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ON THE CONNECTEDNESS OF CERTAIN SETS IN SUMMABILITY THEORY

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ABSTRACT. This note considers the question of the connectedness of the set of limit points of the A-transforms of a sequence, where A is a conservative Hausdorff, quasi-Hausdorff or Meyer-König–Ramanujan type of matrix. New proofs of some known results, as well as some new results are obtained.

§1

Given a conservative matrix A and a sequence s, let $L(A; s)$ denote the set of all limit points of the A-transform of s (if it exists). Several authors have dealt with the question: "When is $L(A, s)$ connected for all $s \in (m)$, the space of bounded sequences?" and considered the cases where A was Hausdorff or was quasi-Hausdorff. (See [1], [2], [3], [8] and also [4].) In the present paper we deal with the same question, and to certain refinements of it; we consider not only Hausdorff and quasi-Hausdorff matrices but also the Meyer-König–Ramanujan type of matrices (S^*, μ) introduced by Ramanujan [7]; we adopt a unified and somewhat novel approach which enables us to deal with matrices $A = (H, \mu)$ or (H^*, μ) or (S^*, μ) more or less simultaneously. The results obtained include and often improve the results obtained by the earlier authors, or are new.

We follow mainly the notation and definitions of Ramanujan [7] and Parameswaran [5], [6]. The proofs of our theorems are based on the following lemmas.

LEMMA 1. *If* $s = \{s_n\}$ *is a bounded sequence and* $a_n \equiv s_n - s_{n-1} = o(1)$, *then* $L(I, s)$ *is connected.* (I = *identity.*) (Barone [1].)

LEMMA 2. (See Parameswaran [5], pp. 52, 56 and 60; Cf. Ramanujan [7], pp. 205, 207 and 211.) *Let* $A = (H, \mu)$, (H^*, μ) *or* (S^*, μ) *be conservative. Then*

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there exists a function $g \in BV[0,1]$ *such that*

$$u_n = \int_0^1 F_n(t,s)\,dg(t) = \left(\int_0^{0+} + \int_{0+}^{1-0} + \int_{1-0}^1\right) dg(t) \tag{1}$$

where $u = \{u_n\} = As$, *and* $\{F_n(t,s)\}$ *denotes the Euler-transform* $\{E_n(t,s)\}$, *of order* t, *of* s *or the 'Taylor-transform'* $\{T_n(t,s)\}$ *of* s *and* $s_n = O(1)$ *or the 'Meyer-König-transform* $\{L_n(t,s)\}$ *of* s *and* $s_n = O(1)$, *according as* $A = (H,\mu)$ *or* (H^*,μ) *or* (S^*,μ), *respectively; further, if* $A = (H^*,\mu)$ *or* (S^*,μ), *the function* $g(t)$ *can be chosen so as to be continuous at* $t = 0$.

LEMMA 3. *Let* A *be a conservative Hausdorff or quasi-Hausdorff matrix. Then there exists a function* $g \in BV[0,1]$ *such that, with the notation* $u = As$,

$$\text{(i)}\quad u_n = \lambda[g(0+)-g(0)]s_0 + \ell[g(1-0)-g(0+)] + s_n[g(1)-g(1-0)] + o(1) \tag{2}$$

for all bounded sequences s *Borel-summable to* ℓ, *and with* $\lambda = 1$ *or* 0 *according as* A *is Hausdorff or not;*

$$\text{(ii)}\quad u_n - u_{n-1} = \ell(\mu_1 - \lim \mu_n) + (s_n - s_{n-1}) \lim \mu_n + o(1) \tag{3}$$

if A *is Hausdorff and* $\{s_n - s_{n-1}\}$ *is bounded and Borel-summable to* ℓ, *or if* $s_n = O(1)$ *and* A *is Hausdorff or quasi-Hausdorff and then indeed* (3) *holds with* $\ell = 0$.

