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PRIMARY DECOMPOSITION IN ENVELOPING ALGEBRAS

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Recently, the first author and, independently, A. V. Jategaonkar have shown that every factor ring of $U(g)$, the universal enveloping algebra of a finite dimensional complex Lie algebra, has a primary decomposition if g is solvable and almost algebraic. On the other hand, a suitable factor ring of $U(SL(2, \mathbb{C}))$ fails to have a primary decomposition (**1**).

In this note, we close the gap between these results by showing that every factor ring of $U(g)$ has a primary decomposition if and only if g is solvable.

The reader is referred to (**2**) for terminology and facts concerning enveloping algebras. Throughout the note we assume that Lie algebras are algebras over the field of complex numbers, $\mathbb{C}$.

In (**4**), Gordon has presented a version of primary decomposition in which he uses the following definition for a primary ideal: An ideal P of the ring R is an *associated prime* of the module M_R if there is a uniform submodule U of M such that P is the *assassinator*, $\text{ass}(U)$, of U, where

$$\text{ass}(U) \equiv \{x \in R \mid \text{ann}_U(Rx) \neq 0\}.$$

A module is *P-primary* if it has a unique associated prime ideal P. If R is right Noetherian, then P is a prime ideal and there is a non-zero submodule V of U with $\text{ann}_R(V) = \text{ann}_R(V') = P$, for any non-zero submodule V' of V. A ring R is *primary* if R_R is a primary module and an ideal I of R is a *primary ideal* if R/I is a primary ring. A ring R has a *primary decomposition* if there are primary ideals $I_1, \ldots, I_n$ of R with $\bigcap_i I_i = 0$.

If I is a right ideal of a ring R then the largest two-sided ideal contained in I is denoted by $\text{bd}(I)$. In any right Noetherian ring R there are right ideals $I_1, \ldots, I_n$ such that each R/I_j is uniform and $0 = \bigcap_j I_j$. Now, obviously, $0 = \bigcap_j \text{bd}(I_j)$; so in order to show that R has a primary decomposition it is sufficient to show that each $R/\text{bd}(I_j)$ is primary.

Theorem. *If g is a finite dimensional solvable Lie algebra and R is a factor ring of $U(g)$ then R has a primary decomposition.*

Proof. The discussion above allows us to assume that there is a right ideal I of R such that R/I is uniform and $\text{bd}(I) = 0$. Let R/I be P-primary, and suppose that K is a right ideal of R such that $P = \text{ann}(K'/I)$ for each right ideal $I \subsetneq K' \leq K \leq R$. Let U be a uniform right ideal of R such that $Q=r(U)$ is a prime ideal. Since $\mathrm{bd}(I)=0$, $RU+I\supsetneqq I$; so $(RU+I)\cap K\supsetneqq I$ and it follows that $Q\subseteq P$. Note that $Q=\mathrm{ann}(RU+I/I)$. Put $M=RU+I/I$ and $N=(RU+I)\cap K/I$. If $Q\subsetneqq P$ then there is an element $y\in P/Q$ such that $yR+Q=Ry+Q$ (**6**, Theorem 3). Hence, $[yR+Q]$ has the AR property in R/Q (**6**, Lemma 8). Now M is a uniform R/Q-module and $Ny=0$; so, as in (**6**, Lemma 8) there is an integer n such that $My^n=0$. Hence $y^n\in Q$ and so $y\in Q$, a contradiction. Thus $Q=P$ and R is P-primary.

A ring R is a *poly-AR* ring if for every pair of prime ideals $Q\subsetneqq P$ of R there is an ideal $Q\subsetneqq A\leqq P$ such that A/Q has the AR property in R/Q. The alert reader will have noticed that the above theorem shows that right Noetherian, poly-AR rings have primary decomposition.

In order to show that factor rings of enveloping algebras of non-solvable Lie algebras do not necessarily have primary decomposition, we need a couple of preliminary results.

