PART VI

MISCELLANEOUS DYNAMICS

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THE PLANAR INVERSE PROBLEM FOR AUTONOMOUS SYSTEMS

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Abstract: We study the general version of the inverse problem for planar trajectories and for autonomous dynamical systems possessing three integrals, i.e., for a given three-parametric family of curves f(x,y,a,b)=c we find the potential V(x,y) for which these curves are orbits of a unit mass. All possible cases, depending on the preassigned function f, are classified and in each case the necessary and sufficient conditions for the existence of a solution are established. Among the examples is the case of the Keplerian conic sections which is studied in detail.

1. INTRODUCTION

During the last few years there appeared a number of papers dealing with the following aspect of the inverse problem: A family of plane curves (depending on one or two parameters) is given in an inertial frame in Cartesian coordinates and required is the potential of a conservative dynamical system with two degrees of freedom (autonomous or not) for which all members of the given family are actual orbits.

For monoparametric families f(x,y)=c and for autonomous systems the answer to the question whether such a potential does exist is, in general, affirmative; the potential V=V(x,y) is given by Szebehely's linear partial differential equation of the first order in V (Shebehely, 1974). This equation is in fact associated with a certain dependence of the total energy E on the function f(x,y) and in examples this dependence E=E(f) has to be given in advance (Broucke, 1979, Broucke and Lass, 1977). Thus, in its general solution, there appear two arbitrary functions (Molnàr, 1981). One also finds in the literature applications, generalizations and modifications of Szebehely's equation. Thus, this equation was generalized by Bozis (1983) in order to include velocity dependent potentials and by Érdi (1982) to study three dimensional orbits. A modification of the same equation was presented by Szebehely and Broucke (1981) to account for non-inertial frames. A modification by Morrison (1976) uses the energy constant as the parameter of the family. Szebehely's equation was written in

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polar coordinates and an application was given recently to a problem of Galactic Dynamics by Szebehely, Lundberg and McGahee (1980).

For preassigned two-parametric families of curves $f(x,y,b)=c$ it is intuitevely expected that the picture regarding the existence of solutions changes. In fact potentials which give rise to such families do or do not exist depending on the function $f(x,y,b)$(Lass, 1972). Concrete criteria for this case were given recently by Bozis (1982) .

In the present paper we face the following problem: Given a three-parametric family of planar curves $f(x,y,a,b)=c$ in Cartesian coordinates x,y is there an autonomous dynamical system for which these curves are actual orbits of a unit mass? No assumption is made in advance for the dependence of the total energy E on the three parameters a,b and c. However, it is understood that when the problem admits a solution $V=V(x,y)$, the total energy $E=T+V$ is constant along each orbit, i.e., eventually $E=E(a,b,c)$.

The motivation for studying this problem is that the totality of the orbits of a two-dimensional autonomous dynamical system possessing three integrals of motion generally is a three-parametric family of planar curves. In this sense this appear to be the most general version of the inverse problem of this sort.

Since we now demand that a larger family of orbits results from a single potential, we expect more conditions to be necessary, and eventually necessary and sufficient, so that this problem admits a solution and, in fact, this is exactly what happens. At a first stage we find a set of necessary conditions for the case at hand to have a chance for an affirmative answer. This set also serves to classify each case at hand. Eventually we establish necessary and sufficient conditions for the problem to admit a solution.

2. ANALYSIS

We consider a three-parametric family of planar curves expressed in the form

$$f(x,y,a,b) = c \tag{1}$$

in the Cartesian x,y plane. We introduce the notation

$$\gamma=f_y/f_x \quad , \qquad \Gamma=\gamma\gamma_x-\gamma_y \quad , \tag{2}$$

$$\lambda=\Gamma^{-1}(-\Gamma_x+\gamma^{-1}\Gamma_y) \quad , \qquad \mu=\lambda\gamma+3\gamma^{-1}\Gamma \quad , \tag{3}$$

where the functions γ,Γ,λ and μ can be derived in a straightforward manner from any given $f(x,y,a,b)$ and the subscripts denote partial differentiations. The function γ vanishes or diverges identically when the family (1) represents the monoparametric families of straight lines which are parallel to the x or the y axis; and of course this is not the case here. Also $\Gamma\neq 0$, because $\Gamma=0$ represents in fact a two-parametric family of straight lines. Obviously, the functions γ,Γ,λ and μ depend on the parameters a and b as well. In fact $\gamma_a\neq 0$ and $\gamma_b\neq 0$