Lemma 3 is essentially contained in [5]; a proof is sketched below for completeness. Part (i) is proved by letting $n \to \infty$ in the righthand side of (1); note that under the conditions stated, $|F_n(t,s)| \le K < \infty$ uniformly in n and t, $\lim_{n\to\infty} F_n(t,s) = \ell$ $(0 < t < 1)$,

$$\lim_{t\to 0+} F_n(t,s) = \lambda s_0 \quad \text{and} \quad \lim_{t \to 1-0} F_n(t,s) = s_n.$$

For part (ii) of Lemma 3, we see that by Lemma 2,

$$\begin{aligned} u_n - u_{n-1} &= \int_0^1 [F_n(t,s) - F_{n-1}(t,s)]\,dg(t) \\ &= \left(\int_0^{0+} + \int_{0+}^{1-0} + \int_{1-0}^1\right) dg(t) \end{aligned} \tag{4}$$

where (a) if A is Hausdorff and $\{a_n\} = \{s_n - s_{n-1}\}$ is bounded and Borel-summable to ℓ, the integrand in (4) reduces to $tE_{n-1}(t;a)$, is uniformly bounded in $0 \le t \le 1$, tends to ℓt for $0 < t < 1$ as $n \to \infty$, and for fixed n, tends to a_n as $t \to 1-0$ and to 0 as $t \to 0+$; and (b) if $s_n = O(1)$, the integrand in (4) is uniformly bounded in $0 \le t \le 1$, tends to 0 as $n \to \infty$ for $0 < t < 1$ and, for fixed n, tends to λs_0 as $t \to 0+$ and to $s_n - s_{n-1}$ as $t \to 1-0$. Then, in each of the cases

(a) and (b), we get the desired result upon letting $n \to \infty$ in the righthand side of (4).

§2

In this section the symbol A may denote equally a conservative Hausdorff matrix (H, μ_n) or a conservative quasi-Hausdorff matrix (H^*, μ_n) except when explicitly specified.

THEOREM 1. (a) *If* $s \in (B)(m)$ [*i.e. s is a Borel-summable bounded sequence*], *then* $L(A, s)$ *is connected if* $L(I, s)$ *is connected.*

(b) *If* $s \in (B)(m)$ *and* $L(A, s)$ *is connected for some* A *with* $\lim \mu_n \neq 0$, *then* $L(I, s)$ *is connected.*

(c) *If* $s \notin (B)(m)$, *then* $L(A, s)$ *is connected for some* A *with* $\lim \mu_n \neq 0 \not\Rightarrow L(I, s)$ *is connected.*

(c)′. *In* (c) *above we may replace the phrase* $\lim \mu_n \neq 0$ by $\lim \mu_n = 0$.

(d) $\lim \mu_n \neq 0 \Leftrightarrow L(A, s)$ *is connected for* (*only*) *almost no sequence of* 0's *and* 1's $\Leftrightarrow L(A, s)$ *is not connected when* $s_n = \frac{1}{2}[1 + (-1)^n]$.

Proof. Parts (a) and (b) follow from Lemmas 1 and 3(i). Part (c) is proved by the example given in Remark (iii) below; (c)′ is a consequence of the fact that if $\lim \mu_n = 0$ then $L(A, s)$ is connected for all $s \in (m)$, a known result which is also included in each of Theorems 2 and 3 below. Part (d) follows from parts (a) and (b), since almost all sequences of 0's and 1's, and in particular $\{\frac{1}{2} + \frac{1}{2}(-1)^n\}$, are Borel-summable.

REMARKS. (i) Theorems 1(a), 1(b) may be compared with earlier results of Ramanujan ([17], Theorems 5, 7) and of the author ([6], theorem 2(i), (ii)) and Theorem 1(d) with another result of the author ([5], Theorem 10) which deal with the A-*summability* of s, i.e. the case when $L(A, s)$ has a unique element. (See also the remark under Theorem 4 below.)

(ii) Theorem 1 (a) has non-trivial content. For, it is known that if we take any function $F(n) \neq o(n^{-1/2})$ with $F(n) \downarrow 0$, e.g. $F(n) = n^{-1/4}$, then there exist Borel-summable bounded divergent sequences s with $s_n - s_{n-1} = O(F(n)) = o(1)$. By Lemma 1, $L(I, s)$ is connected, and hence so is $L(A, s)$ by Theorem 1(a). Thus there are Borel-summable bounded divergent sequences s for which $L(A, s)$ is connected for *every* conservative $A = (H, \mu)$ or (H^*, μ). (See also the corollary to Theorem 3 below.)

