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ON DIRECT SUMS OF INJECTIVE MODULES AND CHAIN CONDITIONS

STANLEY S. PAGE AND YIQIANG ZHOU

ABSTRACT. Let R be a ring and M a right R-module. Let $\sigma[M]$ be the full subcategory of Mod-R subgenerated by M. An M-natural class $\mathcal{K}$ is a subclass of $\sigma[M]$ closed under submodules, direct sums, isomorphic copies, and M-injective hulls. We present some equivalent conditions each of which describes when $\mathcal{K}$ has the property that direct sums of (M-)injective modules in $\mathcal{K}$ are (M-)injective. Specializing to particular M, and/or special subclasses we obtain many new results and known results as corollaries.

1. **Introduction.** Throughout all rings R are associative with identity, and all modules are unitary right R-modules. Given any family $\mathcal{F}$ of right R-modules, and a collection $\{A_i\}_{i\in I}$ with A_i in $\mathcal{F}$ for every $i \in I$, these questions arise: i) Is $\bigoplus_{i\in I} A_i$ in $\mathcal{F}$? ii) Is $\bigoplus_{i\in I} E(A_i)$ in $\mathcal{F}$? (where $E(A_i)$ is the injective hull of A_i.) iii) Is $\bigoplus_{i\in I} E(A_i) = E(\bigoplus_{i\in I} A_i)$? Also, associated with the family $\mathcal{F}$ there is, for each module M, the set of submodules $H_{\mathcal{F}}(M) = \{N \subseteq M : M/N \in \mathcal{F}\}$, with particular interest in the set of right ideals $H_{\mathcal{F}}(R)$. In this paper we address the above questions for families which are called *M-natural classes* for a fixed right R-module M and give the answers to the questions in terms of chain conditions in the sets $H_{\mathcal{K}}(M)$ and $H_{\mathcal{K}}(R)$. More precisely, by a class $\mathcal{K}$ of modules we mean $\mathcal{K}$ is a collection of modules such that $\mathcal{K}$ is closed under isomorphic copies. Mod-R will denote the category of unitary right R-modules. Following Wisbauer [8], for any $M \in$ Mod-R, we denote by $\sigma[M]$ the full subcategory of Mod-R, whose objects are the submodules of M-generated modules. A subclass of $\sigma[M]$ which is closed under submodules, direct sums, and M-injective hulls is called an *M-natural class*. We will see later there do exist many M-natural classes. The results are applied to give direct sum decompositions of certain injective modules which generalize known results.

2. **Results on an M-natural class.** Let M and N be R-modules. N is called *M-singular* if $N \cong L/K$ for some $L \in \sigma[M]$ and $K \leq_e L$. The M-injective hull of N, denoted by $E_M(N)$, is the trace of M in $E(N)$, *i.e.* $E_M(N) = \sum\{f(M) : f \in \operatorname{Hom}(M, E(N))\}$.

LEMMA 1. *Let $N \in \sigma[M]$. Then $N \subseteq E_M(N)$.*

PROOF. Since $N \in \sigma[M]$, there exist an index set I and a module A such that

$$0 \longrightarrow N \xrightarrow{\ell} A; \quad M^{(I)} \xrightarrow{\pi} A \longrightarrow 0,$$

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where ℓ is a monomorphism and π an epimorphism. We have a submodule B of A which is maximal with respect to $\ell(N) \cap B = 0$. Then $\ell(N) \oplus B \leq_e A$. Since B is a closed submodule of A, $\ell(N)$ can embed in A/B as an essential submodule. Thus, without loss of generality, we may assume that $\ell(N) \leq_e A$. For every $\alpha \in I$, let ℓ_α be the homomorphism of M into $M^{(I)}$ sending the element $x \in M$ into the element of $M^{(I)}$ whose value at α is x and whose remaining components are 0. Then $A = \pi(M^{(I)}) = \pi\big(\sum_{\alpha \in I} \ell_\alpha(M)\big) = \sum_{\alpha\in I} \pi \circ \ell_\alpha(M)$. Thus $A \subseteq E_M(A)$. Since ℓ is a monomorphism and $\ell(N) \leq_e A$, there exists an isomorphism $\bar{\ell}\colon E(N) \longrightarrow E(A)$ that extends ℓ. Noting that $\bar{\ell}\big(E_M(N)\big) = \bar{\ell}\Big(\sum\big\{f(M) : f \in \mathrm{Hom}\big(M, E(N)\big)\big\}\Big) = \sum\big\{\bar{\ell} \circ f(M) : f \in \mathrm{Hom}\big(M, E(N)\big)\big\} = \sum\big\{g(M) : g \in \mathrm{Hom}\big(M, E(A)\big)\big\} = E_M(A) \supseteq A \supseteq \bar{\ell}(N)$, we have that $E_M(N) \supseteq N$. ∎

LEMMA 2. *Let M and N be R-modules. If $E_M(N)$ is M-singular, then no non-zero submodules of N can embed in M. The converse is true if M is $M^{(I)}$-projective for all index sets I.*

PROOF. If $E_M(N)$ is not M-singular, then $f(M)$ is not M-singular for some $0 \neq f \in \mathrm{Hom}\big(M, E(N)\big)$. Since $f(M) \cong M/\operatorname{Ker}(f)$ and $M \in \sigma[M]$, $\operatorname{Ker}(f)$ is not essential in M. Hence, $\operatorname{Ker}(f) \cap X = 0$ for some $0 \neq X \subseteq M$. It follows that $X \cong f(X) \subseteq E(N)$. Therefore, the non-zero submodule $f(X) \cap N$ of N can embed in X, and hence in M.

For the converse, we suppose that for some $0 \neq Y \subseteq N$ and $Y \xrightarrow{f} M$. Then there exists a map $g\colon M \longrightarrow E(Y)$ such that $g \circ f$ is the inclusion of Y into $E(Y)$. Then $\operatorname{Ker}(g) \cap f(Y) = 0$. So $\operatorname{Ker}(g)$ is not essential in M. If $g(M)$ is M-singular, then $M/\ker(g) \cong g(M) \cong L/K$ for some $L \in \sigma[M]$ and $K \leq_e L$. Hence we have an epimorphism $h\colon M \longrightarrow L/K$ with $\operatorname{Ker}(h) = \operatorname{Ker}(g)$. Since M is $M^{(I)}$-projective for all I and $L \in \sigma[M]$, it follows that M is L-projective by [2, Proposition 16.12, p. 186]. Therefore, there exists a map $h_1\colon M \longrightarrow L$ such that $\pi \circ h_1 = h$, where π is the natural map from L to L/K. Since $K \leq_e L$, $h_1^{-1}(K) \leq_e M$. But $h_1^{-1}(K) \subseteq \operatorname{Ker}(h) = \operatorname{Ker}(g)$. It follows that $\operatorname{Ker}(g) \leq_e M$, a contradiction. So $g(M)$ is not M-singular. From the fact that $g(M) \subseteq E_M(Y) \subseteq E_M(N)$, it follows that $E_M(N)$ is not M-singular. ∎

COROLLARY 3. *Let M be an* f. g. *quasi-projective module. Then $E_M(N)$ is M-singular iff no non-zero submodules of N can embed in M.*

PROOF. This is because of the fact that if M is f. g. quasi-projective, then M is $M^{(I)}$-projective for all I by [2, Proposition 16.12, p. 186]. ∎

A class $\mathcal{K} \subseteq \sigma[M]$ is said to be an *M-natural class* if $\mathcal{K}$ is closed under submodules, direct sums and M-injective hulls.

EXAMPLES. i) $\sigma[M]$ is an M-natural class.

ii) For an f. g. quasi-projective module M, the class of all modules in $\sigma[M]$ with M-singular M-injective hulls is an M-natural class.

PROOF. It is easy to check by using Corollary 3.