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because otherwise the family (1) would essentially depend on at most two parameters. The above assumptions serve to guarantee that the family (1) is a genuine three-parametric family of planar curves. The force components $X=X(x,y)$ and $Y=Y(x,y)$, inasmuch as they exist, which are derived from a potential $V=V(x,y)$ and give rise to the family of orbits (1), satisfy the linear system of partial differential equations (Bozis 1982 b)

$$X_y = Y_x \tag{4}$$

and

$$-X_x+(1-\gamma^2)\gamma^{-1}X_y+Y_y = \lambda X+\mu Y \ . \tag{5}$$

We demand that the potential, therefore and the force components, are independent of the parameters a,b and c; this expresses the fact that the free parameters of the family (1) enumerate the totality of the orbits admitted by a single potential $V(x,y)$. We seek, therefore, solutions of the system of equations (4) and (5) which satisfy the conditions

$$X_a = X_b = Y_a = Y_b = 0 \ . \tag{6}$$

The analysis which follows is heavily based on and very much facilitated by this requirement.

First we look for necessary conditions on the function γ - which are, in effect, conditions on the function f - for the system of equations (4) and (5) to satisfy the conditions (6). By differentiating equation (5) with respect to a and b we obtain that

$$X_y = \ell X+mY \qquad \text{and} \qquad X_y = LX+MY \tag{7}$$

where

$$\ell = -\frac{\gamma^2}{(1+\gamma^2)\gamma_a}\lambda_a \ , \qquad L = -\frac{\gamma^2}{(1+\gamma^2)\gamma_b}\lambda_b \tag{8}$$

$$m = -\frac{\gamma^2}{(1+\gamma^2)\gamma_a}\mu_a \ , \qquad M = -\frac{\gamma^2}{(1+\gamma^2)\gamma_b}\mu_b \ . \tag{9}$$

Then by differentiating equations (7) with respect to a and b we further obtain that

$$\begin{aligned} &\ell_a X+m_a Y = 0 \ , \qquad \ell_b X+m_b Y = 0 \\ &L_a X+M_a Y = 0 \ , \qquad L_b X+M_b Y = 0 \ . \end{aligned} \tag{10}$$

Finally, by subtracting the two equations (7) we obtain

$$(\ell-L)X+(m-M)Y = 0 \ . \tag{11}$$

Since we disregard as trivial the solutions for which one of the force components vanishes, we demand that the linear homogeneous system of the algebraic equations (10) and (11) admits non-trivial solutions. This system has solutions different from the solution $X = Y = 0$ if and only if the determinant of any two of these five equations equals to zero. In view of the comments which follow, we shall express this requirements in the form

$$\rho = \frac{\ell - L}{m - M} = \frac{\ell_a}{m_a} = \frac{\ell_b}{m_b} = \frac{L_a}{M_a} = \frac{L_b}{M_b} \quad . \tag{12}$$

Comments: (i) The common ratio ρ must be different from zero and from infinity; otherwise one of the equations (10) and (11) would imply that either X or Y vanishes identically.

(ii) If some of the ratios (12) are indeterminate of the form 0/0, these are simply ignored; in fact the corresponding equations (10) and (11) are satisfied identically and give no additional information.

We next consider two cases, depending on whether the ratio ρ (of equation 12) is defined or it is indeterminate.

Case I: The ratio ρ is defined.

