MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES—II

by K. CHANDRASEKHARA RAO

(Received 11 December, 1968; revised 22 September, 1969)

This paper is a continuation of [1]. We begin with the notations for the sequence spaces considered in this paper. Let Γ be the space of sequences $x = \{x_p\}$ of complex numbers such that $|x_p|^{1/p} \to 0$ as $p \to \infty$. Γ can also be regarded as the space of integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$. The sequence space Γ is a vector space over the complex numbers with seminorms

$$q_i = \sup_{|z|=i} \left\{ \left| \sum_{p=1}^{\infty} x_p z^p \right| \right\} \qquad (i = 1, 2, \ldots).$$

Γ is a complete space. If $f(z) = \sum_{p=1}^{\infty} x_p z^p$, as an integral function, belongs to Γ, then Cauchy's inequalities imply that $x_p = x_p(x) = x_p(f)$ is a continuous linear functional on the space Γ, for each fixed p. Thus Γ is an FK space.

Let Γ^* be the space of sequences $s = \{s_p\}$, such that the sequence $\{|s_p|^{1/p}\}$ is bounded. Γ^* may also be considered as the space conjugate to Γ regarded as the space of integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$. Each continuous linear functional $U \in \Gamma^*$ is of the form

$$U(f) = \sum_{p=1}^{\infty} s_p x_p.$$

Let l be the space of sequences $x = \{x_p\}$ such that $\sum_{p=1}^{\infty} |x_p| < \infty$. l is an FK space with the seminorm

$$q(x) = \sum_{p=1}^{\infty} |x_p|.$$

Here the continuity of $x_p = x_p(x)$ follows from the fact that

$$|x_p(x)| \leqq \sum_{p=1}^{\infty} |x_p(x)| < \infty, \quad \text{for each fixed} \quad p.$$

Let $A = (a_{np})$, $(n, p = 1, 2, \ldots)$ be an infinite matrix of complex elements. Then the A transform of $x = \{x_p\}$, $y = \{y_n\}$ is the sequence defined by the equations

$$y_n = \sum_{p=1}^{\infty} a_{np} x_p \qquad (n = 1, 2, \ldots). \tag{1}$$

Here $y = \{y_n\}$ and $x = \{x_p\}$ are both complex sequences.

https://doi.org/10.1017/S0017089500001014 Published online by Cambridge University Press

In this paper we give necessary and sufficient conditions on the matrix A in order that A should transform l into Γ (Theorem 1), and l into Γ^* (Theorem 2).

THEOREM 1. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* Γ *whenever* $\{x_p\}$ *belongs to* l, *it is necessary and sufficient that*

$$|a_{np}|^{1/n} \to 0, \quad as \quad n \to \infty, \quad uniformly\ in \quad p. \tag{2}$$

Proof (Sufficiency). Since $\{x_p\} \in l$, there is a finite $K(\geqq 1)$ such that

$$\sum_{p=1}^{\infty} |x_p| \leqq K. \tag{3}$$

By (2), given $\varepsilon > 0$, we can find $N = N(\varepsilon)$ independent of p such that

$$|a_{np}|^{1/n} < \varepsilon/(2K) \quad \text{for} \quad n > N \quad \text{and all} \quad p. \tag{4}$$

Now we have, by (3) and (4), since $K \geqq 1$,

$$\begin{aligned} |y_n|^{1/n} = \left| \sum_{p=1}^{\infty} a_{np} x_p \right|^{1/n} &\leqq \left(\sum_{p=1}^{\infty} |a_{np}|\ |x_n| \right)^{1/n} \\ &\leqq (\varepsilon/2K)K^{1/n} \\ &\leqq (\varepsilon/2K)K \\ &= \varepsilon/2 < \varepsilon \end{aligned}$$

for $n > N$.

