \ldots, \rho_\ell\}$, with ρ satisfying* (1.1).

(i) *If ρ' also satisfies* (1.1)*, then for irreducible θ,*

$$\begin{aligned}\delta \hookrightarrow \quad &\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{\eta,\eta'}(\rho, \rho'; \sigma))\\ &\qquad\qquad \textit{or}\\ &\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{-\eta,-\eta'}(\rho, \rho'; \sigma))\\ &\qquad\qquad \Updownarrow\\ \delta_0 \hookrightarrow &\,\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho, \nu^{\frac{a'-1}{2}}\rho']; \varsigma_{\eta\eta'}(\rho, \rho'; \Sigma_0)).\end{aligned}$$

(ii) *If ρ' does not satisfy* (1.1)*, then for irreducible θ,*

$$\begin{aligned}\delta \hookrightarrow \quad &\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{\eta,\eta'}(\rho, \rho'; \sigma))\\ &\qquad\qquad \textit{or}\\ &\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho', \nu^{\frac{a'-1}{2}}\rho']; \tau_{-\eta,\eta'}(\rho, \rho'; \sigma))\\ &\qquad\qquad \Updownarrow\\ \delta_0 \hookrightarrow &\,\theta \rtimes \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho], [\nu\rho, \nu^{\frac{a'-1}{2}}\rho']; \varsigma_{\eta'}(\rho, \rho'; \Sigma_0)).\end{aligned}$$

Proof The proof is similar to that of Lemmas 11.4 and 11.5. ■

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In light of Lemma 3.5 and the preceding lemma, we may arbitrarily choose a value $\eta_\rho \in \{\pm 1\}$ for $\varepsilon_{\delta_0}(\rho, a_{\max})$; the remaining values of $\varepsilon_{\delta_0}(\rho', a'_{\max})$ are then defined by the following:

(i) If ρ' satisfies (1.1),

$$\varepsilon_{\delta_0}(\rho', a'_{\max}) = \eta' \Leftrightarrow \delta_0 \hookrightarrow \theta \rtimes \delta\big(\,[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]\,, [\nu\rho, \nu^{\frac{a'_{\max}-1}{2}}\rho']; \varsigma_{\eta_\rho\eta'}(\rho, \rho'; \Sigma_0)\big)$$

for some irreducible θ.

(ii) If ρ' does not satisfy (1.1),

$$\varepsilon_{\delta_0}(\rho', a'_{\max}) = \eta' \Leftrightarrow \delta_0 \hookrightarrow \theta \rtimes \delta\big(\,[\nu\rho, \nu^{\frac{a_{\max}-1}{2}}\rho]\,, [\nu\rho, \nu^{\frac{a'_{\max}-1}{2}}\rho']\,; \varsigma_{\eta'}(\rho, \rho'; \Sigma_0)\big)$$

for some irreducible θ.

We note that the two possible choices for η_ρ give rise to $\sim$-equivalent triples, and hence correspond to the same element of $\mathrm{Trip}_{SO}/\!\sim$.

Having finished the definition of ε_{δ_0}, we now have a well-defined map $\delta_0 \mapsto (Jord(\delta_0), \Sigma_{\delta_0}, \varepsilon_{\delta_0})$ sending discrete series for $SO(2n, F)$ into $\mathrm{Trip}_{SO,adm}/\!\sim$. We now summarize the results.