(iii) Theorem 1(c) shows that Theorem 1(b) is a best possible one. Theorem 1(d) is also a best possible result in the sense that there exists even a regular Hausdorff matrix $A = (H, \mu_n)$ with $\lim \mu_n \neq 0$ and a divergent sequence s of 0's and 1's such that $L(A, s)$ is connected, and there exists also a regular quasi-Hausdorff matrix $G = (H^*, \nu_n)$ with $\lim \nu_n \neq 0$ and a divergent sequence t of 0's and 1's such that $L(G, t)$ is connected, while obviously, neither $L(I, s)$ nor $L(I, t)$ is connected.

To see this we take $A = \frac{1}{2}(I + C_1)$, where C_1, is the Cesaro matrix, and s to be a sequence consisting of alternating bunches of 0's and 1's as follows: $s_0 = 1$, $s_n = 0(n_{2k} < n \le n_{2k+1})$, $s_n = 1(n_{2k+1} < n \le n_{2k+2})$, where $\{n_k\}$ is a sequence of positive integers which increases so rapidly that $L(A, s)$ consists of the interval $[0, 1]$; for instance, we may take $n_k = \exp_{10}(2^k)$.

For the quasi-Hausdorff case, we take $G = \frac{1}{2}(I + C_1^*)$, where $C_1^* = (H^*, \mu_n)$ with $\mu_n = 1/(n+2)$. It is then not difficult to see that there exists a divergent sequence t of 0's and 1's such that $L(G, t)$ is the interval $[0, 1]$.

(iv) Leviatan and Lorch [3] raised the following question: Do the limit points of equivalent transforms of bounded sequences have the same connectedness properties? The matrix $A = \frac{1}{2}(I + C_1)$ considered above is equivalent to I for *all* sequences, and since $L(A, s)$ is connected while $L(I, s)$ is not (where s is as defined in (iii) above), it is seen that the question is answered in the negative for Hausdorff matrices; similarly, for quasi-Hausdorff matrices, the matrices G and I and the sequence t mentioned in (iii) above again provide an answer in the negative.

(v) The examples $u = As$ and $v = Gt$, where A, G, s and t are as in Remark (iii) above, provide yet other proofs of the known fact that the converse of Lemma 1 is not true.

THEOREM 2. *The following statements are equivalent*:

(a) $\lim \mu_n = 0$;

(b) $L(A, s)$ *is connected for some Borel-summable divergent sequences of* 0's *and* 1's;

(c) $L(A, s)$ *is connected for all bounded sequences* s;

(d) A *sums a Borel-summable bounded divergent sequence*;

(e) A *sums all Borel-summable bounded sequences.*

If A is *Hausdorff*, then each of the following is also equivalent to (a):

(f) $L(A, s)$ *is connected for all sequence* $s = \{s_n\}$ *for which* $As \in (m)$, $s_n = o(n^{1/2})$ *and* $s_n - s_{n-1} = O(1)$;

(g) $L(A, s)$ *is connected for all sequences* $s = \{s_n\}$ *for which* $As \in (m)$ *and* $\{s_n - s_{n-1}\}$ *is bounded and Borel-summable to* 0.

Proof. it is well-known that (a)$\Leftrightarrow$(d)$\Leftrightarrow$(e) (Parameswaran [6], Theorem 2); (c)$\Rightarrow$(b), trivially; (b)$\Rightarrow$(a) by Theorem 1(b), and the implication (a)$\Rightarrow$(c) follows from Lemmas 1 and 3(ii).

If A is Hausdorff, then (a)$\Rightarrow$(g)$\Rightarrow$(f) by Lemmas 1 and 3(ii) and the fact that if $s_n = o(n^{1/2})$ then $\{s_n - s_{n-1}\}$ is Borel-summable to 0; since (f)$\Rightarrow$(c) trivially, it follows, from the equivalences already proved that all of (a)–(g) are equivalent.

REMARKS. (i) For regular Hausdorff methods, the relation (a)$\Leftrightarrow$(c) was given by Wells [8] and Erdös and Piranian [2]; Leviatan and Lorch [3] showed that the relation (c)$\Rightarrow$(a) holds for multiplicative Hausdorff matrices, and that (c)$\Leftrightarrow$(a) for conservative quasi-Hausdoff matrices. (ii) The examples considered in Remark (iii) under Theorem 1 show that if the word "Borel-summable" is dropped from the statement (b) of Theorem 2 then the theorem will be true neither for Hausdorff nor for quasi-Hausdorff matrices. (iii) See also the concluding remarks at the end of the paper.