Proposition. *Suppose that the Noetherian ring R is primary and contains a non-zero Artinian ideal. Then R is Artinian.*

Proof. Let U be a uniform right ideal that is Artinian and has a prime annihilator P. Then $P=r(RU)$ and so R/P is Artinian (**5**, Lemma 9). Obviously then, U is P-primary. Since R is primary, it must be P-primary; so $l(P)$ is essential as a right ideal. However, $l(P)$ is an Artinian ideal. Thus, by (**3**), R is Artinian.

Lemma. *Let R be a domain and $B\subsetneqq R$ an ideal of R. If $a_1,\dots,a_n$ are non-zero central elements of R then*

$$a_1R+\dots+a_nR\neq a_1B+\dots+a_nB.$$

Proof. The proof is by induction on n. If $a_1R=a_1B$ then $a_1=a_1b$, for some $b\in B$, and so $a_1(1-b)=0$, a contradiction.

Suppose that $a_1R+\dots+a_nR=a_1B+\dots+a_nB$, and set $X=\sum_{j=2}^{n}a_jB$. Write $a_1=a_1b_1+x_1$, $b_1\in B$, $x_1\in X$ and, for any fixed $i\geqq 2$, $a_i=a_1b_i+x_i$, $b_i\in B$, $x_i\in X$. Premultiply the first equation by b_i, postmultiply the second by b_1 and take the difference, to get $a_1b_i-a_ib_1=b_ix_1-x_ib_1\in X$; so that $a_1b_i\in X$ and $a_i=a_1b_i+x_i\in X$. But then $a_2R+\dots+a_nR\subseteq X=a_2B+\dots+a_nB$, contradicting the inductive hypothesis.

Corollary. *If Z is a central subring of a right Noetherian domain R, and I is an ideal of R such that $I\cap Z\neq 0$, then $(I\cap Z)R\neq(I\cap Z)I$.*

Theorem. *If the finite dimensional Lie algebra g is not solvable then some factor ring of $R=U(g)$ does not have a primary decomposition.*

Proof. Since g is not solvable after factoring out the largest solvable ideal of g we may assume that g is semi-simple (**2**, Proposition 1.4.3). Let $g=\sum_{i=1}^{n}x_i\mathbb{C}$, and set

$I=\sum_{i=1}^{n} x_iR$; so that I is an ideal of R and $R/I\cong\mathbb{C}$. Let Z denote the centre of R and note that $I\cap Z\neq 0$ by (**2**, 4.2.2). By (**2**, Théorème 8.4.3, 8.4.4(i) and 8.5.8), $(I\cap Z)R$ is a prime ideal and the set of prime ideals containing $(I\cap Z)R$ has a unique maximal element, which is obviously I. Thus all prime ideals, and hence all ideals, which contain $(I\cap Z)R$ or $(I\cap Z)I$ are contained in I. Obviously then the only Artinian prime factor ring of $R/(I\cap Z)I$ is R/I. If A is any ideal containing $(I\cap Z)I$ and such that R/A is Artinian then I/A must be the nilpotent radical of R/A. However, $I=I^2$ (**2**, 2.8.8), so the only Artinian factor ring of $R/(I\cap Z)I$ is R/I. Now, by the previous Corollary, $(I\cap Z)R/(I\cap Z)I$ is non-zero, and is nilpotent and Artinian. Thus, if $R/(I\cap Z)I$ has a primary decomposition, at least one of the primary factors must be Artinian, with non-zero nilpotent radical, by the Proposition. This contradicts the observation above, since R/I is simple.

REFERENCES

(**1**) K. A. Brown, Module extensions over Noetherian rings, *J. of Algebra*, to appear.

(**2**) J. Dixmier, *Algèbres Enveloppantes* (Gauthier-Villars, Paris, 1974).

(**3**) S. M. Ginn and P. B. Moss, Finitely embedded modules over Noetherian rings, *Bull. Amer. Math. Soc.* **81** (1975), 709–710.

(**4**) R. Gordon, Primary decomposition in right Noetherian rings, *Comm. in Algebra* **2** (1974), 491–524.

(**5**) G. Krause, T. H. Lenagan and J. T. Stafford, Ideal Invariance and Artinian Quotient rings, *J. of Algebra* **55** (1978), 145–154.

(**6**) J. C. McConnell, Localisation in enveloping rings, *J. London Math. Soc.* **43** (1968), 421–428.

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