Theorem 11.7 *The map $\delta_0 \mapsto (Jord(\delta_0), \Sigma_{\delta_0}, \varepsilon_{\delta_0})$ implements a bijective correspondence between discrete series for all $SO(2n, F)$, $n \neq 1$, and $\mathrm{Trip}_{SO,adm}/\!\sim$. Further, $c\delta_{(Jord,\Sigma_0,\varepsilon)} = \delta_{(Jord,c\Sigma_0,\varepsilon)}$ (where $c \cdot \{\sigma_0, c\sigma_0\}$ is understood to be $\{\sigma_0, c\sigma_0\}$).*

A Extension of a Result of Harish-Chandra

In this appendix, we extend a result of Harish-Chandra (cf. [W, Proposition III.4.1]) to cover the non-connected group $O(2n, F)$. In particular, we show that if an irreducible tempered representation τ has $\tau \hookrightarrow i_{G,M}\delta_1$ and $\tau \hookrightarrow i_{G,M}\delta_2$ with δ_1, δ_2 discrete series of standard parabolic subgroups, then δ_1 and δ_2 (and the corresponding Levi factors) are conjugate. We note that this does not use (directly or indirectly) the results of Goldberg, so we may drop the assumption $charF = 0$ in this appendix and simply assume $charF \neq 2$. However, we need to retain the convention that the trivial representation of $O(2, F)$ is not considered a discrete series.

Lemma A.1 *Let τ be an irreducible tempered representation of $O(2n, F)$. Suppose*

$$\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \delta \quad \textit{and} \quad \tau \hookrightarrow \delta'_1 \times \cdots \times \delta'_\ell \rtimes \delta'$$

with δ_i, δ'_i discrete series for general linear groups and δ, δ' discrete series for orthogonal groups. Then $k = \ell$, and $\delta'_1 \otimes \cdots \otimes \delta'_k \otimes \delta'$ is a Weyl conjugate of $\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta$ or $\delta_1 \otimes \cdots \otimes \delta_k \otimes \hat{c}\delta$. That is, (i) *$\delta'_1, \ldots, \delta'_k, \tilde{\delta}'_1, \ldots, \tilde{\delta}'_k$ is a permutation of $\delta_1, \ldots, \delta_k, \tilde{\delta}_1, \ldots, \tilde{\delta}_k$, subject to the constraint that if $\delta'_i \cong \delta_j$, then $\tilde{\delta}'_i \cong \tilde{\delta}_j$, and* (ii) *$\delta' \cong \delta$ or $\delta' \cong \hat{c}\delta$ (noting that if $\delta = 1$, this requires $\delta' = 1$).*

Remark A.2 Of course, if $\hat{c}\delta \cong \delta$, this already gives the result we want.

Proof Let τ_0 be an irreducible subrepresentation (necessarily tempered) of $\mathrm{Res}^{O}_{SO}\,\tau$. If $\delta_0 \le \mathrm{Res}^{O}_{SO}\,\delta$ is irreducible (necessarily square-integrable), then it follows from Lemma 2.3 that

$$\begin{aligned} \tau_0 \hookrightarrow\ & Res^{O}_{SO}(\delta_1 \times \cdots \times \delta_k \rtimes \delta) \\ \cong\ & \delta_1 \times \cdots \times \delta_k \rtimes \delta_0 \\ & \quad\text{or} \\ & (\delta_1 \times \cdots \times \delta_k \rtimes \delta_0) \oplus c(\delta_1 \times \cdots \times \delta_k \rtimes \delta_0), \end{aligned}$$

and similarly for $\delta'_1, \dots, \delta'_\ell, \delta'$. From [W, Proposition III.4.1] for $SO(2n, F)$, we must have $\ell = k$ and $\delta'_1 \otimes \cdots \otimes \delta'_k \otimes \delta'_0$ a Weyl conjugate of $\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta_0$ or $c(\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta_0)$. It then follows that $\delta'_1 \otimes \cdots \otimes \delta'_k \otimes \delta'$ is a Weyl conjugate of $\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta$ or $\hat{c}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta)$. ■

Lemma A.3 *Suppose θ is an irreducible representation of a general linear group such that*

$$\underbrace{\delta([\nu^b\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^b\rho, \nu^a\rho])}_{k}$$

is a subquotient of $\theta \times \delta([\nu^{c_1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{c_k}\rho, \nu^a\rho])$ with $b \le c_i \le a+1$ for all i. Then $\theta \cong \delta([\nu^b\rho, \nu^{c_1-1}\rho]) \times \cdots \times \delta([\nu^b\rho, \nu^{c_k-1}\rho])$.