For a subclass $\mathcal{F} \subseteq \sigma[M]$, we denote by $C_{\mathcal{F}}$ the class of all modules in $\sigma[M]$ for which no nonzero submodules can embed in any element of $\mathcal{F}$. The following proposition shows that all M-natural classes can be constructed in this way.

PROPOSITION 4. *A class $\mathcal{K}$ of modules is an M-natural class iff $\mathcal{K} = C_{\mathcal{F}}$ for some class $\mathcal{F} \subseteq \sigma[M]$.*

PROOF. First we show that $C_{\mathcal{F}}$ is an M-natural class for any class $\mathcal{F} \subseteq \sigma[M]$. It is easy to see that $C_{\mathcal{F}}$ is closed under submodules. Suppose that $E_M(N) \notin C_{\mathcal{F}}$. Note that $E_M(N) \in \sigma[M]$. There is a $0 \neq N' \subseteq E_M(N)$ such that $N' \hookrightarrow P \in \mathcal{F}$ for some P. Then $0 \neq N \cap N' \hookrightarrow P$. Thus $N \notin C_{\mathcal{F}}$. Let $N = \bigoplus_t N_t$ with all $N_t \in C_{\mathcal{F}}$. Clearly $N \in \sigma[M]$. If $N \notin C_{\mathcal{F}}$, then there is a nonzero cyclic submodule xR of N which is embeddable in some module of $\mathcal{F}$. But $xR \subseteq N_{t_1} \oplus N_{t_2} \oplus \cdots \oplus N_{t_n}$ for some n. Then $N_{t_1} \oplus N_{t_2} \oplus \cdots \oplus N_{t_n}$ is not in $C_{\mathcal{F}}$. Let m be the least number such that some $N_{t_1} \oplus N_{t_2} \oplus \cdots \oplus N_{t_m}$ is not in $C_{\mathcal{F}}$. Then there is a nonzero submodule $N' \subseteq N_{t_1} \oplus N_{t_2} \oplus \cdots \oplus N_{t_m}$ such that N' is embeddable in some module of $\mathcal{F}$. If p_i is the projection of $N_{t_1} \oplus \cdots \oplus N_{t_m}$ onto N_{t_i}, then by the minimality of m, $P_i|_{N'}: N' \longrightarrow N_{t_i}$ is a monomorphism. It follows that $N_{t_i} \notin C_{\mathcal{F}}$, a contradiction. So $C_{\mathcal{F}}$ is closed under direct sums, and hence is an M-natural class.

Conversely, suppose that $\mathcal{K}$ is an M-natural class of modules. Let $\mathcal{F} = C_{\mathcal{K}}$. We show $\mathcal{K} = C_{\mathcal{F}}$. It is easy to see that $\mathcal{K} \subseteq C_{\mathcal{F}}$. Suppose that $N \in C_{\mathcal{F}}$. Then $N \in \sigma[M]$ and $N \notin \mathcal{F}$, and so there exists $0 \neq N' \subseteq N$ such that $N' \in \mathcal{K}$. By Zorn's Lemma, there exists a maximal independent family $\mathcal{X}$ of submodules in $\mathcal{K}$ of N. Let $U = \bigoplus_{N' \in \mathcal{X}} N'$. Then we have a submodule V of N such that $U \cap V = 0$ and $U \oplus V$ is essential in N. If $V \neq 0$, then $V \in C_{\mathcal{F}}$, and we have some $0 \neq P \subseteq V$ with $P \in \mathcal{K}$ just as above. Hence $\mathcal{X} \cup \{P\}$ is an independent family, contradicting the maximality of $\mathcal{X}$. Thus $V = 0$ and U is essential in N. Thus $E(U) = E(N)$, and hence $E_M(U) = E_M(N)$. But since every $N' \in \mathcal{X}$ is in $\mathcal{K}$, it follows that U, and hence $E_M(N)$, and hence $N \in \mathcal{K}$ by our assumptions on $\mathcal{K}$. ∎

Throughout the following, we let $\mathcal{K} = C_{\mathcal{F}}$, where $\mathcal{F} \subseteq \sigma[M]$, be an M-natural class, and $H_{\mathcal{K}}(N) = \{N' \subseteq N : N/N' \in \mathcal{K}\}$.

LEMMA 5. *Suppose that $B_1 \subseteq B_2 \subseteq \cdots$ is a chain of submodules of $N \in \sigma[M]$ such that $B_{i+1}/B_i \in \mathcal{K}$ for all i. Let $B = \bigcup_i B_i$. Then $B/B_i \in \mathcal{K}$ for all i.*

PROOF. Suppose that B/B_i is not in $\mathcal{K}$ for some i. Note that $B/B_i \in \sigma[M]$. Then there exists a nonzero submodule X/B_i of B/B_i ($X \subseteq B$) such that X/B_i is embeddable in an element of $\mathcal{F}$. We have that $0 \neq X/B_i = (X/B_i) \cap (\bigcup_{j \geq i} B_j/B_i) = \bigcup_{j \geq i} (X \cap B_j)/B_i$. It follows that $(X \cap B_j)/B_i \neq 0$ for some $j > i$. Since $(X \cap B_i)/B_i = 0$, we may assume that $X \cap B_{j-1}/B_i = 0$. Then $\big((X \cap B_j) + B_{j-1}\big)/B_{j-1} \cong (X \cap B_j)/(X \cap B_{j-1}) = (X \cap B_j)/B_i \subseteq X/B_i$. This shows that B_j/B_{j-1} has a nonzero submodule which is embeddable in an element of $\mathcal{F}$. Thus B_j/B_{j-1} is not in $\mathcal{K}$, a contradiction. ∎

LEMMA 6. *Let $B_1 \subseteq B_2 \subseteq \cdots$ be a chain as in Lemma 5. Then there exists $K \subseteq N$ such that $N/K \in \mathcal{K}$, $B_1 \subseteq K$, and $(B_{i+1} + K)/(B_i + K) \cong B_{i+1}/B_i$ for all $i \geq 1$.*

PROOF. If $N/B_1 \in \mathcal{K}$, then let $K = B_1$ and we are done. Assume $N/B_1 \notin \mathcal{K}$, and let $B = \bigcup_i B_i$. Then there exists a submodule X/B_1 of N/B_1 ($X \subseteq N$) which is maximal with respect to $(B/B_1) \cap (X/B_1) = 0$. Let $K = X$. Then clearly $K \cap B = B_1$. It follows that $K \cap B_i = B_1$ for all i. Hence, $(B_{i+1}+K)/(B_i+K) \cong B_{i+1}/\big(B_{i+1} \cap (B_i+K)\big) = B_{i+1}/\big(B_i+(B_{i+1} \cap K)\big) = B_{i+1}/(B_i+B_1) = B_{i+1}/B_i$. Note that K/B_1 is a closed submodule of N/B_1. By using of [5, Proposition 1.4, p. 18], we have that B/B_1 can embed in N/K as an essential submodule. Therefore, $E(B/B_1) \cong E(N/K)$. It follows that $E_M(B/B_1) \cong E_M(N/K)$. By Lemma 5, $B/B_1 \in \mathcal{K}$. Thus $E_M(B/B_1)$, hence $E_M(N/K)$ is in $\mathcal{K}$ since $\mathcal{K}$ is closed under M-injective hulls. But $N/K \in \sigma[M]$. We have that $E_M(N/K) \supseteq N/K$ by Lemma 1. It follows that $N/K \in \mathcal{K}$ since $\mathcal{K}$ is closed under submodules. ∎

LEMMA 7. *The following are equivalent for a module $N \in \sigma[M]$ and an M-natural class $\mathcal{K}$:*

(a) Every chain of submodules of N: $B_1 \subseteq B_2 \subseteq \cdots$ with all $B_{i+1}/B_i \in \mathcal{K}$, terminates;
(b) $H_{\mathcal{K}}(N)$ has a. c. c.

