ASYMMETRY IN THE LATTICE OF KERNEL FUNCTORS

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Much of the research done by different authors on the lattice of kernel functors (equivalently, linear topologies) has been summarized by Golan in [**2**]. More recently, the rings whose lattices of kernel functors are linearly ordered were introduced in [**3**] as a categorical generalization of valuation rings in the non-commutative case. Results (and examples) in [**3**] show that there is an abundance of non-commutative rings R whose lattices $\mathbb{K}(R)$, both in Mod-R and R-Mod, are simultaneously linearly ordered; however, the question of the symmetry of this condition remained open. Here we will prove that, for every natural number $n \geq 3$, there exists a ring R_n such that $\mathbb{K}(\text{Mod-}R_n)$ is a linearly ordered lattice of n elements, whereas $\mathbb{K}(R_n\text{-Mod})$ is not linearly ordered.

Throughout, familiarity with [**3**] is assumed but, instead of $\mathbb{K}(R)$ we are denoting by $\mathbb{K}(\text{Mod-}R)$ (respectively $\mathbb{K}(R\text{-Mod})$) the lattice of kernel functors on Mod-R (respectively R-Mod), to specify sides.

We begin by considering the case $n = 3$.

Let K be a field, $f: K \to K$ a field monomorphism which is not onto and let L denote $f(K)$. Consider the ring S of twisted power series, that is, $S = \{\sum x^j a_j; a_j \in K\} = K[[x;f]]$ with the usual addition and $ax = xf(a)$, for every a in K.

The only right ideals of S are $S \supset xS \supset x^2S \supset \ldots$, and thus $\mathbb{K}(\text{Mod-}S)$ is linearly ordered by [**3**, Lemma 7]. Let R be the ring S/x^2S. Therefore $R = \{a + xb; a, b \in K\}$, where $kx = xf(k)$ and $x^2 = 0$. Clearly R has only three right ideals, R, xR, and (0), and $\mathbb{K}(\text{Mod-}R)$ is linearly ordered.

It is obvious that for every L-subspace V of K the set xV is a left ideal of R. Moreover, these are the only proper left ideals of R. In fact, given ${}_RI \subsetneq R$ and $a + xb \in I$, it follows that $a = 0$ (otherwise an inverse can be found since f is a monomorphism) and therefore $I \subseteq xK$. Let V denote $\{u \in K; xu \in I\}$. This set is an L-subspace of K since given $u \in K$ and $t \in L$, $t = f(k)$ for a certain $k \in K$; so $x(tu) = xf(k)u = k(xu) \in kI \subseteq I$.

Once we have obtained the left ideals of R we are in a position to prove the following lemma.

LEMMA. *For the ring R as above, the following conditions are equivalent*:

(a) $\mathbb{K}(R\text{-Mod})$ *is linearly ordered*;

(b) *for any pair V, W of L-subspaces of K, there exists a finite set $\{x_1, \ldots, x_s\}$ of non-zero elements of K such that either $Vx_1 \cap \ldots \cap Vx_s \subseteq W$ or $Wx_1 \cap \ldots \cap Wx_s \subseteq V$.*

Proof. (a)$\Rightarrow$(b). We may assume V and W are not comparable, since otherwise $x_1 = 1$ will do. Set $I = xV$ and $J = xW$. By [**3**, Proposition 1], there exists in R a finite set $\{r_i\}_1^p = \{a_i + xb_i\}_1^p$ such that $I \supseteq (J:\{r_i\}_1^p)$, for instance. If $a_i = 0$ for every i then $xKr_i = xKxb_i \in x^2R = (0)$ and $xK \subseteq (J:\{r_i\}_1^p) \subseteq I = xV$, which implies $V = K \supseteq W$, a contradiction. So one may assume that $a_1, \ldots, a_s$ are non-zero ($s \leq p$) and $a_{s+1} = \ldots = a_p = 0$, and proceed to prove that $Wa_1^{-1} \cap \ldots \cap Wa_s^{-1} \subseteq V$. In fact, $k \in \bigcap_{j=1}^{s} Wa_j^{-1}$ implies

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$k = w_j a_j^{-1}$ for some $w_j \in W$, *and therefore*

$$xkr_i = \begin{cases} xw_i a_i^{-1} a_i \in xW = J & \text{if } i \leq s, \\ xkxb_i = 0 \in J & \text{if } s < i. \end{cases}$$

Hence $xk \in (J : \{r_i\}_1^p) \subseteq I = xV$ and so $k \in V$.

