J. Austral. Math. Soc. 22 (Series A) (1976), 264–273.

A NOTE ON UPPER RADICALS IN RINGS

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(Received 16 July 1974; revised 26 March 1975)

Abstract

A class (called *c-radicals*) is defined such that, given a c-radical P, there is in any class M' a certain internal criterion that its upper radical $UM' = P$. For P a non-c-radical (called a *q-radical*) there exists no smallest class M such that $UM = P$, and P is a q-radical if and only if for some M with $P = UM$ there exists $0 \neq R \in M$ such that when an image $\bar{R}$ of R has a non-zero image in M there exists an infinite chain of epimorphisms $\bar{R} \to R_1 \to R_2 \to \cdots$ with all $R_i \in M$ and no R_i the image of any R_j with $j > i$. Several examples of such rings are constructed including a ring all of whose images are primitive. Thus all radicals contained in the Jacobson radical are q-radicals.

1. Introduction

All rings to be considered are associative and it will be assumed that a class of rings always contains all rings isomorphic to any member of the class. For a ring R write $I < R$ to mean I is an ideal of R and let $\bar{R}$ always designate some non-zero homomorphic image of R. For a class M of rings let

$$UM = \{R \mid \text{every } \bar{R} \notin M\},$$

$$SM = \{R \mid \text{for every } 0 \neq I < R \text{ we have } I \notin M\}.$$

It is known by Theorem 1 of Enersen and Leavitt (1973) that UM is radical (in the Kurosh-Amitsur sense; see Divinsky (1965), page 4) if and only if every $0 \neq R \in M$ has some $\bar{R} \in SUM$, and when this is the case UM is the "upper radical" defined by M. Note that in defining UM it makes no difference whether $0 \in M$ or not. For convenience we will assume that 0 is a member of all classes. An upper radical UM is said to have "property (Int) relative to M" if for R an arbitrary ring, $UM(R)$ is the intersection of a set $\{I_i\}$ of ideals of R for which $R/I_i \in M$ (equivalently, if every $R \in SUM$ is a subdirect sum of members of M). A class M which is closed under taking ideals is called "hereditary".

It is well-known that classes $M' \neq M$ can exist for which $UM = UM'$. For example, the prime radical (Lower Baer radical) is equal to both UM and UM'

when M is the class of all prime rings and M' is the class of all semiprime rings. One of the unsolved problems of radical theory is to find, for a given class M, internal criteria on a class M' so that $UM' = UM$. We do have criteria in certain special cases. For example, it was shown in Enersen and Leavitt (1973) that

PROPOSITION 1. *A sufficient condition that* $UM = UM'$ *is that every* $0 \neq R \in M \cup M'$ *has a non-zero homomorphic image in* $M \cap M'$ *and if either* M *or* M' *is homomorphically closed or contains nothing but Noetherian rings then the condition is also necessary.*

LEMMA 2. *If every* $0 \neq R \in M$ *has a non-zero image in* M' *then* $UM' \subseteq UM$.

PROOF. If $R \notin UM$ then R has a non-zero image in M and so a non-zero image in M'. Thus $R \notin UM'$ and we conclude that $UM' \subseteq UM$.

This implies an easy criterion in the special case $M' \subseteq M$, namely:

PROPOSITION 3. *For a given class* M, *if* $M' \subseteq M$ *then* $UM' = UM$ *if and only if every* $0 \neq R \in M$ *has some* $\bar{R} \in M'$.

PROOF. The necessity is clear and the sufficiency follows from Proposition 1 (or Lemma 2).

Remark that neither of the criteria of Propositions 1 or 3 can be properly called "internal" criteria on M' since reference needs to be made to conditions on rings outside M'.

The author wishes to express his thanks to the referee who pointed out a serious gap in the first version of this paper (namely, that the "cycles" of the next section cannot be ruled out).