Since ρ must be equal, from equations (10) and (11), to $-Y/X$, it must also be independent of the parameters a and b. We obtain, therefore, the conditions

$$\rho_a = 0 \quad \text{and} \quad \rho_b = 0 \quad , \tag{13}$$

which, however, are immediate consequences of equations (12). In this case all equations (10) and (11) reduce to the single equation

$$\rho X + Y = 0 \tag{14}$$

which must be combined with one of the two equations (7), say

$$\ell X + mY = X_y \ . \tag{15}$$

Note that, in view of equations (11) and (15), the second of equations (7) is also satisfied. We have to satisfy, therefore, the system of equations (4),(5),(14) and (15). Depending on the form of the system of equations (14) and (15), we distinguish two subcases:

Subcase Ia: The determinant $\delta = \rho m - \ell$ is different from zero

We have, in this case, that

$$X = -\delta^{-1} X_y \quad , \qquad Y = \rho\delta^{-1} X_y \ . \tag{16}$$

By combining now equations (4),(5),(14) and (15) we readily obtain that

$$X_x = \eta X \quad \text{and} \quad X_y = -\delta X, \tag{17}$$

where

$$\eta = (\gamma^2 - 1)\gamma^{-1}\delta + \rho\delta - \rho_y - \lambda + \mu\rho \ . \tag{18}$$

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Provided that the integrability condition

$$\delta_x+\eta_y = 0 \tag{19}$$

of the system of equations (17) is satisfied, they can be solved and determine X uniquely, up to a multiplicative constant; the second of equations (16) then can be used to determine Y algebraically. The force components so obtained satisfy equation (4) and therefore they arise from a potential. The solutions are acceptable provided that they are independent of the parameters a and b of the family of orbits. This is satisfied provided that

$$\delta_a = \delta_b = \eta_a = \eta_b = 0 . \tag{20}$$

However, it is straightforward to show that the conditions (20) are immediate consequences of the conditions (12) and (13) and therefore they are not additional necessary conditions. Note that in this case we also have that $\rho\neq\ell/m\neq L/M\neq\rho$. The final conclusion is that in the subcase Ia the necessary and sufficient conditions are the conditions (12) and (19). Whenever these conditions are satisfied, the force components are determined uniquely up to a constant factor.

Subcase Ib: The determinant $\delta=\rho m-\ell$ equals to zero

We have, in this case, that

$$\rho=\ell/m = L/M \tag{21}$$

as well. It is seen from equations (4),(14) and (15) that $X_y=Y_x= 0$ and therefore the force components are of the form

$$X=X(x) \quad \text{and} \quad Y=Y(y) \; ; \tag{22}$$

the corresponding potential is separable. The equations that have to be satisfied reduce in this case to the equations

$$-X_x+Y_y = \lambda X+\mu Y \quad \text{and} \quad \rho X+Y = 0 . \tag{23}$$

By differentiating the first of these equations with respect to a and b we obtain that

$$\lambda_a X+\mu_a Y = 0 = \lambda_b X+\mu_b Y \tag{24}$$

from which we obtain the necessary conditions

$$\rho = \lambda_a/\mu_a = \lambda_b/\mu_b \tag{25}$$

which however are satisfied by virtue of the equations (8),(9),(12) and (21). The two equations (23) now give that

$$X_x+\theta X = 0 \tag{26}$$

where

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$$\theta = \lambda - \mu\rho + \rho_y \quad . \tag{27}$$

Equation (26) determines uniquely, up to a multiplicative constant, an acceptable solution of our problem provided that

$$\theta_y = 0 \tag{28}$$

and

$$\theta_a = \theta_b = 0 \quad . \tag{29}$$

The conditions (29) are immediate consequences of the conditions (25) while equation (28) gives an additional independent necessary condition. Then Y is determined algebraically from the second equation (23) and the resulting solution is acceptable, provided that $Y_x = 0$ which, since $Y = -\rho X$, leads to the additional and independent necessary condition

$$\rho_x = \rho\,\theta \quad . \tag{30}$$

The final conclusion then is in the subcase Ib that the necessary and sufficient conditions are given by equations (12),(21),(28) and (30) and that, whenever these conditions are satisfied, the force components are determined uniquely, up to a constant factor.

Case II: The ratio ρ is indeterminate

Since all the ratios of equations (12) are of the form 0/0, the equations (10) and (11) are satisfied and therefore they can be omitted. We are left with equations (4),(5) and one, say the first, of equations (7). These are written as follows

$$\begin{aligned} -X_x + Y_y &= \{\lambda + (\gamma^2 - 1)\gamma^{-1}\ell\}X + \{\mu + (\gamma^2 - 1)\gamma^{-1}m\}Y \ , \\ X_y &= \ell X + mY \ , \\ X_y &= Y_x \ . \end{aligned} \tag{31}$$

The coefficients of X and Y in the right hand sides of the first two of equations (31) are independent of the parameters a and b. Therefore, all the solutions of the above system are independent of a and b and, as such, they are acceptable solutions of our problem. The necessary and sufficient conditions, therefore, in this case are

$$\ell - L = m - M = \ell_a = \ell_b = m_a = m_b = 0 \quad . \tag{32}$$

The example (35) of the next section shows that generally the force components are not uniquely determined in this case.