PROOF. (a) ⇒ (b). This follows because $\mathcal{K}$ is closed under submodules.

(b) ⇒ (a). Suppose there exists a strictly ascending chain of submodules of N: $B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots$ such that B_{i+1}/B_i is in $\mathcal{K}$ for every i. We show that this leads to a contradiction by constructing a strictly ascending chain $K_1 \subset K_2 \subset \cdots$ with every $N/K_i \in \mathcal{K}$. By Lemma 6, there is a $K_1 \subseteq N$ such that $N/K_1 \in \mathcal{K}$, $B_1 \subseteq K_1$, and $(B_{i+1}+K_1)/(B_i+K_1) \cong B_{i+1}/B_i$ for all i. Then $B_1 \subseteq K_1 \subset B_2+K_1$ and $B_2+K_1 \subset B_3+K_1 \subset \cdots$ is a strictly ascending chain with all $(B_{i+1}+K_1)/(B_i+K_1)$ in $\mathcal{K}$ for $i \geq 2$. Suppose we have constructed $K_1, K_2, \ldots, K_n$ such that all N/K_i are in $\mathcal{K}$, $K_1 \subset K_2 \subset \cdots \subset K_n$, $K_n \subset B_{n+1}+K_n$ and $B_{n+1}+K_n \subset B_{n+2}+K_n \subset \cdots$ is a strictly ascending chain with $(B_{i+1}+K_n)/(B_i+K_n)$ in $\mathcal{K}$ for all $i \geq n+1$. Applying Lemma 6 to the chain $B_{n+1}+K_n \subset B_{n+2}+K_n \subset \cdots$, wehave a $K_{n+1} \subseteq N$ such that $N/K_{n+1} \in \mathcal{K}$, $B_{n+1}+K_n \subseteq K_{n+1} \subset B_{n+2}+K_{n+1}$ and $B_{n+2}+K_{n+1} \subset B_{n+3}+K_{n+1} \subset \cdots$ is a strictly ascending chain with $(B_{i+1}+K_{n+1})/(B_i+K_{n+1})$ in $\mathcal{K}$ for all $i \geq n+2$. The induction principle implies that there exists a sequence $\{K_i : i \in \mathbf{N}\}$ with all N/K_i in $\mathcal{K}$ and $K_1 \subset K_2 \subset \cdots K_n \subset \cdots$ is a strictly ascending chain. The lemma is proved. ∎

COROLLARY 8. *Let $\mathcal{K}$ be an M-natural class, and let Y be a submodule of $X \in \sigma[M]$. Then $H_{\mathcal{K}}(X)$ has* a. c. c. *iff both $H_{\mathcal{K}}(Y)$ and $H_{\mathcal{K}}(X/Y)$ has* a. c. c..

PROOF. The necessity follows from Lemma 7. To show the sufficiency, we assume that $X_1 \subseteq X_2 \subseteq \cdots$ is an ascending chain of submodules of X with each $X/X_i \in \mathcal{K}$. Then we have $X_1 \cap Y \subseteq X_2 \cap Y \subseteq \cdots \subseteq Y$ and $(X_1+Y)/Y \subseteq (X_2+Y)/Y \subseteq \cdots \subseteq X/Y$. Since $(X_{i+1} \cap Y)/(X_i \cap Y) \hookrightarrow X_{i+1}/X_i$, we have that $(X_{i+1} \cap Y)/(X_i \cap Y) \in \mathcal{K}$ for each i. Since $H_{\mathcal{K}}(Y)$ has a. c. c., by Lemma 7, there exists a positive integer m such that $(X_{m+s} \cap Y) = (X_m \cap Y)$ for all s. Then for any $j \geq m$, we have that $[(X_{j+1}+Y)/Y]/[(X_j+Y)/Y] \cong (X_{j+1}+Y)/(X_j+Y) \cong X_{j+1}/[(X_j+(X_{j+1} \cap Y)] = X_{j+1}/X_j \in \mathcal{K}$. Because $H_{\mathcal{K}}(X/Y)$ has a. c. c., by using Lemma 7 again, we can find a positive integer n ($n \geq m$) such that $X_{n+t}+Y = X_n+Y$ for all t. Then it is clear that $X_{n+k} = X_n$ for all k. ∎

PROPOSITION 9. *The following are equivalent for an M-natural class $\mathcal{K}$:*

(a) Every direct sum of M-injective modules in $\mathcal{K}$ is M-injective;

(b) Every direct sum of M-injective hulls of modules in $\mathcal{K}$ is M-injective;

(c) For every cyclic (or finitely generated) submodule xR of M, any chain of submodules of xR: $B_1 \subseteq B_2 \subseteq \cdots$ such that all B_{i+1}/B_i are in $\mathcal{K}$, terminates.

PROOF. (a) $\Rightarrow$ (b). This is because of the fact that if $N \in \mathcal{K}$, then $E_M(N)$ is M-injective, and in $\mathcal{K}$.

(b) $\Rightarrow$ (c). Suppose that for a cyclic submodule $xR \subseteq M$, there is a strictly ascending chain of submodules of xR: $B_1 \subset B_2 \subset \cdots$ such that all $B_{i+1}/B_i \in \mathcal{K}$. Then $E = \bigoplus_i E_M(B_{i+1}/B_i)$ is M-injective by (b). Let $B = \bigcup B_i$ and p_i the natural map from B_{i+1} onto B_{i+1}/B_i. Since $E_M(B_{i+1}/B_i) \supseteq B_{i+1}/B_i$ by Lemma 1, we let ℓ_i be the inclusion of B_{i+1}/B_i to $E_M(B_{i+1}/B_i)$. Because $E_M(B_{i+1}/B_i)$ is M-injective, there exists a homomorphism $f_i: M \longrightarrow E_M(B_{i+1}/B_i)$ that extends $\ell_i \circ p_i$. Define a map $f: B \longrightarrow E$ via $\pi_i \circ f(b) = f_i(b)$, where π_i is the projection of E onto $E_M(B_{i+1}/B_i)$. Then f is well-defined. Since E is M-injective and $B \subseteq xR \subseteq M$, there exists $g: xR \longrightarrow E$ that extends f. We have that $g(M) \subseteq \bigoplus_{i=1}^{m} E_M(B_{i+1}/B_i)$ for some m. Then $\pi_i \circ f = 0$ for all $i > m$. If $b \in B_{m+1}$, then $0 = \pi_{m+1} \circ f(b) = f_{m+1}(b) = b + B_m$. This implies that $B_{m+1} = B_m$, a contradiction.

(c) $\Rightarrow$ (a). By [6, Theorem 1.7, p. 3], to show (a), it suffices to show that every direct sum of countable M-injective modules in $\mathcal{K}$ is M-injective. So let $N = \bigoplus_{i=1}^{\infty} N_i$, where each N_i is M-injective, and in $\mathcal{K}$. Let B be a submodule of xR with $x \in M$ and $f: B \longrightarrow N$ a homomorphism. Let $B_k = \{b \in B : f(b) \in \bigoplus_{i=1}^{k} N_i\}$. Then $B_1 \subseteq B_2 \subseteq \cdots$, and

$$B_{k+1}/B_k \overset{\phi}{\hookrightarrow} \Big(\bigoplus_{i=1}^{k+1} N_i\Big) \Big/ \Big(\bigoplus_{i=1}^{k} N_i\Big) \cong N_{k+1} \in \mathcal{K} \quad \text{via}$$

$$\phi(b + B_k) = f(b) + \bigoplus_{i=1}^{k} N_i.$$

Since $\mathcal{K}$ is closed under submodules, $B_{k+1}/B_k \in \mathcal{K}$. By (c), there exists m such that $B_{m+i} = B_m$ for all i. Thus $f(B) \subseteq \bigoplus_{i=1}^{m} N_i$. Since $\bigoplus_{i=1}^{m} N_i$ is M-injective, there exists $g: xR \longrightarrow N_1 \oplus \cdots \oplus N_m \subseteq N$ which extends f. Hence N is xR-injective for every $x \in M$. By [6, Proposition 1.4, p. 2], N is M-injective. ■

THEOREM 10. *The following are equivalent for an M-natural class $\mathcal{K}$:*

(a) Every direct sum of M-injective modules in $\mathcal{K}$ is M-injective;

(b) Every direct sum of M-injective hulls of modules in $\mathcal{K}$ is M-injective;

(c) For any cyclic (or finitely generated) submodule $A \subseteq M$, $H_{\mathcal{K}}(A)$ has a. c. c.