2. c-Radicals and s-radicals

A pair $\{R, K\}$ of rings is called a *cycle* if there exist epimorphisms $R \to K$ and $K \to R$, and when this is the case we will say R and K are equivalent (written $R \sim K$). Note that a set $\{R_1, \cdots, R_n\}$ of rings is a "general cycle", that is there exist epimorphisms $R_1 \to R_2 \to \cdots \to R_n \to R_1$ if and only if all $R_i \sim R_j$. Also note that the relation $\sim$ is an equivalence relation. Two classes M, N of rings will be called *equivalent* (written $M \sim N$) if every $R \in M$ is equivalent to some $K \in N$ and conversely. For a class M of rings we define the *cycle closure* $\bar{M} = \{K \mid K \sim R \text{ for some } R \in M\}$. Clearly $\bar{M}$ is cycle-closed (that is, $\bar{\bar{M}} = \bar{M}$) and also $\bar{M} \sim M$. Then we have:

PROPOSITION 4. *For* M, N *arbitrary classes,* $M \sim N$ *implies* $UM = UN$. *Thus* $UM = U\bar{M}$ *for any class* M.

https://doi.org/10.1017/S1446788700014725 Published online by Cambridge University Press

PROOF. Since every non-zero $R \in M$ has some $\bar{R} \in N$ it follows from Lemma 2 that $UN \subseteq UM$, and symmetrically $UM \subseteq UN$.

A ring R in a class M will be said to have *property* (c) *relative to* M if $R \sim \bar{R}$ whenever an image $\bar{R} \in M$.

REMARK 1. If R has property (c) relative to M then:
(i) R has property (c) relative to any $M' \subseteq M$, and
(ii) R has property (c) relative to $\bar{M}$.

A class M will be called a *c-class* if every $R \in M$ has property (c) relative to M and a radical P is called a *c-radical* if $P = UM$ for some c-class M. By Remark 1 every c-class M is contained in a (cycle-closed) c-class $\bar{M}$.

THEOREM 5. *If M is a c-class and M' an arbitrary class, then $UM' = UM$ if and only if (i) every $0 \neq R \in M'$ has some $\bar{R} \in \bar{M} \cap M'$, and (ii) $\bar{M} \cap M' \sim \bar{M}$.*

PROOF. The sufficiency is clear since Propositions 3 and 4 imply $UM' = U(\bar{M} \cap M') = U\bar{M} = UM$. On the other hand suppose that $UM' = U\bar{M}$. If $0 \neq R \in M'$ then $R \notin UM' = U\bar{M}$ so there is some $\bar{R} \in \bar{M}$. But then $\bar{R}$ has an image $0 \neq K \in M'$ which in turn has an image $0 \neq H \in \bar{M}$. Then property (c) implies $\bar{R} \sim H$ and so $\bar{R} \sim K$. Since $\bar{M}$ is cycle-closed $K \in \bar{M} \cap M'$. Also for any $R \in \bar{M}$ the same argument produces some $K \in M'$ such that $R \sim K$ and so $K \in \bar{M} \cap M'$, that is $\bar{M} \sim \bar{M} \cap M'$.

COROLLARY 6. *Let M_1 be a class of rings such that $UM_1 = P$ for a c-radical P, then there exists a c-class $N_1 \subseteq M_1$ such that $UN_1 = UM_1$. For any other class M_2 of rings containing a c-class N_2 such that $UN_2 = UM_2$ we have $UM_2 = P$ if and only if $\bar{N}_2 = \bar{N}_1$.*

PROOF. If $P = UM$ for a c-class M then from Theorem 5 we can take $N_1 = M_1 \cap \bar{M} \sim \bar{M}$ whence $\bar{N}_1 = \bar{M}$. Then if $P = UM_2 = UN_2$ where N_2 is a c-class we have $N_2 \cap \bar{M} \sim \bar{M}$ and $\bar{N}_2 \cap M \sim \bar{N}_2$, and so $\bar{N}_2 = \bar{M} = \bar{N}_1$. Also if $\bar{N}_2 = \bar{N}_1$ then $UN_2 = U\bar{N}_2 = U\bar{N}_1 = UN_1 = P$.

We now consider a special case of property (c). We will say a ring $R \in M$ has *property* (s) *relative to* M if $R \cong \bar{R}$ whenever an image $\bar{R} \in M$. A class M is an *s-class* if every $R \in M$ has property (s) relative to M and a radical P is an *s-radical* if $P = UM$ for some s-class M.

COROLLARY 7. *If M is a class of simple rings then $P = UM$ is an s-radical. For this case, $\bar{M} = M$ so $UM' = P$ for a class M' if and only if $M \subseteq M'$ and every $0 \neq R \in M'$ has some $\bar{R} \in M$.*

REMARK 2. From Corollary 7 many well-known radicals (such as the Brown-McCoy radical) are s-radicals (hence c-radicals). There are also many s-radicals which are not upper radicals of classes of simple rings, such as UM for a hereditary class M containing none of the proper (i.e. non-isomorphic) images of any of its members. For example, $M = \{I \mid I < Z\}$ where Z = the integers.