3. EXAMPLES

(3A): As an example for the subcase Ia of the analysis of the previous section we consider the (planar) Kepler problem: We assume that a system admits as orbits all the members of the three-parametric family of

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ellipses in the same plane and with common one of their focal points, but with arbitrary eccentricity, magnitude, and orientation of their major axes.

To describe this family we choose a Cartesian coordinate system whose origin coincides with the common focal point. The family then is described by the equation

$$f=(x^2+y^2)\{1+e\cos(\vartheta-\vartheta_0)\}^2 = \text{const.} \quad , \tag{33}$$

where e and ϑ_0 are the two parameters and $\tan\vartheta=y/x$. From the expression (33) we obtain that

$$\gamma=(\sin\vartheta+a)/(\cos\vartheta+b) \ , \tag{34}$$

where, here and henceforth, we shall consider, instead of e and ϑ_0, the $a = e\sin\vartheta_0$ and $b = e\cos\vartheta_0$ as the two free parameters of the family. To simplify the computations, we introduce the notation

$$\tan\omega=(b\sin\vartheta-a\cos\vartheta)/(1+a\sin\vartheta+b\cos\vartheta) \tag{35}$$

in which γ simplifies to

$$\gamma=\tan(\vartheta-\omega) \ . \tag{36}$$

Note that the parameters a and b appear in γ only through the combination ω. Then we obtain that

$$\gamma+{}^{x}/_{y} = \cos\omega/\sin\vartheta\cos(\vartheta-\omega), \tag{37}$$

$$\Gamma=\mp(x^2+y^2)^{-1/2}\cos^2\omega\cos^{-2}(\vartheta-\omega)(\cos\vartheta+b)^{-1} \ . \tag{38}$$

In order to obtain the last expression we have used equation (A.4) of the appendix and that

$$\vartheta_x-\omega_x=-\sin\vartheta\cos\omega\cos(\vartheta-\omega)(x^2+y^2)^{-1/2}(\cos\vartheta+b)^{-1} \ . \tag{39}$$

From the first of equations (3) we obtain,after a lengthy calculation, that

$$\lambda=3\{\sin(\vartheta-\omega)-b\sin\omega\}/(x^2+y^2)^{1/2}(\cos\vartheta+b)\sin(\vartheta-\omega) \ , \tag{40}$$

and a similar expression for μ.

Because it is very complicated to evaluate all the quantities which appear in the ratios (12), we follow an indirect approach to establish the validity of the necessary and sufficient conditions (12),(13) and (19).

It turns out that the combination

$$x\lambda+y\mu=y\lambda(\gamma+xy^{-1})+3y\Gamma\gamma^{-1}=-3\sin2\omega/\sin(2\vartheta-2\omega) \tag{41}$$

is very simple and it depends on the parameters a and b only through

ω. Hence from equations (36) and (41) we immediately obtain that

$$(x\lambda+y\mu)_a = -6\sin 2\vartheta \sin^{-2}(2\vartheta-2\omega)\omega_a$$
$$\gamma^2(1+\gamma^2)^{-1}=\sin^2(\vartheta-\omega) \ , \ -\gamma^2(1+\gamma^2)^{-1}\gamma_a^{-1}=\sin^2(2\vartheta-2\omega)(4\omega_a)^{-1} \tag{42}$$

and therefore that

$$x\ell+ym = -\frac{\gamma^2(x\lambda+y\mu)_a}{(1+\gamma^2)\gamma_a} = -\frac{3}{2}\sin 2\vartheta. \tag{43}$$

Obviously, we similarly obtain that

$$xL+yM = -\frac{3}{2}\sin 2\vartheta \ . \tag{44}$$

Hence $x\ell+ym=xL+yM$ which implies that

$$\rho = \frac{\ell-L}{m-M} = -\frac{y}{x} \ . \tag{45}$$

Since the right hand sides of equations (43) and (44) are independent of the parameters a and b we also have that

$$x\ell_a+ym_a=x\ell_b+ym_b=xL_a+yM_a=xL_b+yM_b= 0 \tag{46}$$

which, combined with equation (45), implies the validity of the conditions (12) and (13).