PROOF. By Lemma 7 and Proposition 9. ■

COROLLARY 11. *The following are equivalent for an* f. g. *module M, and an M-natural class $\mathcal{K}$:*

(a) Every direct sum of M-injective modules in $\mathcal{K}$ is M-injective;

(b) Every direct sum of M-injective hulls of modules in $\mathcal{K}$ is M-injective;

(c) $H_{\mathcal{K}}(M)$ has a. c. c.

PROOF. By Theorem 10 and Corollary 8. ∎

By applying Theorem 10 to the M-natural class $\sigma[M]$, we have another consequence:

COROLLARY 12. *The following are equivalent for a module M:*
(a) Every direct sum of M-injective modules is M-injective;
(b) Every direct sum of M-injective modules in $\sigma[M]$ is M-injective;
(c) Every direct sum of M-injective hulls of modules is M-injective;
(d) M is a locally Noetherian module, i.e. every cyclic (or finitely generated) submodule of M is a Noetherian module.

PROOF. The equivalence of (a) $\Leftrightarrow$ (d) is [6, Theorem 1.11]. And (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follow from Theorem 10. ∎

LEMMA 13. *Let $N \in \sigma[M]$. If N is M-injective, then N is quasi-injective.*

PROOF. If N is M-injective, then N is $M^{(I)}$-injective by [6, Proposition 1.5, p. 2]. If $N \in \sigma[M]$, then N is a submodule of a homomorphic image of some $M^{(I)}$. Thus N is N-injective by [6, Proposition 1.3, p. 1]. ∎

An R-module N is said to be an *extending module*, if every closed submodule of N is a summand. Recall that a family $\{N_i : i \in I\}$ of submodules of a module N is said to be a *local summand* if the sum $\sum_{i\in I} N_i$ is direct, and $\bigoplus_{i\in F} N_i$ is a direct summand for every finite subset F of I.

LEMMA 14. *Let $\mathcal{K}$ be an M-natural class, $N \in \mathcal{K}$ an extending module. If $H_{\mathcal{K}}(M)$ has* a. c. c., *then every local summand of N is a summand.*

PROOF. Let $\Gamma = \{X_\lambda : \lambda \in \Lambda\}$ be a local summand of N and $X = \sum_{\lambda\in\Lambda} X_\lambda$. Since N is an extending module, $X \leq_e Y \subseteq^{\oplus} N$ for some Y. Then Y is an extending module and in $\mathcal{K}$. So, without loss of generality, we can assume that $X \leq_e N$. We need to show that $X = N$. Suppose that $X \neq N$. Note that $N \subseteq E_M(N) = \sum\big\{f(M) : f \in \operatorname{Hom}\big(M, E(N)\big)\big\}$. There exists a least number t such that, for some $x \in N \setminus X$ and some $f_i \in \operatorname{Hom}\big(M, E(N)\big)$ $(i = 1, \ldots, t)$, $x \in f_1(M) + \cdots + f_t(M)$. For $g_i \in \operatorname{Hom}\big(M, E(N)\big)$ $(i = 1, \ldots, t)$, we denote by $\bigoplus_{i=1}^t g_i$ the map $M^{(t)} \longrightarrow E(N)$ which sends $(x_1, \ldots, x_t)$ to $g_1(x_1) + \cdots + g_t(x_t)$. Then, as a submodule of $E_M(N)$, $\operatorname{Im}(\bigoplus_{i=1}^t g_i)$ is in $\mathcal{K}$. Now, let $\Omega = \big\{\operatorname{Ker}(\bigoplus_{i=1}^t g_i)$: there exist $y \in N \setminus X$ and $g_i \in \operatorname{Hom}\big(M, E(N)\big)$ $(i = 1, \ldots, t)$, such that $y \in g_1(M) + \cdots + g_t(M)\big\}$. By the choice of t, we see that Ω is a non-empty subset of $H_{\mathcal{K}}(M^{(t)})$. Since $H_{\mathcal{K}}(M)$ has a. c. c., $H_{\mathcal{K}}(M^{(t)})$ has a. c. c. by Corollary 8, and thus there exist $y \in N \setminus X$ and $g_i \in \operatorname{Hom}\big(M, E(N)\big)$ $(i = 1, \ldots, t)$, such that $y \in g_1(M) + \cdots + g_t(M)$ and $\operatorname{Ker}(\bigoplus_{i=1}^t g_i)$ is a maximal element in Ω. Since $X \leq_e N$, then $0 \neq yr \in X$ for some $r \in R$. Let $yr \in \bigoplus_{i=1}^n X_{\lambda_i}$. By assumption, $N = (\bigoplus_{i=1}^n X_{\lambda_i}) \oplus Z$ for some $Z \subseteq N$. Then $E(N) = E(\bigoplus_{i=1}^n X_{\lambda_i}) \oplus E(Z)$. Let p_Z be the projection of $E(N)$ onto $E(Z)$. Write $y = y_1 + y_2$, where $y_1 \in \bigoplus_{i=1}^n X_{\lambda_i}$, $y_2 \in Z$. Clearly, $y_2 \notin X$ and $yr = y_1 r$. Let $h_i = p_Z \circ g_i$ $(i = 1, \ldots, t)$. Then $y_2 = p_Z(y) \in \sum_{i=1}^t h_i(M) \subseteq E(Z)$. Therefore, $\operatorname{Ker}(\bigoplus_{i=1}^t h_i) \in \Omega$. It is easy to check that $\operatorname{Ker}(\bigoplus_{i=1}^t g_i) \subseteq \operatorname{Ker}(\bigoplus_{i=1}^t h_i)$. Choose $a_i \in M$ $(i = 1, \ldots, t)$ such that

$g_1(a_1) + \cdots + g_t(a_t) = y$. Then $\sum_{i=1}^t g_i(a_i r) = yr \neq 0$, but $\sum_{i=1}^t h_i(a_i r) = p_Z(yr) = 0$. Thus, $(a_1 r, \ldots, a_t r) \in \mathrm{Ker}(\bigoplus_{i=1}^t h_i) \setminus \mathrm{Ker}(\bigoplus_{i=1}^t g_i)$. Hence $\mathrm{Ker}(\bigoplus_{i=1}^t g_i) \subset \mathrm{Ker}(\bigoplus_{i=1}^t h_i)$, which contradicts the maximality of $\mathrm{Ker}(\bigoplus_{i=1}^t g_i)$. ∎

LEMMA 15. *Let $N \in \sigma[M]$. Then*

(a) *If $N_1, \ldots, N_m$ is an independent set of submodules of N, then $E_M(N_1 \oplus \cdots \oplus N_m) = E_M(N_1) \oplus \cdots \oplus E_M(N_m)$;*

(b) *If $X \subseteq N$, then $E_M(N) = E_M(X) \oplus E_M(Y)$ for some $Y \subseteq N$.*

PROOF. (a) It suffices to show that $E_M(N_1 \oplus N_2) = E_M(N_1) \oplus E_M(N_2)$. We have that $E(N_1 \oplus N_2) = E(N_1) \oplus E(N_2)$. Let p_1, p_2 be the projections of $E(N_1 \oplus N_2)$ onto $E(N_1)$ and $E(N_2)$, respectively. It is clear that $E_M(N_1) \oplus E_M(N_2) \subseteq E_M(N_1 \oplus N_2)$. Define a homomorphism $\sigma\colon E_M(N_1 \oplus N_2) \longrightarrow E_M(N_1) \oplus E_M(N_2)$ via $\sigma(x) = p_1(x) + p_2(x)$. The restriction of σ on $E_M(N_1) \oplus E_M(N_2)$ is equal to the identity map of $E_M(N_1) \oplus E_M(N_2)$. Therefore $E_M(N_1) \oplus E_M(N_2)$ is a direct summand of $E_M(N_1 \oplus N_2)$. But by Lemma 1, $N_1 \oplus N_2 \subseteq E_M(N_1) \oplus E_M(N_2)$, implying that $E_M(N_1) \oplus E_M(N_2) \leq_e E_M(N_1 \oplus N_2)$. It follows that $E_M(N_1 \oplus N_2) = E_M(N_1) \oplus E_M(N_2)$.