Proof For a representation π of a general linear group, let us write $\tau \le r_{min}\pi$ if τ is supercuspidal (not necessarily unitary) and $\tau \le r_{M,G}\pi$ for some standard Levi factor M (*i.e.*, $r_{min}\pi$ consists of the minimal nonzero Jacquet modules). Observe that

$$r_{min}\,\underbrace{\delta([\nu^b\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^b\rho, \nu^a\rho])}_{k} \ge \underbrace{\nu^a\rho \otimes \cdots \otimes \nu^a\rho}_{k} \otimes \underbrace{\nu^{a-1}\rho \otimes \cdots \otimes \nu^{a-1}\rho}_{k} \otimes \cdots \otimes \underbrace{\nu^b\rho \otimes \cdots \otimes \nu^b\rho}_{k}.$$

Let k_i be the number of times $\nu^i\rho$ appears in the supercuspidal support of $\delta([\nu^{c_1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{c_k}\rho, \nu^a\rho])$, *i.e.*, $k_i = |\{j \mid c_j \le i \le a\}|$. Since $r_{min}(\theta \times \delta([\nu^{c_1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{c_k}\rho, \nu^a\rho]))$ consists of shuffles of $r_{min}\delta([\nu^{c_1}\rho, \nu^a\rho]) = \nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{c_1}\rho, \dots, r_{min}\delta([\nu^{c_k}\rho, \nu^a\rho]) = \nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{c_k}\rho$, $r_{min}\theta$, we see that

$$r_{min}\theta \ge \underbrace{\nu^a\rho \otimes \cdots \otimes \nu^a\rho}_{k-k_a} \otimes \underbrace{\nu^{a-1}\rho \otimes \cdots \otimes \nu^{a-1}\rho}_{k-k_{a-1}} \otimes \cdots \otimes \underbrace{\nu^b\rho \otimes \cdots \otimes \nu^b\rho}_{k-k_b}.$$

The only irreducible representation with this property is $\delta([\nu^b\rho, \nu^{c_1-1}\rho]) \times \cdots \times \delta([\nu^b\rho, \nu^{c_k-1}\rho])$, as needed. ■

Lemma A.4 *Suppose τ is an irreducible tempered representation of $O(2n, F)$. Write*

$$\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \delta,$$

with $\delta_1,\dots,\delta_k$ discrete series for general linear groups and δ a discrete series for an even orthogonal group. Suppose θ is a discrete series for a general linear group with $\theta \not\cong \delta_1,\dots,\delta_k,\tilde{\delta}_1,\dots,\tilde{\delta}_k$.

(i) *If $\theta \cong \tilde{\theta}$, then $\mu^*(\underbrace{\theta\times\cdots\times\theta}_{\ell}\rtimes\tau)$ contains $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\tau$ with multiplicity 2^ℓ and no other terms of the form $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\lambda$.*

(ii) *If $\theta \not\cong \tilde{\theta}$, then $\mu^*(\underbrace{\theta\times\cdots\times\theta}_{\ell}\rtimes\tau)$ contains $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\tau$ with multiplicity 1 and no other terms of the form $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\lambda$.*

Proof We start with (i). Writing $\theta = \delta([\nu^{-a}\rho,\nu^a\rho])$, an easy calculation, noting that $\theta\cong\tilde{\theta}$ requires $\rho\cong\tilde{\rho}$–gives

$$M^*\left(\delta([\nu^{-a}\rho,\nu^a\rho])\right) = \sum_{i=-a}^{a+1}\sum_{j=i}^{a+1}\delta([\nu^{-i-1}\rho,\nu^a\rho])\times\delta([\nu^j\rho,\nu^a\rho])\otimes\delta([\nu^i\rho,\nu^{j-1}\rho]).$$