Finally for $\rho=-y/x$ we readily obtain that

$$\delta=-(x\ell+ym)/x = 3y/(x^2+y^2) \tag{47}$$

and

$$\eta=(y^2-2x^2)/x(x^2+y^2) \tag{48}$$

from which the last condition (19) is also verified. For these expression for δ and η, equations (17) and (14) are readily integrated and give

$$X=kx(x^2+y^2)^{-3/2} \ , \ Y=ky(x^2+y^2)^{-3/2} \ , \ k \ \text{ a constant}, \tag{49}$$

i.e., the Newtonian force. It should be noted that we have here derived Newton's force law by using only the first of the three laws of Kepler's and the weaker assumption that the motion is conservative, not central as stated by Kepler's second law.

(3B). As an example of a three parametric family of curves which arise from a separable potential we consider the family

$$f=b\sqrt{ax^2-1} - a\sqrt{by^2-1} = c \ . \tag{50}$$

For this family we readily obtain that

$$\gamma=-yx^{-1}(ax^2-1)^{1/2}(by^2-1)^{-1/2}, \quad \Gamma=Ax^{-1}(by^2-1)^{-1} \tag{51}$$

where

$$A=y^2x^{-2}-(ax^2-1)^{1/2}(by^2-1)^{-1/2} \tag{52}$$

and, after a long calculation, that

$$\lambda = \frac{1}{x} + \frac{2y^2}{x^3A} - \frac{2x}{A(y^2-x^2A)} + \frac{3bx}{\sqrt{(ax^2-1)(by^2-1)}} + \frac{1}{xA}\{\frac{y^2-x^2A}{x^2} - \frac{x^2}{y^2-x^2A}\} . \tag{53}$$

What turns out to be very simple is the combination

$$y^3\lambda+x^3\mu=(y^3+x^3\gamma)\lambda+3x^3\Gamma\gamma^{-1} = 3(y^4-x^4)(xy)^{-1} . \tag{54}$$

Therefore this family satisfies the relationships

$$\lambda_a/\mu_a = \lambda_b/\mu_b = - x^3/y^3 . \tag{55}$$

In addition we readily obtain that

$$y^3\ell+x^3m=0=y^3L+x^3M \tag{56}$$

and therefore all the ratios (12) are equal to $\rho = -x^3/y^3$ and the conditions (12),(13) and (21) are satisfied. Moreover, we obtain that $\theta=3/x$ which checks the validity of the final conditions (28) and (30). The family (50) represents the totality of orbits of the autonomous conservative system with potential

$$U = k(x^{-2}+y^{-2}),$$

where k is a constant.

(3Γ). For the case II we present a two-parametric worth of examples, characterized by the two arbitrary constants p and q which are distinct from the parameters a,b and c of the family of orbits.

The three-parametric family of curves is

$$f(x,y,a,b) = y+\int \{pg^3-3qg^2-3pg+q\}^{-1}dg = c , \tag{58}$$

where $g=g(x,y,a,b)$ is any two-parametric family of solutions of the equation

$$gg_x-g_y= pg^3-3qg^2-3pg+q . \tag{59}$$

We shall give the presentation in three steps.

(i) First we show that the family (58) satisfies the conditions (32). By using equations (58) and (59) we readily obtain that the function $\gamma=f_y/f_x$ of the present family equals to the solution g of equation (59) i.e., that $\gamma=g$. Therefore, $\Gamma=pg^3-3qg^2-3pg+q$ from which we can readily obtain that

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$$\lambda=-\gamma^{-1}\, d\Gamma/d\gamma = -3g^{-1}(pg^2-2qg-p)\ ,$$
$$\mu=3\Gamma\gamma^{-1}-\, d\Gamma/d\gamma = -3g^{-1}(qg^2+2pg-q)\ . \qquad (60)$$

It is the fact that λ and μ depend on the parameters a and b only implicitly, through the function g, which makes the evaluation of ℓ,L,m and M rather simple. In fact we obtain that

$$\ell=L=3p \quad , \qquad m=M=3q \qquad (61)$$

and therefore the necessary and sufficient conditions (32) are satisfied.