(b) If $X \subseteq N$, then $X \oplus Y \leq_e N$ for some $Y \subseteq N$. Then by (a), $E_M(N) = E_M(X \oplus Y) = E_M(X) \oplus E_M(Y)$. ∎

THEOREM 16. *The following are equivalent for an* f. g. *module M and an M-natural class $\mathcal{K}$:*

(a) *Every direct sum of M-injective hulls of modules in $\mathcal{K}$ is M-injective;*

(b) *Every M-injective hull of a module in $\mathcal{K}$ is a direct sum of uniform modules;*

(c) *Every M-injective hull of a module in $\mathcal{K}$ has a decomposition that complements direct summands;*

(d) *Every extending module in $\mathcal{K}$ is a direct sum of uniform modules.*

PROOF. (a) $\Rightarrow$ (d). If (a) holds, then we have a. c. c. on $H_{\mathcal{K}}(M)$ by Corollary 11. Then, if $N \in \mathcal{K}$ is an extending module, every local summand of N is a summand by Lemma 14. Hence, by [6, Theorem 2.17, p. 25], N is a direct sum of indecomposable modules. But every extending indecomposable module is a uniform module, and thus N is a direct sum of uniform modules.

(d) $\Rightarrow$ (b). By Lemma 13, every M-injective hull of a module in $\mathcal{K}$ is an extending module.

(b) $\Leftrightarrow$ (c). By Lemma 13 and [6, Theorem 2.22, p. 27].

(c) $\Rightarrow$ (a). Let D be a direct sum of M-injective hulls of modules in $\mathcal{K}$. Since every M-injective hull of a module in $\mathcal{K}$ is a direct sum of uniform modules, every M-injective hull of a module in $\mathcal{K}$ is, in fact, a direct sum of M-injective hulls of uniform modules in $\mathcal{K}$. Therefore we can write $D = \bigoplus\{E_M(D_\alpha) : \alpha \in B_t, t \in T\}$ such that all D_α are uniform modules in $\mathcal{K}$, and if $\alpha \in B_t$, $\alpha' \in B_{t'}$, then $E_M(D_\alpha) \cong E_M(D_{\alpha'})$ if and only if $t = t'$. For each t, choose one $\alpha_t \in B_t$ and let $F = \{\alpha_t : t \in T\}$ and $D_1 = \bigoplus_{\alpha \in F} E_M(D_\alpha)$. Let $E = E_M(D_1^{(\mathbf{N})})$. Then E is in $\mathcal{K}$ since $\mathcal{K}$ is an M-natural class. So we can write $E = \bigoplus_A E_\alpha$, a decomposition that complements direct summands. For each $\alpha \in F$, let

$A(\alpha) = \{\beta \in A : E_\beta \cong E_M(D_\alpha)\}$. Now for each $n > 0$, the module $E_M(D_\alpha)^{(n)}$ is isomorphic to a submodule of E, and hence isomorphic to a direct summand of E by Lemma 15. So by [2, 12.2, p. 142], $\mathrm{Card}\big(A(\alpha)\big) \geq n$. And hence $A(\alpha)$ is infinite. Let $B = \bigcup_{\alpha\in F} A(\alpha)$. Then $D_1^{(\mathbf{N})}$ is isomorphic to a summand of the direct summand $\bigoplus_B E_\alpha$ of E. Therefore $D_1^{(\mathbf{N})}$ is M-injective. Let B' is the disjoint union of $\{B_t : t \in T\}$. Then by [6, Theorem 1.7, p. 3], $D_1^{(B')}$ is M-injective. From this it follows that $D \cong \bigoplus\{E_M(D_\alpha) : \alpha \in B_t, t \in T\} \cong \bigoplus_{t\in T} E_M(D_{\alpha_t})^{(B_t)} \subseteq^{\oplus} \bigoplus_{t\in T} E_M(D_{\alpha_t})^{(B')} \cong \left(\bigoplus_{t\in T} E_M(D_{\alpha_t})\right)^{(B')} \cong D_1^{(B')}$. This shows that D is M-injective. ∎

COROLLARY 17. *The following are equivalent for an* f. g. *module M:*

(a) M is a Noetherian module;

(b) For every N, $E_M(N)$ is a direct sum of uniform modules;

(c) For every N, $E_M(N)$ has a decomposition that complements direct summands;

(d) Every extending module in $\sigma[M]$ is a direct sum of uniform modules. ∎

If M is an f. g. quasi-projective module, then the class $\{N \in \sigma[M] : E_M(N)$ is M-singular$\}$ is a M-natural class.

COROLLARY 18. *The following are equivalent for an* f. g. *quasi-projective module M:*

(a) Every direct sum of M-injective modules with M-singular M-injective hulls is M-injective;

(b) Every direct sum of M-singular M-injective hulls of modules is M-injective;

(c) Every chain of submodules of M: $M_1 \subseteq M_2 \subseteq \cdots$ such that each $E_M(M_{i+1}/M_i)$ is M-singular, terminates;

(d) For any module A, if $E_M(A)$ is M-singular, then $E_M(A)$ is a direct sum of uniform modules;

(e) For any module A, if $E_M(A)$ is M-singular, then $E_M(A)$ has a decomposition that complements direct summands;

(f) Every extending module with M-singular M-injective hull is a direct sum of uniform modules. ∎.

3. **A special case: $M = R$.** If $M = R$, then the M-singular submodule and the M-injective hull of a module N coincide, respectively, with the singular submodule and injective hull of the module N in the usual sense. The M-natural classes are just the classes of R-modules which are closed under submodules, direct sums, and injective hulls . We simply call such classes natural classes.

EXAMPLES. i) Mod-R is a natural class.

ii) The class of all modules with singular injective hulls is a natural class.

iii) For any hereditary torsion theory τ, the τ-torsionfree class is a natural class.

iv) For any hereditary stable torsion theory τ, the τ-torsion class is a natural class.

PROPOSITION 19. *Every natural class is closed under extensions of modules.*

PROOF. By Proposition 4, we may suppose that $\mathcal{K} = C_{\mathcal{F}}$ is a natural class, and $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ an exact sequence with $N, M/N$ both in $\mathcal{K}$. If $M \notin \mathcal{K}$, then there is $0 \neq N_1 \subseteq M$ such that N_1 is embeddable in some module in $\mathcal{F}$. Since $N \in \mathcal{K}$, $N \cap N_1 = 0$. It follows that N_1 can embed in M/N, implying $M/N \notin \mathcal{K}$. This is a contradiction. ■

A natural class is, in general, not a hereditary torsion class or a hereditary torsionfree class. For example, the class of torsion $\mathbf{Z}$-modules is a natural class, but is not closed under products, and hence is not a hereditary torsionfree class. An example of a natural class, but not of a hereditary torsion class is provided in [7, Remarks ii].

For a submodule N of a module M_R and $x \in M$, we denote by $(N : x)$ the set $\{r \in R : xr \in N\}$. In particular, we let $x^{\perp} = (0 : x)$.