REMARK 3. There are s-classes M for which UM is not radical. For example $M = \{R\}$ for any ring R is (trivially) an s-class but UM would rarely be radical. One can also construct c-classes which are not s-classes but whether or not there exists a c-radical which is not an s-radical is open.

PROPOSITION 8. *Let $M' \subseteq M$ and $UM' = UM$. Then M' is an s-class if and only if no proper subclass $M'' \subset M'$ has the property $UM'' = UM$.*

PROOF. Let M' be an s-class. If M'' is a proper subclass of M' then $UM' \subseteq UM''$ and there is some $R \in M'$, $R \notin M''$. If any $\bar{R} \in M''$ then R would have a proper image in M' contradicting M' being an s-class. Thus $R \in UM''$ whereas $R \notin UM'$. On the other hand, if M' is not an s-class then some $R \in M'$ has a proper image $\bar{R} \in M'$. Let $M'' = M' \backslash R$ then Proposition 3 implies $UM'' = UM'$.

REMARK 4. If M is any cycle-closed c-class then M can be partitioned into equivalence classes relative to the relation $\sim$. Each equivalence class is a set, and it may happen that there exists a class M_1 containing exactly one representative from each of these equivalence sets. (For example, if M is itself a set then the axiom of choice can be used.) When this is the case M_1 is an s-class for which $UM_1 = UM$. Even when M is too big to be a set, one might only be concerned with the radicals in a certain set of rings. In this case one could take a universal class V containing the rings in question but small enough to be a set, and consider the upper radical relative to V, namely $U_V(M \cap V) = (UM) \cap V$, whose radical would coincide with $UM(R)$ in all $R \in V$. Again there would exist a smallest class (that is, an s-class) $M_1 \subseteq M \cap V$ such that $U_V(M_1) = U_V(M \cap V)$.

3. q-Radicals

A radical will be called a *q-radical* if it is not a c-radical, and in this section we will consider the problem of characterizing such radicals. For a class M of rings a sequence $\{R_1, R_2, \cdots\}$ will be called a *chain in M* if: (1) all $R_i \in M$, (2) for all $n \geqq 1$ there exists an epimorphism $R_n \to R_{n+1}$, and (3) for all $m, n \geqq 1$ there exists no epimorphism $R_{m+n} \to R_n$. We will say that a chain $R_1 \to R_2 \to \cdots$ is initiated by a ring R if there exists an epimorphism $R \to R_1$.

A ring $R \in M$ will be said to have *property* (q) *relative to* M if for every $\bar{R}$ which has an image $0 \neq K \in M$ there exists an infinite chain in M initiated by $\bar{R}$.

PROPOSITION 9. *If $R \in M$ does not have property (q) relative to M then R has an image $0 \neq K \in M$ with property (c) relative to M.*

PROOF. Let $R \in M$ with $\bar{R}$ an image which has a non-zero image in M but does not initiate any infinite chain in M. Now if every chain $R_1 \to R_2 \to \cdots \to R_n$ of length n initiated by $\bar{R}$ were extendable to a longer chain in M then one could define a sequence $\{R_1\} \subset \{R_1, R_2\} \subset \cdots$ whose union would be an infinite chain in M initiated by $\bar{R}$. Thus there must exist some $R_n \in M$ such that when $\bar{R}_n \in M$ there exists an epimorphism $\bar{R}_n \to R_j$ for some $j \leqq n$. But then $R_n \sim \bar{R}_n$, that is R_n has property (c) relative to M.

THEOREM 10. *A radical P is a q-radical if and only if $P = UM$ for some class M containing a ring with property (q) relative to M.*

PROOF. Let P be a c-radical, that is $P = UN$ for some c-class N. Let M be any class for which $P = UM$. If $R \in M$ then there exists some $\bar{R} \in N$ where $\bar{R}$ has some image $0 \neq K \in M$. Suppose $\bar{R} \to R_1 \to R_2$ where $R_1, R_2 \in M$. Then R_2 has an image $0 \neq H \in N$ so property (c) in N implies $\bar{R} \sim H$. Therefore, $R_1 \sim R_2$ and so $\bar{R}$ cannot initiate any chain in M. Thus M contains no elements with property (q) relative to M.