(ii) Second we determine the corresponding potential for a given choice of p and q. The force components satisfy the linear system of equations

$$X_x-Y_y+6qX-6pY=0 \quad , \qquad (62)$$

$$X_y=3pX+3qY \quad , \qquad (63)$$

$$X_y=Y_x\ . \qquad (64)$$

By taking the x derivative of eq. (62) and using equation (64) to eliminate Y we obtain a second order linear partial differential equation in X with constant coefficients which generally is of the irreducible type. Therefore its solutions are of the form

$$X=e^{3Ax+3By} \qquad (65)$$

for suitably chosen constants A and B. Equation (63) then gives that

$$Y=(B-p)q^{-1}e^{3Ax+3By} \qquad (66)$$

while equations (62) and (64) give two algebraic equations in A and B which, after some manipulations, become

$$A^3-3(p^2+q^2)A+2q(p^2+q^2)=0, \qquad B=pA(A-q)^{-1}\ . \qquad (67)$$

It turns out that the first of equations (67) has three different real roots when $pq\neq 0$, and obviously, to any of these roots there corresponds an acceptable solution of our problem. We conclude, therefore, that in this case the force components are not determined uniquely from the three-parametric family of curves. In fact, since the equations (62)-(64) are linear, we can also consider arbitrary superpositions of solutions; thus for a given choice of p and q (with $pq\neq 0$) we can construct a two-parametric family of force components which accept the three parametric family of orbits (58), where we have not counted the arbitrary overall multiplication factor as a free parameter. The corresponding potential is

$$V = -(3A)^{-1}e^{3Ax+3By} \quad . \tag{68}$$

(iii) Third, we describe how the family (58) was obtained. The condition $\ell=L$ demands that $\lambda_a/\gamma_a = \lambda_b/\gamma_b$ which is satisfied when $\lambda=\lambda(\gamma)$. This last condition is satisfied when $\Gamma=\Gamma(\gamma)$, which also guarantees that $m=M$. In this case we obtain that

$$\ell = L = \frac{\gamma\ddot{\Gamma}-\dot{\Gamma}}{1+\gamma^2} \quad , \qquad m = M = \frac{\gamma^2\ddot{\Gamma}-3\gamma\dot{\Gamma}+3\Gamma}{1+\gamma^2} \quad , \tag{69}$$

where the dots denote differentiations with respect to γ. The only way for ℓ and m to be independent of a and b is that the two expressions in (69) are constants, say $3p$ and $3q$ respectively. By solving the resulting equations we obtain that $\Gamma=p\gamma^3-3q\gamma^2-3p\gamma+q$. The family described by equations (58) and (59) is obtained by reconstructing f from a given Γ.

(3Δ). Finally as an example of a three parametric family of curves which does not arise from any autonomous conservative system we consider the family of all possible circles (with arbitrary center and radius) in the plane

$$f=f(x,y,a,b)=(x-a)^2+(y-b)^2 = c \; . \tag{70}$$

For this family we easily obtain that

$$\gamma = \frac{y-b}{x-a} \quad , \quad \Gamma = -\frac{f}{(x-a)^3} \quad , \quad \lambda = \frac{3}{x-a} \quad , \quad \mu = -\frac{3}{y-b} \tag{71}$$

and therefore

$$\ell = -\frac{3(y-b)}{f} \; , \qquad m = 0, \qquad L = 0, \qquad M = -\frac{3(x-a)}{f} \quad . \tag{72}$$

Obviously some of the ratios (12) become infinite, so no solution to our problem exists

4. INVARIANCE PROPERTIES OF THE THEORY

The analysis of the present paper and the necessary and sufficient conditions (12), (19), (21), (28), (30) and (32) at which we arrive on section 2, depend explicitly on the parameters a and b of the family of orbits (1). However, it is intuitively expected that one should have the freedom to reparametrize the original family in an arbitrary manner, say,

$$\tilde{a} = \tilde{a}(a,b) \quad \text{and} \quad \tilde{b} = \tilde{b}(a,b) \tag{73}$$

and that this reparametrization will not alter the classification and the conclusions of the analysis of section 2. The freedom in the choice of the parameters of the family (1) represents the gauge freedom of the pro-

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blem considered in this paper. We now show that the above mentioned conditions are indeed gauge invariant.