For a natural class $\mathcal{K}$, $M \in \mathcal{K}$ iff $x^{\perp} \in H_{\mathcal{K}}(R)$ for all $x \in M$. Therefore we have that $\mathcal{K}$ forms a hereditary torsion class iff $H_{\mathcal{K}}(R)$ is an idempotent filter.

PROPOSITION 20. *Let $\mathcal{K}$ be a natural class. Then*
(a) $\mathcal{K}$ forms a hereditary torsion class iff $I \subseteq J$ with $I \in H_{\mathcal{K}}(R)$ implies $J \in H_{\mathcal{K}}(R)$;
(b) $\mathcal{K}$ is a hereditary torsionfree class iff $H_{\mathcal{K}}(R)$ is closed under arbitrary intersections.

PROOF. (a) One direction is obvious. Suppose that $H_{\mathcal{K}}(R)$ is closed under super sets. Since $I \in H_{\mathcal{K}}(R)$ implies $(I : a) \in H_{\mathcal{K}}(R)$ for all $a \in R$, we only need to show that: If $(I : a) \in H_{\mathcal{K}}(R)$ for any $a \in J \in H_{\mathcal{K}}(R)$, then $I \in H_{\mathcal{K}}(R)$. Consider the exact sequence $0 \longrightarrow (I + J)/I \longrightarrow R/I \longrightarrow R/(I + J) \longrightarrow 0$. Clearly $R/(I + J) \in \mathcal{K}$. For any $a \in J$, $(I : a) = (I \cap J : a) \in H_{\mathcal{K}}(R)$, and so $[aR + (I \cap J)]/(I \cap J) \cong R/(I \cap J : a) \in \mathcal{K}$ for all $a \in J$. Then $(I + J)/I \cong J/(I \cap J) \in \mathcal{K}$. By Proposition 19, $\mathcal{K}$ is closed under extensions, and hence $R/I \in \mathcal{K}$, *i.e.* $I \in H_{\mathcal{K}}(R)$.

(b) Suppose that $H_{\mathcal{K}}(R)$ is closed under arbitrary intersections. Let $M = \prod_t M_t$ with every $M_t \in \mathcal{K}$. For any $x \in M$, write $x = (x_t)$ with $x_t \in M_t$ for all t. We have $x_t^{\perp} \in H_{\mathcal{K}}(R)$ since $R/x_t^{\perp} \cong x_t R \subseteq M_t$ and M_t is in $\mathcal{K}$. Then $x^{\perp} = \bigcap_t x_t^{\perp} \in H_{\mathcal{K}}(R)$. It follows that $M \in \mathcal{K}$. Therefore we have shown that $\mathcal{K}$ is closed under products, and hence $\mathcal{K}$ forms a hereditary torsionfree class.

For the converse, suppose that $\mathcal{K}$ is a hereditary torsionfree class. Then it is closed under products. Let $\{I_t : t \in A\} \subseteq H_{\mathcal{K}}(R)$. Since $R/\bigcap_t I_t \hookrightarrow \prod_{t \in A} R/I_t$ and $\prod_{t \in A} R/I_t$ is in $\mathcal{K}$, then $\bigcap_{t \in A} I_t \in H_{\mathcal{K}}(R)$. Hence $H_{\mathcal{K}}(R)$ is closed under intersections. ■

A result of Miller and Teply states that for any hereditary torsion theory τ, d. c. c. on τ-closed right ideals implies a. c. c. on τ-closed right ideals. A question related to this is that for any natural class $\mathcal{K}$, does d. c. c. on $H_{\mathcal{K}}(R)$ imply that a. c. c. on $H_{\mathcal{K}}(R)$? Even though the answer to the question is 'Yes', there is no possibility to improve the result of Miller and Teply at this point as the following proposition shows:

PROPOSITION 21. *A natural class $\mathcal{K}$ with* d. c. c. *on $H_{\mathcal{K}}(R)$ must be a hereditary torsionfree class.*

PROOF. Let $H_1 \subseteq H_{\mathcal{K}}(R)$, and H_2 the set of all finite intersections of elements in H_1. Then $H_1 \subseteq H_2 \subseteq H_{\mathcal{K}}(R)$. If d. c. c. on $H_{\mathcal{K}}(R)$, then d. c. c. on H_2. Hence there is a minimal element, $I_1 \cap \cdots \cap I_m$ say (with every $I_i \in H_{\mathcal{K}}(R)$), in H_2. Then for any $I \in H_1$, $I \cap I_1 \cap \cdots \cap I_m = I_1 \cap \cdots \cap I_m$ by the choice of $I_1 \cap \cdots \cap I_m$, implying that $I_1 \cap \cdots \cap I_m \subseteq I$ for all $I \in H_1$. Hence $\bigcap\{I : I \in H_1\} = I_1 \cap \cdots \cap I_m \in H_{\mathcal{K}}(R)$. Thus, $H_{\mathcal{K}}(R)$ is closed under arbitrary intersections. From Proposition 20, it follows that $\mathcal{K}$ is a hereditary torsionfree class. ∎

For a class $\mathcal{K}$, let $\mathcal{T}_{\mathcal{K}} = \{T : \mathrm{Hom}(T, C) = 0$, for all $C \in \mathcal{K}\}$, and $\mathcal{F}_{\mathcal{K}} = \{F : \mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}_{\mathcal{K}}\}$. The pair of classes $(\mathcal{T}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}})$ forms a torsion theory, and $\mathcal{F}_{\mathcal{K}}$ is the smallest torsionfree class containing $\mathcal{K}$. We will say that the torsion theory $(\mathcal{T}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}})$ is cogenerated by $\mathcal{K}$. It is easy to see that if $\mathcal{K}$ is a natural class, then $(\mathcal{T}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}})$ is a hereditary torsion theory.

PROPOSITION 22. *If $\mathcal{K}$ is a natural class, then the following are equivalent:*
(a) Every direct sum of injective modules in $\mathcal{K}$ is injective;
(b) If $I \subseteq R_R$, then there is an f. g. *right ideal $J \subseteq I$ such that $I/J \in \mathcal{T}_{\mathcal{K}}$.*

PROOF. (a) ⇒ (b). Suppose there is $I \subseteq R_R$ such that for any f. g. right ideal $J \subseteq I$, $I/J \notin \mathcal{T}_{\mathcal{K}}$. Choose $0 \neq a_1 \in I$. Then $I/(a_1R) \notin \mathcal{T}_{\mathcal{K}}$, and so there exists $I_1 \subseteq R_R$ such that $a_1R \subseteq I_1 \subset I$ and $I/I_1 \in \mathcal{K}$. Since $I_1 \subset I$, there is an $a_2 \in I$ but $a_2 \notin I_1$. By the assumption on I, $I/(a_1R + a_2R) \notin \mathcal{T}_{\mathcal{K}}$. Therefore, there exists I_2 such that $a_1R + a_2R \subseteq I_2 \subset I$ and $I/I_2 \in \mathcal{K}$. By a simple induction, we can choose a sequence $\{a_i \in R : i \in \mathbf{N}\}$ and a sequence $\{I_i : i \in \mathbf{N}\}$ of right ideals of R such that $a_1R + \cdots + a_nR \subseteq I_n \subset I$, $a_{n+1} \notin I_n$, and $I/I_n \in \mathcal{K}$ for all $n \in \mathbf{N}$. Thus, $E(I/I_n) \in \mathcal{K}$ for all n. Set $E = \bigoplus_{i \in \mathbf{N}} E(I/I_i)$. Then by (a), E is injective. Let $K = \sum_{i=1}^{\infty} a_iR$. We have a homomorphism $f: K \longrightarrow E$ defined by $\pi_i \circ f(a) = a + I_i$, where π_i is the projection of E onto $E(I/I_i)$. Since E is injective, there exists some $x \in E$ such that $f(a) = xa$ for all $a \in K$. Then there is some positive m such that $\pi_i \circ f = 0$ for all $i > m$. Therefore, $0 = \pi_{m+1} \circ f(a_{m+2}) = a_{m+2} + I_{m+1}$, implying that $a_{m+2} \in I_{m+1}$, a contradiction.