On the other hand suppose $P = UM$ where M contains no ring with property (q) relative to M. Let $M_1 = \{K \in M \mid K$ has property (c) relative to $M\}$. Then $UM \subseteq UM_1$ and by Proposition 9 if $R \in M$ then R has a non-zero image in M_1. Thus $UM = UM_1$ and so P is a c-radical.

Note that the last part of this proof shows, in fact, that if there exists any M with $P = UM$ and M q-free then P is a c-radical. Thus we have:

COROLLARY 11. *A radical P is a q-radical if and only if every class M for which $P = UM$ contains a ring with property (q) relative to M.*

We can say even more, namely that for any q-radical there is a ring which is "universal" for the (q) property:

THEOREM 12. *A radical P is a q-radical if and only if there exists a ring R such that, for any class M, if $P = UM$ then there is some $\bar{R} \in M$ with property (q) relative to M.*

PROOF. The sufficiency is obvious, so suppose P is a q-radical then by Theorem 10 there exists a class N such that $P = UN$ containing a ring R with property (q) relative to N. Let M be any class for which $UM = P$. Then there is some $\bar{R} \in M$ and if $\bar{R}$ does not have property (q) relative to M then by Proposition 9 it has an image $0 \neq K \in M$ with property (c) relative to M. But K has an image in N so since R has property (q) relative to N, there is in N an infinite chain $R_1 \to R_2 \to \cdots$ initiated by K. But $R_2 \in N$ implies R_2 has an image $0 \neq H \in M$ and since K has property (c) relative to M, we have $K \sim H$ so $R_1 \sim R_2$. From this contradiction we conclude that indeed $\bar{R}$ has property (q) relative to M.

Note, in fact, that if P is a q-radical there must be such a ring "universal" for the (q) property residing in every class M such that $P = UM$.

4. Some constructions of q-radicals

EXAMPLE 1. Let $\{K_1, K_2, \cdots\}$ be a sequence of simple rings without unit of increasing infinite ($\geqq \aleph_0$) cardinality $c_1 < c_2 < \cdots$ and each ring of characteristic 0. Let $\{B_j\}$ be the set of all ideals of the direct sum $B = \bigoplus^{\infty} K_i$ and let $\{F_j\}$ be the set of all such ideals which are finite direct sums. We will write $K_i \subseteq B_j$ to mean K_i is one of the summands appearing in B_j. Now B is an algebra over the rationals Q and so we may construct the split direct sum $R = B + Q$, and we will write $R_j = R/F_j$. Note that an arbitrary $\bar{R}$ has form $R/B_j \cong B_i + Q$, where any $K_r \subseteq B_i$ if and only if $K_r \not\subseteq B_j$. Also notice that this way of representing R/B_j is unique for if $K_r \subseteq B_i$ and say, $K_r \not\subseteq B_s$, then $B_i + Q \not\cong B_s + Q$ since the first has an ideal of cardinality c_r whereas all ideals of $B_s + Q$ have cardinality greater than or less than c_r.

LEMMA 13. *If there exists an epimorphism $R/B_j \to R/B_k$ then $B_j \subseteq B_k$.*

PROOF. We have $R/B_j \cong B_s + Q$ where any $K_i \subseteq B_s$ if and only if $K_i \not\subseteq B_j$. Thus if R/B_j has an image $R/B_k \cong (B_s + Q)/B_u \cong B_t + Q$ then any $K_i \subseteq B_t$ if and only if $K_i \subseteq B_s$ but $K_i \not\subseteq B_u$, that is $B_t \subseteq B_s$. But this implies $B_j \subseteq B_k$.

We now proceed with our construction of a q-radical, defining the class $N = \{R, R_j, B_j\}$. Since the only proper ideals of R or R_j are members of $\{B_j\}$ the class N is hereditary and so UN is radical. Then we have:

THEOREM 14. *The ring R of Example 1 has property (q) relative to the class N so UN is a q-radical.*

PROOF. If $\bar{R} = R/B_j$ has an image $R_j = R/F_j \in N$ then from Lemma 13 it follows that $B_j \subseteq F_j$, that is B_j is a finite direct sum. Thus $B_j = F_{j_1}$ and there exists an infinite properly ascending chain $F_{j_1} \subset F_{j_2} \subset \cdots$ of members of $\{F_j\}$. Thus $\bar{R}$ initiates an infinite chain $R_{j_1} \to R_{j_2} \to \cdots$ in N.