(Necessity). Suppose that (2) is not satisfied. Then for some $\varepsilon > 0$ there exists no N such that $|a_{np}|^{1/n} < \varepsilon$ for $n > N$ and $p = 1, 2, \ldots$. That is, for this ε and any N there is an $n > N$ and a p such that

$$|a_{np}|^{1/n} \geqq \varepsilon \tag{5}$$

If A transforms l into Γ, then A transforms l into l. So, by the Knopp–Lorentz theorem [2], $\sup_p \sum_{n=1}^{\infty} |a_{np}| < \infty$. Hence we have, by writing $w_n = \sup_p |a_{np}|$,

$$|w_n| \leqq Q/2 \quad \text{for all } n \text{ and } Q > 0, \tag{6}$$

and (6) implies that

$$\{a_{np}\} \quad \text{is bounded for each fixed} \quad n. \tag{7}$$

Also, we have

$$|a_{np}|^{1/n} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each fixed} \quad p. \tag{8}$$

We shall construct a sequence $\{x_p\}$ with the supplementary condition

$$|x_p| \leqq 1 \quad \text{for all values of} \quad p \tag{9}$$

and show that the corresponding $\{y_n\}$ does not belong to Γ, using (5) to (8).

https://doi.org/10.1017/S0017089500001014 Published online by Cambridge University Press

First choose n_1 and p_1, by (5), such that

$$|a_{n_1p_1}|^{1/n_1} > \varepsilon/2; \tag{10}$$

choose $n_2 > n_1$ sufficiently large and $p_2 > p_1$ such that

$$|Q/2^{n_2}| < (\varepsilon/8)^{n_1}, \tag{11}$$

and, by (5) and (8), that

$$|a_{n_2p_2}|^{1/n_2} > \varepsilon/2, \tag{12}$$

$$|a_{n_2p_1}|^{1/n_2} < \varepsilon/16. \tag{13}$$

Next choose $n_3 > n_2$ sufficiently large and $p_3 > p_2$ such that

$$|Q/2^{n_3}| < (\varepsilon/16)^{n_2}, \tag{14}$$

and, by (5) and (8), that

$$|a_{n_3p_3}|^{1/n_3} > \varepsilon/2, \tag{15}$$

$$|a_{n_3p_2}|^{1/n_3} < \varepsilon/24, \qquad |a_{n_3p_1}|^{1/n_3} < \varepsilon/24. \tag{16}$$

Then choose $n_4 > n_3$ sufficiently large and $p_4 > p_3$ such that

$$|Q/2^{n_4}| < (\varepsilon/24)^{n_3}, \tag{17}$$

and, by (5) and (8), that

$$|a_{n_4p_4}|^{1/n_4} > \varepsilon/2, \tag{18}$$

$$\left.\begin{aligned} &|a_{n_4p_3}|^{1/n_4} < \varepsilon/32, \qquad |a_{n_4p_2}|^{1/n_4} < \varepsilon/32, \\ &|a_{n_4p_1}|^{1/n_4} < \varepsilon/32, \end{aligned}\right\} \tag{19}$$

and so on. We set

$$\left.\begin{aligned} &x_{p_1} = 1/2^{n_1}, \qquad x_{p_2} = 1/2^{n_2}, \qquad x_{p_3} = 1/2^{n_3}, \ldots \\ &x_p = 0 \quad \text{for} \quad p \neq p_1, p_2, p_3, \ldots \end{aligned}\right\} \tag{20}$$

and have, by (10),

$$\begin{aligned} |y_{n_1}|^{1/n_1} &\geqq (\tfrac{1}{2})|a_{n_1p_1}|^{1/n_1} - \left|\sum_{j=2}^{\infty} a_{n_1p_j}x_{p_j}\right|^{1/n_1} \\ &> (\tfrac{1}{4})\varepsilon - \left|\sum_{j=2}^{\infty} a_{n_1p_j}x_{p_j}\right|^{1/n_1} \\ &> (\tfrac{1}{4})\varepsilon - (\tfrac{1}{8})\varepsilon = (\tfrac{1}{8})\varepsilon, \end{aligned}$$

https://doi.org/10.1017/S0017089500001014 Published online by Cambridge University Press

since

$$\left|\sum_{j=2}^{\infty} a_{n_1 p_j} x_{p_j}\right|^{1/n_1} \leqq |2w_{n_1}/2^{n_2}|^{1/n_1}$$