(b) ⇒ (a). Let $M = \oplus M_t$, where each $M_t \in \mathcal{K}$ and is injective. Let $f: I \longrightarrow M$ be a homomorphism, where I is a right ideal of R. By (b), there exists an f. g. right ideal $J \subseteq I$ such that $I/J \in \mathcal{T}_{\mathcal{K}}$. Then $f(J) \subseteq M_{t_1} \oplus \cdots \oplus M_{t_n}$ for some n. Write $f(I) + M_{t_1} + \cdots + M_{t_n} = M_{t_1} \oplus \cdots \oplus M_{t_n} \oplus X$ for some $X \subseteq M$. Now f induces an epimorphism $\bar{f}: I/J \longrightarrow \big(f(I) + M_{t_1} + \cdots + M_{t_n}\big)/(M_{t_1} + \cdots + M_{t_n}) \cong X$, where $\bar{f}(a + J) = f(a) + (M_{t_1} + \cdots + M_{t_n})$. Since $I/J \in \mathcal{T}_{\mathcal{K}}$, we have $X \in \mathcal{T}_{\mathcal{K}}$. Thus $\pi_t(X) \in \mathcal{T}_{\mathcal{K}}$ for all t. But since $\pi_t(X) \subseteq M_t \in \mathcal{K}$, we have $\pi_t(X) \in \mathcal{K} \cap T_{\mathcal{K}}$, implying $\pi_t(X) = 0$ for all t. Hence $X = 0$. Then we have that $f(I) \subseteq M_{t_1} \oplus \cdots \oplus M_{t_n}$. Since $M_{t_1} \oplus \cdots \oplus M_{t_n}$ is injective, we can apply Baer's Injective Lemma. ∎

THEOREM 23. *The following are equivalent for a natural class $\mathcal{K}$:*
(a) Every direct sum of injective modules in $\mathcal{K}$ is injective;
(b) Every injective module in $\mathcal{K}$ is a direct sum of uniform modules;
(c) Every injective module in $\mathcal{K}$ has a decomposition that complements direct summands;

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(d) Every extending module in $\mathcal{K}$ is a direct sum of uniform modules.

(e) $H_{\mathcal{K}}(R)$ has a. c. c.

PROOF. By Corollary 11 and Theorem 16. ∎

A module M is said to be *weakly-injective* if for any f. g. submodule $N \subseteq E(M)$, there exists some $X \subseteq E(M)$ such that $N \subseteq X \cong M$. The following theorem generalizes a result of Al-Huzali, S. K. Jain and Lopez-Permouth [1].

THEOREM 24. *The following are equivalent for a natural class $\mathcal{K}$:*

(a) Every cyclic module in $\mathcal{K}$ has finite Goldie dimension;

(b) Every finitely generated module in $\mathcal{K}$ has finite Goldie dimension;

(c) Every direct sum of injective modules in $\mathcal{K}$ is weakly-injective;

(d) Every direct sum of weakly-injective modules in $\mathcal{K}$ is weakly- injective.

PROOF. (a) $\Rightarrow$ (b). Because $\mathcal{K}$ is a natural class, the proof of Camillo [3, Proposition] can be applied.

(b) $\Rightarrow$ (c). Let $M = \bigoplus_{t\in A} E_t$, where every E_t is injective and is in $\mathcal{K}$. Let N be a finitely generated submodule of $E(M)$. Then $N \in \mathcal{K}$. By (b), there exist uniform submodules $U_1, \ldots, U_n$ of N such that $U_1 \oplus \cdots \oplus U_n$ is essential in N. Since M is essential in $E(M)$, we can choose $0 \neq x_i \in U_i \cap M$ for every i. Then $\bigoplus_{i=1}^n x_iR \subseteq E_{t_1} \oplus \cdots \oplus E_{t_m}$ for some m. Hence $E = E(\bigoplus_{i=1}^n x_iR) \subseteq M$, and we have that $M = E \oplus K$ for some K. By noting that $\bigoplus_{i=1}^n x_iR \leq_e U_1 \oplus \cdots \oplus U_n \leq_e N$, we have that $E(N) = \bigoplus_{i=1}^n E(U_i) = \bigoplus_{i=1}^n E(x_iR) \cong E$. Since $\bigoplus_{i=1}^n x_iR \leq_e E(N)$, it follows that $E(N) \cap K = 0$. Thus $E(N) \oplus K \subseteq E(M)$, and $N \subseteq E(N) + K = E(N) \oplus K \cong E \oplus K = M$.

(c) $\Rightarrow$ (d). Suppose that $M = \bigoplus_{i\in A} M_i$ such that every $M_i \in \mathcal{K}$ is weakly-injective. Let N be a finitely generated submodule of $E(M)$. Then every $E(M_i) \in \mathcal{K}$. Hence $\bigoplus_{i\in A} E(M_i)$ is weakly-injective by (c). Since $M \subseteq \bigoplus_{i\in A} E(M_i)$, there exists a submodule $Y \subseteq E(M)$ such that $N \subseteq Y \cong \bigoplus_{i\in A} E(M_i)$. Write $Y = \bigoplus_{i\in A} E(Y_i)$ such that $M_i \cong Y_i$ for all $i \in A$. Then we have $N \subseteq \bigoplus_{i\in F} E(Y_i)$ for a finite subset F of A. Since $\bigoplus_{i\in F} E(Y_i)$ is weakly-injective, there exists $X_1 \subseteq E(\bigoplus_{i\in F} E(Y_i)$ such that $N \subseteq X_1 \cong \bigoplus_{i\in F} Y_i \cong \bigoplus_{i\in F} M_i$. Then we have $N \subseteq X_1 \oplus (\bigoplus_{i\notin F} Y_i) = X \cong M$ with $X \subseteq E(M)$.

(d) $\Rightarrow$ (a). Let $xR \in \mathcal{K}$. Suppose xR is not finite dimensional. Then xR contains an essential submodule which is a direct sum of infinitely many nonzero submodules $\bigoplus_{t\in A} N_t$. Then $E(xR) = E(\bigoplus_{t\in A} N_t) = E\big(\bigoplus_{t\in A} E(N_t)\big)$. Clearly, all $E(N_t) \in \mathcal{K}$. Then (d) implies that $\bigoplus_{t\in A} E(N_t)$ is weakly-injective. Therefore we have $xR \subseteq Y \cong \bigoplus_{t\in A} E(N_t)$ for some $Y \subseteq E(xR)$. Write $Y = \bigoplus_{t\in A} E(Y_t)$ with each $Y_t \cong N_t$. Then $xR \subseteq \bigoplus_{t\in F} E(Y_t)$ for a finite subset F of A. Note that $xR \subseteq Y \subseteq E(xR)$. Hence xR is essential in Y, showing that $Y_t = 0$ for all $t \notin F$. But, then $N_t = 0$ for all $t \notin F$, a contradiction. ∎

Next, we apply the previous results to: i) A hereditary stable torsion class; ii) A hereditary torsionfree class.

Consider a hereditary stable torsion theory τ. Then $\mathcal{K} =$ the τ-torsion class is a natural class, and $H_{\mathcal{K}}(R)$ is the set of all right τ-dense ideals. By Theorem 23, we have

COROLLARY 25. *Let τ be a hereditary stable torsion theory. The following are equivalent for a ring R:*

(a) R has a. c. c. *on right τ-dense ideals;*
(b) Every direct sum of τ-torsion injective modules is injective;
(c) Every τ-torsion injective module is a direct sum of uniform modules;
(d) Every τ-torsion injective module has a decomposition that complements direct summands;
(e) Every τ-torsion extending module is a direct sum of uniform modules. ∎

REMARK. The implication of (a) $\Rightarrow$ (c) is a result of [4, Proposition (41.12), p. 390].