Write $\mu^*(\tau)=\sum_h \xi_h\otimes\eta_h$. Then

$$\begin{aligned}
&\mu^*(\underbrace{\theta\times\cdots\times\theta}_{\ell}\rtimes\tau)\\
&=\sum_{i_1=-a}^{a+1}\sum_{j_1=i_1}^{a+1}\cdots\sum_{i_\ell=-a}^{a+1}\sum_{j_\ell=i_\ell}^{a+1}\sum_h\Big(\delta\big([\nu^{-i_1-1}\rho,\nu^a\rho]\big)\times\delta([\nu^{j_1}\rho,\nu^a\rho])\\
&\times\cdots\times\delta\big([\nu^{-i_\ell-1}\rho,\nu^a\rho]\big)\times\delta([\nu^{j_\ell}\rho,\nu^a\rho])\times\xi_h\big)\\
&\otimes\Big(\delta([\nu^{i_1}\rho,\nu^{j_1-1}\rho]\big)\times\cdots\times\delta\big([\nu^{i_\ell}\rho,\nu^{j_\ell-1}\rho]\big)\rtimes\eta_h\Big).
\end{aligned}$$

First, suppose $\xi_h=1$ (so $\eta_h=\tau$). This gives rise to the 2^ℓ copies of

$$\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\tau:$$

for each m, we can have either (i) $i_m=a+1$ (so $j_m=a+1$ and $\delta([\nu^{-i_m-1}\rho,\nu^a\rho])=\theta$) or (ii) $i_m=-a$ and $j_m=-a$ (so $\delta([\nu^{j_m}\rho,\nu^a\rho])=\theta$). Any other choice for the i_m, j_m produces less than ℓ copies of $\nu^{-a}\rho$ in the supercuspidal support, so cannot contribute a copy of

$$\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\tau.$$

Now, suppose $\xi_h\neq 1$. Suppose there were a term of the form $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\lambda$. The contribution to $\underbrace{\theta\times\cdots\times\theta}_{\ell}$ from $M^*(\underbrace{\theta\times\cdots\times\theta}_{\ell})$ has the form $\delta([\nu^{k_1}\rho,\nu^a\rho])\times\cdots\times$

$\delta([\nu^{k_{\ell'}}\rho,\nu^a\rho])$, where $\ell'\le\ell$ and $k_1,\dots,k_{\ell'}$ are the values from $-i_1-1,\dots,-i_\ell-1,j_1,\dots,j_\ell$ that are less than or equal to a. Note that since $\xi_h\ne 1$, we must have either $\ell'<\ell$ or at least one $k_m>-a$ (or both). By Lemma A.3, ξ_h must contribute

$$\delta([\nu^{-a}\rho,\nu^{k_1-1}\rho])\times\cdots\times\delta([\nu^{-a}\rho,\nu^{k_{\ell'}}\rho])\times\underbrace{\delta([\nu^{-a}\rho,\nu^a\rho])\times\cdots\times\delta([\nu^{-a}\rho,\nu^a\rho])}_{\ell-\ell'}$$

(and this is a nontrivial contribution from above). If $\delta([\nu^{-a}\rho,\nu^{k_1-1}\rho])\times\cdots\times\delta([\nu^{-a}\rho,\nu^{k_{\ell'}}\rho])$ is nontrivial, this contradicts the Casselman criterion for the temperedness of τ (cf. Section 2). Therefore,

$$\xi_h=\underbrace{\delta([\nu^{-a}\rho,\nu^a\rho])\times\cdots\times\delta([\nu^{-a}\rho,\nu^a\rho])}_{\ell-\ell'}.$$

However, this contradicts $\theta\ncong\delta_1,\dots,\delta_k,\tilde{\delta}_1,\dots,\tilde{\delta}_k$. Thus there is no term of the form $\underbrace{\theta\times\cdots\times\theta}_{\ell}\otimes\lambda$ when $\xi_h\ne 1$.