(b)$\Rightarrow$(a). We are given two left R-ideals I and J, which may be taken of the form xV, xW, respectively, for certain subspaces V and W. By hypothesis, there exist non-zero elements $k_1, \ldots, k_s$ in K such that $Wk_1 \cap \ldots \cap Wk_s \subseteq V$, for instance. Set $C = \{k_1^{-1}, \ldots, k_s^{-1}\}$ and let us check that $I \supseteq (J :_R C)$. In fact, if $a + xb \in (J : C)$ then $ak_i^{-1} + xbk_i^{-1} = (a + xb)k_i^{-1} \in xW$ for every i, and so $a = 0$ and $bk_i^{-1} \in W$ for every i. Since $b \in \bigcap Wk_i \subseteq V$, it follows that $a + xb = xb \in xV = I$. Now [**3**, Proposition 1] ensures that $\mathbb{K}(R\text{-Mod})$ is linearly ordered.

Observe that we will be done if a particular choice of K and f enables us to violate condition (b).

In the rational function field in infinitely many indeterminates $K = \mathbb{Q}(x_1, \ldots, x_n, \ldots)$, take the ring monomorphism $f : K \to K$ given by $f(x_j) = x_{j+1}$. Therefore $f(K) = \mathbb{Q}(x_2, x_3, \ldots) = L$ and $K = L(x_1) = L(t)$ if we set $x_1 = t$. Consider the L-subspaces $V = L[t]$ and $W = t^{-1}L[t^{-1}]$, and notice that, given arbitrary non-zero elements $f_1/g_1, \ldots, f_s/g_s$ of K, the polynomial $f = \prod_1^s f_i \notin W$. However $f = \left(\prod_{i \neq j} f_i\right) g_j f_j / g_j \in Vf_j/g_j$ for every j, which implies that $\bigcap_1^s Vf_j/g_j \nsubseteq W$. On the other hand, pick an integer $m > \sum_1^s \deg(f_i) + \deg(g_i)$ to obtain the element $f/t^m \notin V$. Moreover

$$f/t^m = \frac{\left(\prod_{i \neq j} f_i\right) g_j}{t^m} f_j/g_j \in Wf_j/g_j$$

for every j, and therefore $\bigcap_1^s Wf_j/g_j \nsubseteq V$.

So far we have constructed a ring R such that $\mathbb{K}(\text{Mod-}R)$ is a linearly ordered lattice of three elements, whereas $\mathbb{K}(R\text{-Mod})$ is not linearly ordered.

Let us denote by R_3 the ring just constructed and proceed to tackle the case of arbitrary n. Consider S as before and define $R_n = S/x^{n-1}S$; this is a right chain ring and $\mathbb{K}(\text{Mod-}R_n)$ is a linearly ordered lattice of n elements, by [**3**, Corollary 7]. However, if we choose K and f as above, $\mathbb{K}(R_n\text{-Mod})$ linearly ordered would force $\mathbb{K}(R_3\text{-Mod})$ to be linearly ordered, as R_3 is an epimorphic image of R_n, and [**3**, Proposition 2] would apply.

COMMENTS. Generalizations to the non-artinian case would be obtained by considering non-finite ordinal numbers τ and a ring of type τ as in [**1**, p. 312]. The argument still applies since R_3 is an epimorphic image of such a ring. Details are omitted. Finally, the case $n = 2$ has to be ruled out since $\mathbb{K}(\text{Mod-}R) = \{\mathbb{0}, \mathbb{00}\}$ if and only if R is a simple artinian ring, if and only if $\mathbb{K}(R\text{-Mod}) = \{\mathbb{0}, \mathbb{00}\}$.

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