Since $P = UN$ is a q-radical we know from Proposition 8 that there cannot exist a minimal class M such that $P = UM$. We can say even more, namely:

THEOREM 15. *For the radical $P = UN$ of Example 1 if N' is a class such that $P = UN'$ has property (Int) relative to N' then there is a class N'' properly contained in N' such that $P = UN''$ has property (Int) relative to N''.*

PROOF. If $UN = UN'$ then some $\bar{R} \in N'$ and since $\bar{R}$ has an image in N, we have as above that $\bar{R} \cong R/F_j$ for some finite direct sum F_j. We can therefore find K_r, K_s with $K_r \neq K_s$ and neither a summand of F_j. We can regard K_r and K_s as ideals of $\bar{R}$ with $K_r \cap K_s = 0$. Thus $\bar{R}$ is a subdirect sum of $\bar{R}/K_r$ and $\bar{R}/K_s$ each a proper image of $\bar{R}$. Each of these is in $N \subseteq SP$ and since $P = UN'$ has property (Int) relative to N' each of them is a subdirect sum of members of N' none of which, by Lemma 13, could be isomorphic with $\bar{R}$. Therefore any subdirect sum of members of N' is also a subdirect sum of members of $N'' = N' \backslash \bar{R}$. Thus $P = UN''$ also has property (Int) relative to N''.

REMARK 5. Any radical P has (trivially) property (Int) relative to SP. Thus starting with SUN, this theorem provides an infinite properly descending sequence of classes $\{N_i\}$ such that $UN = UN_i$ has property (Int) relative to N_i.

REMARK 6. Notice that every ring in the hereditary class N has a simple image. Thus $N \subseteq SUM$ where M is the class of all simple rings. We therefore have an s-radical whose semisimple class contains a class not containing an s-class (in fact, a class N such that UN is a q-radical).

REMARK 7. In Example 1 if $B_r \notin \{F_j\}$ then R/B_r has no images in N so $R/B_r \in UN$. But any such ring has some $B_j \in N$ as an ideal so UN is not a hereditary radical. Note that if the $\{K_i\}$ are constructed by the method of Heyman and Leavitt (using algebras over fields of increasing cardinality) then each K_i contains an idempotent so is primitive. Thus the Jacobson radical $J \subset UN$ and since all $R/B_r \notin J$ the inclusion is proper. On the other hand, the $\{K_i\}$ could be constructed by the Sąsiada method; see Sąsiada and Cohn (1967) (using formal power series over fields of increasing cardinality) and in this case, UN would be incomparable with J.

We now show that J is a q-radical by constructing a ring with property q relative to the class of all primitive rings.

EXAMPLE 2. Let $\{r_0, r_1 \cdots\}$ be a sequence of ordinals defined as follows: $r_0 = 0$ and for all $n \geqq 1$, $r_n = s_n + 1$ where s_n is defined by $\aleph_u = 2^{\aleph_v}$, where $u = s_n$

https://doi.org/10.1017/S1446788700014725 Published online by Cambridge University Press

and $v = r_{n-1}$. Let $h = \lim r_n$ and let V be the vector space over Z_2 with basis a set W of cardinality $\aleph_h$. Our ring R will be the set of all linear transformations of V of rank $< \aleph_h$. It is easy to show (and well-known) that the ideals of R are precisely the sets $I_m = \{$all linear transformations of rank $< \aleph_m\}$ for all $0 \leqq m \leqq h$. Note that for all $m < h$, R/I_m is subdirectly irreducible with simple heart I_{m+1}/I_m.

Lemma 16. *For all integers $n \geqq 1$ there does not exist an epimorphism $R/I_m \to R/I_{r_n}$ for any $m \geqq r_{n+1}$.*

Proof. If such an epimorphism did exist then for some $k \geqq m \geqq r_{n+1}$ we would have an isomorphism $R/I_k \cong R/I_{r_n}$. For notational simplicity let us write $r_n = r$, $\aleph_r = a$ and $\aleph_k = b$. Such an isomorphism would imply isomorphic hearts, namely $I_{k+1}/I_k \cong I_{r+1}/I_r$. Let S be a subset of the basis W of cardinality a. Let $\alpha \in R$ be the identity on S and zero on the complement $S' = W \backslash S$. Then $\alpha \in I_{r+1}$, $\alpha \notin I_r$ and if $\bar{\alpha}$ is the image of α in I_{r+1}/I_r there would be some corresponding $\bar{\beta} \in I_{k+1}/I_k$. Let β be any preimage of $\bar{\beta}$ in I_{k+1} so β has rank b. Thus if T is a basis for $V\beta$ then T has cardinality b and we may extend to a basis $T \cup T'$ of V. Since $\bar{\beta}$ is idempotent, $\beta^2 - \beta = \gamma \in I_k$. Thus the matrix of β relative to $T \cup T'$ has form