Part (ii) is similar (but a bit easier). Note that in this case, $\rho\ncong\tilde{\rho}$. ■

Theorem A.5 *Let τ be an irreducible tempered representation of $O(2n,F)$. Suppose*

$$\tau\hookrightarrow\delta_1\times\cdots\times\delta_k\rtimes\delta\quad\text{and}\quad\tau\hookrightarrow\delta'_1\times\cdots\times\delta'_\ell\rtimes\delta'$$

with δ_i,δ'_i discrete series for general linear groups and δ,δ' discrete series for orthogonal groups. Then $k=\ell$ and $\delta'_1\otimes\cdots\otimes\delta'_k\otimes\delta'$ is a Weyl conjugate of $\delta_1\otimes\cdots\otimes\delta_k\otimes\delta$. That is, (i) *$\delta'_1,\dots,\delta'_k,\tilde{\delta}'_1,\dots,\tilde{\delta}'_k$ is a permutation of $\delta_1,\dots,\delta_k,\tilde{\delta}_1,\dots,\tilde{\delta}_k$, subject to the constraint that if $\delta'_i\cong\delta_j$, then $\tilde{\delta}'_i\cong\tilde{\delta}_j$, and* (ii) *$\delta'\cong\delta$ (noting that if $\delta=1$, this requires $\delta'=1$).*

Proof From Lemma A.1 and Remark A.2, all that needs to be shown is that $\delta'\ncong\hat{c}\delta$ when $\hat{c}\delta\ncong\delta$. If $\hat{c}\delta\ncong\delta$, it follows from the previous lemma that $\mu^*(\delta_1\times\cdots\times\delta_k\rtimes\delta)$ contains $\delta_1\times\cdots\times\delta_k\otimes\delta$ but not $\delta_1\times\cdots\times\delta_k\otimes\hat{c}\delta$. It then follows from Frobenius reciprocity that we cannot have $\delta'\cong\hat{c}\delta$, as needed. ■

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References

[Aub] A.-M. Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique.* Trans. Amer. Math. Soc. **347**(1995), no. 6, 2179–2189. doi:10.2307/2154931 *Erratum.* Trans. Amer. Math. Soc. **348**(1996), no. 11, 4687–4690. doi:10.1090/S0002-9947-96-01776-X

[B1] D. Ban, *Parabolic induction and Jacquet modules of representations of $O(2n,F)$.* Glas. Mat. Ser. III **34(54)**(1999), no. 2, 147–185.

[B2] ———, *Self-duality in the case of $SO(2n,F)$.* Glas. Mat. Ser. III **34(54)**(1999), no. 2, 187–196.

[B-J1] D. Ban and C. Jantzen, *The Langlands classification for non-connected p-adic groups.* Israel J. Math. **126**(2001), 239–261. doi:10.1007/BF02784155

[B-J2] ———, *Degenerate principal series for even orthogonal groups.* Represent. Theory **7**(2003), 440–480. doi:10.1090/S1088-4165-03-00166-3

[B-Z] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I.* Ann. Sci. École Norm. Sup. **10**(1977), no. 4, 441–472.

[B-W] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Annals of Mathematics Studies, 94, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1980.

[C] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, http://www.math.ubc.ca/~cass/research.html

[G-K] S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group.* Adv. in Math. **43**(1982), no. 2, 101–121. doi:10.1016/0001-8708(82)90030-5

[G1] D. Goldberg, *Reducibility of induced representations for $Sp(2n)$ and $SO(n)$.* Amer. J. Math. **116**(1994), no. 5, 1101–1151. doi:10.2307/2374942

[G2] ———, *Reducibility for non-connected p-adic groups with G^0 of prime index.* Canad. J. Math. **47**(1995), no. 2, 344–363.

[J1] C. Jantzen, *On supports of induced representations for symplectic and odd-orthogonal groups.* Amer. J. Math. **119**(1997), no. 6, 1213–1262. doi:10.1353/ajm.1997.0039

[J2] ———, *On square-integrable representations of classical p-adic groups.* Canad. J. Math. **52**(2000), no. 3, 539–581.