It is straightforward to see that under the change of gauge (73) the functions f, γ, λ and μ remain invariant while ℓ, L, m and M change according to

$$\tilde{\ell} = \ell-(\ell-L)\gamma_b{}^{b\tilde{a}}/\gamma_{\tilde{a}} \quad , \quad \tilde{L}=L+(\ell-L)\gamma_a{}^{a\tilde{b}}/\gamma_{\tilde{b}}$$
$$\tilde{m}=m-(m-M)\gamma_b{}^{b\tilde{a}}/\gamma_{\tilde{a}} \quad , \quad \tilde{M}=M+(m-M)\gamma_a{}^{a\tilde{b}}/\gamma_{\tilde{b}} \tag{74}$$

from which we readily obtain that

$$\tilde{\rho} = \frac{\tilde{\ell}-\tilde{L}}{\tilde{m}-\tilde{M}} = \frac{\ell-L}{m-M} = \rho \quad . \tag{75}$$

Hence the condition (13) implies that $\tilde{\rho}_{\tilde{a}} = \tilde{\rho}_{\tilde{b}} = 0$ which means that it is a gauge invariant condition. Then by using equations (12) and (74) we can easily show that

$$\tilde{\ell}_{\tilde{a}}/\tilde{m}_{\tilde{a}} = \tilde{\ell}_{\tilde{b}}/\tilde{m}_{\tilde{b}} = \tilde{L}_{\tilde{a}}/\tilde{M}_{\tilde{a}} = \tilde{L}_{\tilde{b}}/\tilde{M}_{\tilde{b}} = \rho = \tilde{\rho} \ , \tag{76}$$

which establishes the gauge invariance of the condition (12). Next we easily see that

$$\tilde{\delta}=\tilde{\rho}\tilde{m}-\tilde{\ell}=\rho m-\ell=\delta \tag{77}$$

which, among others, shows the gauge invariance of the classification of section 2 and of condition (21). Finally, equations (75) and (77) imply that $\tilde{\eta}=\eta$ and $\tilde{\theta}=\theta$ which establish the gauge invariance of the conditions (19),(28),(30) and (32), Q.E.D.

The original definitions, given by equation (3), of the two basic quantities λ and μ seem unrelated. However, the subsequent analysis is completely symmetrical in λ and μ. Here we establish the existence of a simple relationship between λ and μ which explains the symmetrical form of the theory in λ and μ. Precisely we shall show that under the change of coordinates

$$\tilde{x} = y \quad , \quad \tilde{y} = \varepsilon x, \tag{78}$$

where $\varepsilon^2=1$, the quantities λ and μ transform according to

$$\tilde{\lambda}= - \mu \quad , \quad \tilde{\mu}= -\varepsilon\lambda \quad . \tag{79}$$

(For $\varepsilon=+1$ the transformation (78) represents the interchange of the x and y axis, while for $\varepsilon=-1$ it represents a rotation in the x-y plane by 90^o degrees).

The proof is straightforward. For the same family of curves $\tilde{f}=f=$ $=$constant we obtain that

$$\tilde{\gamma} = f_{\tilde{y}}/f_{\tilde{x}} = \varepsilon\gamma^{-1} \tag{80}$$

and therefore that

$$\tilde{\Gamma}=\tilde{\gamma}\tilde{\gamma}_{\tilde{x}}-\tilde{\gamma}_{\tilde{y}} = \Gamma\gamma^{-3} \quad . \tag{81}$$

By using equations (80) and (81) and performing the required differentiations we obtain that

$$\tilde{\lambda}=\tilde{\Gamma}^{-1}(-\tilde{\Gamma}_{\tilde{x}}+\tilde{\gamma}^{-1}\tilde{\Gamma}_{\tilde{y}}) = -\mu \tag{82}$$

and

$$\tilde{\mu}=\tilde{\lambda}\tilde{\gamma}+3\tilde{\gamma}^{-1}\tilde{\Gamma}=-\varepsilon\lambda, \tag{83}$$

Q.E.D

5. DISCUSSION

Newton's law of gravitation is derived in the literature on the assumptions that (i) the orbits are ellipses with common focal point (Kepler's first law) and (ii) the areal velocity is constant (Kepler's second law). As a byproduct of the present analysis we have derived Newton's law by using only the first law and the assumption that the forces are conservative, which is weaker than being central .