The next two theorems follow readily from Lemmas 1 and 3(ii) and yield conditions that are sufficient in order that $L(A, s)$ be connected for a given pair A and s.

THEOREM 3. *Let* $s \in (m)$ *and* A *be given. Then* $L(A, s)$ *is connected if* $\mu_n a_n \equiv \mu_n(s_n - s_{n-1}) = o(1)$.

COROLLARY. *There exist bounded divergent sequences s which are not Borel-summable and such that* $L(A, s)$ *is connected for every conservative* $A = (H, \mu)$ *or* (H^*, μ). *(Cf. Remark* (ii) *under Theorem* 1.)

For, we can take any bounded sequence $\{s_n\}$ *which is not Borel-summable and for which* $s_n - s_{n-1} = o(1)$.

THEOREM 4. *Let* $A = (H, \mu)$ *be conservative Hausdorff and* $\{s_n\}$ *a sequence that* $As \in (m)$ *and* $\{a_n\} \equiv \{s_n - s_{n-1}\}$ *is bounded and Borel-summable to* ℓ. *Then* $L(A, s)$ *will be connected if any one of the following conditions holds*:

(i) $a_n = o(1)$;
(ii) $a_n \to \ell$ *and* $\mu_1 = 0$;
(iii) $\mu_n \to \ell = 0$;
(iv) $\mu_n \to \mu_1 = 0$.

REMARK. The example of A and s given in Remark (iii) under Theorem 1 show that none of the conditions given in Theorems 3 or 4 is necessary for $L(A, s)$ to be connected. However, Theorem 3 is best possible in the sense that we cannot replace the small o by a large O, as is seen from Theorem 1(d).

§3

In an earlier paper (Parameswaran [5], Theorem 8) it is proved that a conservative matrix $A = (S^*, \mu)$ of the Meyer-König–Ramanujan type sums all Borel-summable bounded sequences, irrespective of whether $\lim \mu_n$ is 0 or not. The contrast with the Hausdorff and quasi-Hausdorff matrices is reflected also in the following result.

THEOREM 5. *If* $A = (S^*, \mu)$ *is conservative, then* $L(A, s)$ *is connected for all bounded sequences s.*

Proof. If $u = As$ where A, s are as in the theorem, then

$$(5) \qquad u_n \int_{0+}^{1-0} L_n(t, s)\, dg(t) + s_0[g(1) - g(1-0)]$$

by (1), since $\lim_{t \to 1-0} L_n(t, s) = s_0$; hence

$$(6) \qquad u_n - u_{n-1} = \int_{0+}^{1-0} [L_n(t, s) - L_{n-1}(t, s)] = o(1)$$

since the integrand is uniformly bounded and tends to 0 as $n \to \infty$ for $0 < t < 1$; (these are given in [5], p. 60 and p. 51 respectively). The theorem now follows from Lemma 1.

Concluding remarks. The Authors thanks the referee for drawing his attention to Liu and Rhoades [4] which appeared after the first version of the present paper was submitted. Liu and Rhoades consider the regular 'generalized' Hausdorff and quasi-Hausdorff matrices $(A^{(\alpha)}, \mu_n) = (H^{(\alpha)}, \mu_n)$ or $(H^{*(\alpha)}, \mu_n)$ for $\alpha \geq 0$. However each conservative generalized matrix $A^{(\alpha)}$, whether it is a matrix of one of the above two types or is a 'generalized' Meyer-König–Ramanujan matrix $(S^{*(\alpha)}, \mu_n)$ is absolutely equivalent for bounded sequences to an ordinary matrix of the same kind. (This result and some others on the $(A^{(\alpha)}, \mu)$-matrices will appear elsewhere.) The theorems of the present paper are therefore true also for the generalized matrices $A = (H^{(\alpha)}, \mu)$ or $(H^{*(\alpha)}, \mu)$ or $(S^{*(\alpha)}, \mu)$, $\alpha \geq 0$; thus they include, and extend, the results of Liu and Rhoades [4].

Finally, the author thanks the referee for many comments which improved the presentation of the results.

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