$$\beta = \begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix} + G,$$

where G is a matrix of rank $< b$ and B is some submatrix in the columns corresponding to T. We now rearrange T as the union of a set $\{T_i\}$ of b (disjoint) sets each of cardinality b. Then relative to the basis $(\cup T_i) \cup T'$ the matrix becomes

$$\beta = \begin{bmatrix} \cdot & & & 0 & & & \\ & \cdot & & & & & \\ & & I_i & & & & \\ & 0 & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ \cdot & \cdot & \cdot & B_i & \cdot & \cdot & \cdot & 0 \end{bmatrix} + G',$$

where I_i is the identity on T_i and B_i is formed from the columns of B corresponding to the T_i. Also G' (similar to G) has rank $< b$. If we let

$$\beta_i = \begin{bmatrix} & 0 & \\ 0 & I_i & 0 \\ & 0 & \\ 0 & B_i & 0 \end{bmatrix},$$

we obtain a set $\{\beta_i\}$ of b orthogonal idempotents with the property that $\beta\beta_i\beta - \beta_i \in I_k$. There would then exist a set $\{\bar{\alpha}_i\}$ of orthogonal idempotents in I_{r+1}/I_r satisfying the relations $\bar{\alpha}\bar{\alpha}_i\bar{\alpha} = \bar{\alpha}_i$. Let $\{\alpha_i\}$ be any set of preimages in I_{r+1} of the $\{\bar{\alpha}_i\}$. Since α is the identity on S and zero on S' it follows that the matrix of α_i relative to $S \cup S'$ has form

$$\alpha_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} + H,$$

where A_i has non-zero elements only in the rows and columns relative to S and H has rank $< a$. Now the matrix A_i has a elements so the number of possible choices of such a matrix is $2^a = \aleph_{s_{n+1}}$. But $k \geqq m \geqq r_{n+1} = s_{n+1} + 1$ so $b = \aleph_k > 2^a$. Thus for some $i \neq j$ we would have $A_i = A_j$ whence $\alpha_i - \alpha_j$ would be of rank $< a$ contradicting $\bar{\alpha}_i \neq \bar{\alpha}_j$.

Remark 8. It has come to our attention that Divinsky (1975) has given a proof that the images of the ring of all linear transformations of rank $< \aleph_\omega$ of an $\aleph_\omega$-dimensional space are non-isomorphic. However, the construction is different from that given here and also the author assumes the generalized continuum hypothesis.

Remark 9. The field F over which the space V of Example 2 is defined is immaterial provided V has sufficiently high dimension. That is, if F has cardinality $\aleph_h$ then one can begin with $r_0 = h$ and proceed exactly as in the example.

Theorem 17. *J is a q-radical.*

Proof. It suffices to show that the ring R of Example 2 has property (q) relative to the class of all primitive rings. Note that every image $\bar{R}$ of R is subdirectly irreducible and every ideal contains an idempotent, so every $\bar{R}$ is primitive. Now $R = I_h$ and if $\bar{R} = R/I$ is non-zero then $I = I_t$ for some $t < h$. But $h = \lim r_n$ so $t \leqq r_n$ for some n. Then by Lemma 16 there is an infinite chain $R/I_{r_n} \to R/I_{r_{n+1}} \to \cdots$ initiated by $\bar{R}$.

Corollary 18. *All radicals contained in J are q-radicals.*

PROOF. If P is a radical such that $P \subseteq J$ then $SJ \subseteq SP$. Since SJ contains all primitive rings, the ring R of Example 2 is in SP and has property (q) relative to SP.

Note that we can construct (q)-radicals larger than J by restricting the class of primitive rings as follows:

PROPOSITION 19. *Let M be the class of all homomorphic images of all ideals of the ring R of Example* 2. *Then UM is a q-radical properly containing J.*

PROOF. M is a hereditary class so UM is a radical such that R has property (q) relative to M. All rings in M are primitive so $M \subseteq SJ$ and thus $J \subseteq UM$. However, there are many non-Jacobson radical rings (such as any ring with unit) having no image in M. Thus the inclusion is proper.

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