[J3] ———, *Square-integrable representations of classical p-adic groups. II.* Represent. Theory **4**(2000), 127–180. doi:10.1090/S1088-4165-00-00081-9

[J4] ———, *Duality and supports of induced representations for orthogonal groups.* Canad. J. Math. **57**(2005), no. 1, 159–179.

[J5] ———, *Jacquet modules of induced representations for p-adic special orthogonal groups.* J. Algebra, **305**(2006), no. 2, 802–819. doi:10.1016/j.jalgebra.2005.12.026

[K] T. Konno, *A note on the Langlands classification and irreducibility of induced representations of p-adic groups.* Kyushu J. Math. **57**(2003), no. 2, 383–409. doi:10.2206/kyushujm.57.383

[Mœ1] C. Mœglin, *Normalisation des opérateurs d'entrelacement et réductibilité des induites de cuspidales; le cas des groupes classiques p-adiques.* Ann. of Math. **151**(2000), no. 2, 817–847. doi:10.2307/121049

[Mœ2] ———, *Sur la classification des séries discrètes des groupes classiques: paramètres de Langlands et exhaustivité.* J. Eur. Math. Soc. **4**(2002), no. 2, 143–200. doi:10.1007/s100970100033

[M-T] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups.* J. Amer. Math. Soc. **15**(2002), no. 3, 715–786. doi:10.1090/S0894-0347-02-00389-2

[Mu1] G Muić, *Composition series of generalized principal series; the case of strongly positive discrete series.* Israel J. Math. **140**(2004), 157–202. doi:10.1007/BF02786631

[Mu2] ———, *Reducibility of generalized principal series.* Canad. J. Math. **57**(2005), no. 3, 616–647.

[Mu3] ———, *Construction of Steinberg type representations for reductive p-adic groups.* Math. Z. **253**(2006), no. 3, 635–652. doi:10.1007/s00209-006-0946-6

[S-S] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building.* Inst. Hautes Études Sci. Publ. Math. **85**(1997), 97–191.

[Sh1] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups.* Ann. of Math. **132**(1990), no. 2, 273–330. doi:10.2307/1971524

[Sh2] ———, *Twisted endoscopy and reducibility of induced representations for p-adic groups.* Duke Math. J. **66**(1992), no. 1, 1–41. doi:10.1215/S0012-7094-92-06601-4

[S1] A. J. Silberger, *The Langlands quotient theorem for p-adic groups.* Math. Ann. **236**(1978), no. 2, 95–104. doi:10.1007/BF01351383

[S2] ———, *Special representations of reductive p-adic groups are not integrable.* Ann. of Math. **111**(1980), no. 3, 571–587. doi:10.2307/1971110

[T1] M. Tadić, *Representations of p-adic symplectic groups.* Compositio Math. **90**(1994), no. 2, 123–181.

[T2] ———, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups.* J. Algebra **177**(1995), no. 1, 1–33. doi:10.1006/jabr.1995.1284

[T4] ———, *On regular square integrable representations of p-adic groups.* Amer. J. Math. **120**(1998), no. 1, 159–210. doi:10.1353/ajm.1998.0007

[T5] ———, *On classification of some classes of irreducible representations of classical groups.* In: Representations of real and p-adic groups, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 2, Singapore University Press, Singapore, 2004, pp. 95–162.

[T6] ———, *On invariants of discrete series representations of classical p-adic groups.* Manuscripta Math, to appear. http://www.hazu.hr/~tadic

[W] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra).* J. Inst. Math. Jussieu **2**(2003), no. 2, 235–333. doi:10.1017/S1474748003000082

[Z] A. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of $GL(n)$*. Ann. Sci. École Norm. Sup. **13**(1980), no. 2, 165–210.

[Zh] Y. Zhang, *L-packets and reducibilities.* J. Reine Angew. Math. **510**(1999), 83–102.

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