Let τ is a hereditary torsion theory and $\mathcal{K}$ is the τ-torsionfree class. Then $\mathcal{K}$ is a natural class and $H_{\mathcal{K}}(R)$ is the set of all right τ-closed ideals.

COROLLARY 26. *The following are equivalent for a ring R and a hereditary torsion theory τ:*

(a) R has a. c. c. *on right τ-closed ideals;*
(b) Every direct sum of τ-torsionfree injective modules is injective;
(c) For any right ideal I of R, there is an f. g. *right ideal $J \subseteq I$ such that I/J is a τ-torsion module;*
(d) Every τ-torsionfree injective module is a direct sum of uniform modules;
(e) Every τ-torsionfree injective module has a decomposition that complements direct summands;
(f) Every τ-torsionfree extending module is a direct sum of uniform modules.

REMARK. The equivalences of (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) are contained in [4, Proposition (20.17), p. 182].

PROOF. By Proposition 22 and Theorem 23. ∎

COROLLARY 27. *A ring R is a right Noetherian ring iff for some hereditary stable torsion theory τ, R has* a. c. c. *on τ-closed right ideals and* a. c. c. *on τ-dense right ideals.*

PROOF. One direction is obvious. For any injective right module E, $E\big(\tau(E)\big) = \tau(E)$, and hence $E = \tau(E) \oplus X$. Then X is a τ-torsionfree injective module. If R has a. c. c. on τ-closed ideals, and a. c. c. on τ-dense ideals, then both $\tau(E)$ and X are direct sums of uniform modules by Corollary 26 and 27. Hence E is a direct sum of uniform modules. It follows that R is right Noetherian. ∎

It is natural to ask if the above corollary may be generalized to any hereditary torsion theory. In the following, we provide an example of ring R and a non-stable hereditary torsion theory τ such that R has a. c. c. on τ-closed right ideals and a. c. c. on τ-dense right ideals, but R is not a right Noetherian ring.

EXAMPLE 28. Let A be a ring, $\{M_\alpha : \alpha \in I\}$ a set of $A - A$ bimodules. Let $R = A \oplus (\bigoplus_{\alpha \in I} M_\alpha)$. R will become a ring under the following '+', and '$\circ$':

$$(a;\ldots,x_\alpha,\ldots) + (b;\ldots,y_\alpha,\ldots) = (a+b;\ldots,x_\alpha + y_\alpha,\ldots)$$
$$(a;\ldots,x_\alpha,\ldots) \circ (b;\ldots,y_\alpha,\ldots) = (ab;\ldots,ay_\alpha + x_\alpha b,\ldots).$$

We take $A = \mathbf{Z}$, and $M_i = \mathbf{Z}/(p_i)$, where p_i is the i-th prime number, and $i = 1, 2, \ldots$. It is easy to check that $\mathrm{Soc}(R_R) = \oplus_{i=1}^{\infty} M_i$ is an essential right ideal of R, and R is not a right Noetherian ring. Let $\mathcal{K} = \{M_i : i = 1, 2, \ldots\}$ and $\tau = (\mathcal{T}, \mathcal{F})$ be the torsion theory generated by $\mathcal{K}$, *i.e.*,

$$\mathcal{F} = \{F \in \text{Mod-}R : \mathrm{Hom}(C, F) = 0 \text{ for all } C \in \mathcal{K}\}$$
$$\mathcal{T} = \{T \in \text{Mod-}R : \mathrm{Hom}(T, F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

It is easy to see that τ is hereditary. If $I \in H_{\mathcal{F}}(R)$, then $\mathrm{Hom}(C, R/I) = 0$ for all $C \in \mathcal{K}$. Hence $\mathrm{Soc}(R_R) \subseteq I$. Thus $I = (n) \oplus \mathrm{Soc}(R_R)$ for some non-negative integer n. Write $n = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}$. Then we have a R- module decomposition $R/I \cong \mathbf{Z}/(p_1^{t_1}) \oplus \mathbf{Z}/(p_2^{t_2}) \oplus \cdots \oplus \mathbf{Z}/(p_m^{t_m})$. If some $t_i > 0$, then $\mathbf{Z}/(p_i)$ can embed in $\mathbf{Z}/(p_i^{t_i})$, hence in R/I, as R-modules. This contradicts the fact that $R/I \in \mathcal{F}$. So all $t_i = 0$. We have $n = 0$ or 1. Hence $H_{\mathcal{F}}(R) = \{\mathrm{Soc}(R_R), R\}$. So R has a. c. c. on τ-closed right ideals.

Note that every right ideal I can be expressed as $I = (n) \oplus X$ for some $X \subseteq \mathrm{Soc}(R_R)$. Let $(n_1) \oplus X_1 \subseteq (n_2) \oplus X_2 \subseteq \cdots \subseteq (n_k) \oplus X_k \subseteq \cdots$ be a chain of elements in $H_{\mathcal{T}}(R)$. Then $(n_1) \subseteq (n_2) \subseteq \cdots$. Hence $(n_i) = (n_s)$ for some s, for all $i \geq s$ since Z is a Noetherian ring. Suppose that $n_s = 0$. Then $(n_s) \oplus X_s = X_s \in H_{\mathcal{T}}(R)$, and hence $R/X_s \in \mathcal{T}$. It follows that $R/\,\mathrm{Soc}(R_R) \in \mathcal{T}$. We have shown that $R/\,\mathrm{Soc}(R_R) \in \mathcal{F}$. But, then $R = \mathrm{Soc}(R_R)$, a contradiction. Hence $n_s \neq 0$. Let m be a positive integer such that $(n_s, p_j) = 1$ for all $j \geq m$. Then $X_i \supseteq M_m \oplus M_{m+1} \oplus \cdots$ for all $i \geq s$. Therefore, the chain $X_s \subseteq X_{s+1} \subseteq \cdots$ must terminate. Hence we have that the chain $(n_1) \oplus X_1 \subseteq (n_2) \oplus X_2 \subseteq \cdots$ terminates. Thus R has a. c. c. on $H_{\mathcal{T}}(R)$. ∎

PROPOSITION 29. *A ring R is a right Noetherian ring iff for some hereditary torsion theory τ, R has* a. c. c. *on τ-closed right ideals and every ascending chain of right ideals of R: $I_1 \subseteq I_2 \subseteq \cdots$ such that each I_{i+1}/I_i is τ-torsion, terminates.*

PROOF. If $I_1 \subseteq I_2 \subseteq \cdots$ is a chain of right ideals of R. We have a natural homomorphism $f_i \colon R/I_i \longrightarrow R/I_{i+1}$, where $f_i(a + I_i) = a + I_{i+1}$. Write $\tau(R/I_i) = K_i/I_i$. Then $f_i\big(\tau(R/I_i)\big) \subseteq \tau(R/I_{i+1})$, implying that $K_i \subseteq K_{i+1}$. Since every K_i is τ-closed and R has a. c. c. on τ-closed right ideals, there exists an m such that $K_j = K_m$ for all $j \geq m$. If $j > m$, then $I_{j+1}/I_j \subseteq K_{j+1}/I_j = K_j/I_j = \tau(R/I_i)$ is τ-torsion. By our assumption, the chain $I_m \subseteq I_{m+1} \subseteq \cdots$ terminates. Therefore the chain $I_1 \subseteq I_2 \subseteq \cdots$ terminates. So R is a right Noetherian ring. ∎

REMARK. Applying our results to the class of all modules with singular injective hulls, we can reestablish the various characterizations of rings for which direct sums of singular injective modules are injective, which appear in [7].

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Mathematics Department
The University of British Columbia
Vancouver, British Columbia
V6T 1Z2