$$\leqq |Q/2^{n_2}|^{1/n_1} < (\tfrac{1}{8})\varepsilon,$$

by using (6) and (11). We also have, by (12),

$$|y_{n_2}|^{1/n_2} \geqq (\tfrac{1}{2})|a_{n_2 p_2}|^{1/n_2} - |a_{n_2 p_1} x_{p_1}|^{1/n_2} - \left|\sum_{j=3}^{\infty} a_{n_2 p_j} x_{p_j}\right|^{1/n_2}$$

$$> (\tfrac{1}{4})\varepsilon - |a_{n_2 p_1} x_{p_1}|^{1/n_2} - \left|\sum_{j=3}^{\infty} a_{n_2 p_j} x_{p_j}\right|^{1/n_2}$$

$$> (\tfrac{1}{4})\varepsilon - (\tfrac{1}{16})\varepsilon - (\tfrac{1}{16})\varepsilon = (\tfrac{1}{8})\varepsilon,$$

since, by (9) and (13),

$$|a_{n_2 p_1} x_{p_1}|^{1/n_2} \leqq |a_{n_2 p_1}|^{1/n_2} \leqq (\tfrac{1}{16})\varepsilon$$

and, by (6) and (14),

$$\left|\sum_{j=3}^{\infty} a_{n_2 p_j} x_{p_j}\right|^{1/n_2} \leqq |2w_{n_2}/2^{n_3}|^{1/n_2}$$

$$\leqq |Q/2^{n_3}|^{1/n_3} < (\tfrac{1}{16})\varepsilon.$$

Also, we have

$$|y_{n_3}|^{1/n_3} \geqq (\tfrac{1}{2})|a_{n_3 p_3}|^{1/n_3} - |a_{n_3 p_2} x_{p_2}|^{1/n_3} - |a_{n_3 p_1} x_{p_1}|^{1/n_3} - \left|\sum_{j=4}^{\infty} a_{n_3 p_j} x_{p_j}\right|^{1/n_3}$$

$$> (\tfrac{1}{4})\varepsilon - |a_{n_3 p_2} x_{p_2}|^{1/n_3} - |a_{n_3 p_1} x_{p_1}|^{1/n_3} - \left|\sum_{j=4}^{\infty} a_{n_3 p_j} x_{p_j}\right|^{1/n_3}$$

$$> (\tfrac{1}{4})\varepsilon - (\tfrac{1}{24})\varepsilon - (\tfrac{1}{24})\varepsilon - (\tfrac{1}{24})\varepsilon = (\tfrac{1}{8})\varepsilon,$$

since

$$|a_{n_3 p_2} x_{p_2}|^{1/n_3} \leqq |a_{n_3 p_2}|^{1/n_3} < (\tfrac{1}{24})\varepsilon$$

by (9) and (16); similarly

$$|a_{n_3 p_1} x_{p_1}|^{1/n_3} < (\tfrac{1}{24})\varepsilon$$

and

$$\left|\sum_{j=4}^{\infty} a_{n_3 p_j} x_{p_j}\right|^{1/n_3} \leqq |2w_{n_3}/2^{n_4}|^{1/n_3} \leqq |Q/2^{n_4}|^{1/n_3} < (\tfrac{1}{24})\varepsilon,$$

by (6) and (17).

https://doi.org/10.1017/S0017089500001014 Published online by Cambridge University Press

Proceeding in this way we construct a sequence $\{x_p\}$ satisfying (20) and (9) that belongs to l and for which the corresponding A transform $\{y_n\}$ does not belong to Γ. This contradiction establishes the necessity of (2). This completes the proof.

THEOREM 2. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* Γ^* *whenever* $\{x_p\}$ *belongs to* l, *it is necessary and sufficient that*

$$|a_{np}|^{1/n} \leqq M \quad \textit{independently of} \quad n, p. \tag{21}$$

The proof is similar to that of Theorem 1.

My thanks are due to Professor V. Ganapathy Iyer for his helpful guidance during the preparation of this paper.

REFERENCES

1. K. Chandrasekhara Rao, Matrix transformations of some sequence spaces, *Pacific J. Math*, **31** (1969), 171–174.

2. K. Knopp and G. G. Lorentz, Beiträge zur absoluten Limitierung, *Arch. Math.* **2** (1949), 10–16.

LOYOLA COLLEGE, MADRAS
and
KERALA UNIVERSITY, TRIVANDRUM