A possible generalization of the analysis of this paper, on which we are presently working, refers to non-conservative dynamical systems. In the corresponding analysis we no longer have equation (4), while equation (5) is slightly modified. Since the majority of the necessary and sufficient conditions derived in section 2 arises from the successive differentiations of equation (5) with respect to the parameters a and b, we expect that the lack of information which results from the omission of equation (4) will be easily substituted from the information arising from the remaining equations.

Currently there is a lot of interest in the precise determination of the gravitational field of the earth from the observed motions of artificial satellites (Szebehely 1980). The theory developed in the present paper might be modified to account for such trajectories. Some preparatory numerical work would of course be necessary to fit into the theory.

APPENDIX

The evaluation of the basic quantities λ and μ of the theory of the present paper for a typical three-parametric family of planar curves

$$f(x,y,a,b) = c \tag{A.1}$$

is rather lengthy. We here obtain some useful general expressions for them with the additional assumption that the function f of equation (A.1) is homogeneous in x and y of degree n. When $n=0$ equation (A.1) represents straight lines passing through the origin and this is rather uninteresting. When $n=0$ the degree of homogeneity is irrelevant since it can be altered by raising eq. (A.1) to a suitable power. It is

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expected therefore that n will not appear explicitly in the expressions for λ and μ.

In fact it can be argued that any three-parametric family of curves can be put in the form (A.1) with f homogeneous. Indeed, by expressing the equation of the family of curves in polar coordinates and solving it for "r" one can always write it in the form

$$r = cg(\vartheta, a, b) \tag{A.2}$$

where one of the constants (c) is made to determine the scaling of r. Since $r = \sqrt{x^2+y^2}$ and $\vartheta = \tan^{-1}(y/x)$ are homogeneous of degree one and zero respectively, the family (A.2) is of the form (A.1) with f homogeneous of degree one. It should be pointed out, however, that the use of the homogeneous form of a given family of curves is not always the most convenient computationally. For instance, in the Example (3B) it was found more convenient to consider the non-homogeneous presentation (50) of the family of curves.

For a homogeneous f one can use Euler's theorem (stating that $xf_x+yf_y = nf$) to simplify some of the computations. Moreover in this case the function $\gamma = f_y/f_x$ is homogeneous of zero degree and therefore it can be viewed as a function of the single independent variable $z=y/x$. Hence

$$\gamma_x = -z\dot{\gamma}x^{-1} \quad , \qquad \gamma_y = \dot{\gamma}x^{-1} \quad , \tag{A.3}$$

where the dot denotes differentiation with respect to z. By expressing all the partial derivatives in terms of ordinary derivatives of γ it is straightforward to obtain that

$$\Gamma = \gamma\gamma_x - \gamma_y = (\gamma + xy^{-1})\gamma_x = -(\gamma z+1)\dot{\gamma}x^{-1} \tag{A.4}$$

and then that

$$\lambda = (x\gamma)^{-1}\{(\gamma z+1)\ddot{\gamma}\dot{\gamma}^{-1} + z\dot{\gamma} + 2\gamma\} \tag{A.5}$$

and

$$\mu = x^{-1}\{(\gamma z+1)(\ddot{\gamma}\dot{\gamma}^{-1} - 3\dot{\gamma}\gamma^{-1}) + z\dot{\gamma} + 2\gamma\} . \tag{A.6}$$

Equations (A.5) and (A.6) imply the useful relation

$$x\lambda + y\mu = (\gamma z+1)\gamma^{-1}\{(\gamma z+1)\ddot{\gamma}\dot{\gamma}^{-1} - 2z\dot{\gamma} + 2\gamma\} . \tag{A.7}$$

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