S. Tsuyumine
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RINGS OF MODULAR FORMS
ON EICHLER'S PROBLEM

SHIGEAKI TSUYUMINE

In his paper [4] or lecture note [3], Eichler asked the problem when the ring of modular forms is Cohen-Macaulay. We shall try to investigate it for the Hilbert or Siegel modular case.

When the dimension n is one, any ring of modular forms for an arithmetic group is Cohen-Macaulay, indeed a normal (graded) ring of Krull dimension two is always Cohen-Macaulay. So we consider the case $n > 1$. Unfortunately rings of modular forms do not always have this nice property. In the case of (symmetric or not) Hilbert modular forms it is essentially Freitag's result (see 7.1 Satz [6] and Proposition A in § 1.1). Let $A(\Gamma) = \oplus_k A(\Gamma)_k$ be the ring of Hilbert modular forms for a group Γ. Then the same question for $A(\Gamma)^{(2)} = \oplus_{k\equiv 2(0)} A(\Gamma)_k$ with $n = 2$ was raised by Thomas and Vasquez [20], in which it is shown by using the criterion due to Stanley [18], [19] that $A(\Gamma)^{(2)}$ is also Gorenstein if it is Cohen-Macaulay under some condition on Γ. Also Eichler derived some consequence of the 'hypothesis' of $A(\Gamma)^{(2)}$ being Cohen-Macaulay with $n = 2$ in [3].

In this paper we shall show this affirmatively, and moreover get when $A(\Gamma)^{(r)}$ is Cohen-Macaulay for general n and $r \geq 2$, as well as the case of symmetric Hilbert modular forms (Theorem 1). Furthermore if $n = 2$ and if Γ acts freely on H^2, the necessary and sufficient condition for $A(\Gamma)$ to be Cohen-Macaulay is given as

$$(1) \qquad \dim A(\Gamma)_1 = \tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)\cdot a + \chi + h)$$

where K is a corresponding real quadratic field, ζ_K is its zeta function, $a = [SL_2(O_K); \Gamma]$, O_K being the ring of integers of K, h is the number of the cusps and χ is the arithmetic genus of the non-singular model of the Hilbert modular surface.

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Let us refer to the case of Siegel modular forms, and let $A(\Gamma)$ denote the ring of Siegel modular forms for an arithmetic group Γ. When the degree n of the Siegel space is two, if Γ possesses, as its normal subgroup, the principal congruence subgroup $\Gamma_2(2)$ of level two, then $A(\Gamma)$ is Cohen-Macaulay. But the Cohen-Macaulayness does not always hold, indeed $A(\Gamma_2(\ell))$ is such an example if $\ell \geq 6$. When $n = 3$, $A(\Gamma_3(\ell))$, $\ell \geq 3$, is no longer Cohen-Macaulay. We shall show these by disproving the Serre duality theorem for $\mathrm{Proj}\,(A(\Gamma_n(\ell)))$ which should hold if $A(\Gamma_n(\ell))$ would be Cohen-Macaulay.

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§ 1. Preliminaries

1. Let k be a field, and R be a noetherian k-algebra. We call R Cohen-Macaulay if any ideal generated by a regular sequence has no embedded prime. A k-scheme is called Cohen-Macaulay if all of its local rings are Cohen-Macaulay.

When R is a graded algebra, we have the following (for the detail see Serre [16] Theorem 2 IV-20, Hochster and Robert [13] § 1 (d));

Proposition A. *Let $R = \oplus_{m\geq 0} R_m$ $(R_0 = k)$ be a normal noetherian graded k-algebra of dimension $N+1$. Then the following conditions are equivalent*:

(i) *R is Cohen-Macaulay.*

(ii) *For some (equivalently any) system of homogeneous elements $x_0, \cdots, x_N$ such that R is integral over $k[x_0, \cdots, x_N]$, R is free over it.*

(iii) *Let $X = \mathrm{Proj}\,(R)$, and $\mathcal{O}_X$ be its structure sheaf. Then the cohomology group $H^\nu(X, \mathcal{O}_X(m))$ vanishes for $1 \leq \nu \leq N-1$ and for every $m \in \mathbf{Z}$, where $\mathcal{O}_X(m)$ is Serre's twisting sheaf.*

As an easy consequence of this, we get the following;

Corollary. *Let R be as in the proposition, and let r be an integer. Then $R^{(r)} = \oplus_{m\equiv 0(r)} R_m$ is Cohen-Macaulay if and only if $H^\nu(X, \mathcal{O}_X(m)) = 0$ for $1 \leq \nu \leq N-1$, $m \equiv 0 \bmod r$.*

Next one is a part of the famous results in [12].

PROPOSITION B. *Let G be a finite group acting k-linearly on R. Suppose either* char $(k)=0$, *or the order of G is coprime to* char (k). *If R is Cohen-Macaulay, then so is the invariant subring R^G.*

If we use the notation of Proposition A, then X is a Cohen-Macaulay scheme if and only if $H^\nu(X, \mathcal{O}_X(m))$ vanishes for $\nu < N$, $m \ll 0$ (see for example, the proof of Theorem 7.6, Chap. III, Hartshorne [9]). So if R is a Cohen-Macaulay algebra, then $X = \operatorname{Proj}(R)$ is a Cohen-Macaulay scheme. The converse is not necessarily true, indeed an N-dimensional projective manifold over $\boldsymbol{C}$ carrying non-trivial holomorphic p-forms $(0 < p < N)$ is such an example.

2. We shall prepare two lemmas for the later use.

LEMMA 1. *Let D be a domain in $\boldsymbol{C}^n$ and S be a finite group acting on D as holomorphic automorphisms. Let $\pi\colon D \to Y = D/S$ be the quotient. Take an automorphy factor $\rho(g, z)$ $g \in S$, $z \in D$ and consider an action on $\mathcal{O}_D$ as*

$$f(z) \longmapsto \rho(g, z)^{-1} f(gz) . \tag{2}$$

If $\mathscr{F}$ denotes the invariant subsheaf of $\pi_*(\mathcal{O}_D)$ under this action, then we have

$$i_*(\mathscr{F}|_{Y_0}) = \mathscr{F}$$

where Y_0 is the regular open subset of Y with the inclusion map i.

Proof. Since all non-zero sections f of $\mathscr{F}$ over an open subset V have V as their supports, i.e., $\{P \in V \mid f_P = 0 \text{ in } \mathscr{F} \otimes \mathcal{O}_{Y,P}\} = \phi$, $\mathscr{F}$ is a subsheaf of $i_*(\mathscr{F}|_{Y_0})$. $Y' = Y - Y_0$ is of codimension ≥ 2, since Y is a normal complex space. Hence any holomorphic function g on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$ is extendable to whole $\pi^{-1}(V)$, and moreover if g satisfies (2) on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$, then the extension of g also satisfies (2) on $\pi^{-1}(V)$. This shows that the injection of $\mathscr{F}$ to $i_*(\mathscr{F}|_{Y_0})$ is surjective. q.e.d.

The following is an easy consequence of Corollary to Proposition 5.2.3 Grothendieck [8] (see also Théorème 5.3.1 and its Corollary).

LEMMA 2*. *Let Y be a separated scheme over $\boldsymbol{C}$, and let $\mathscr{F}$ be a coherent sheaf over Y. Let G be a finite group acting on Y, $\mathscr{F}$ compatibly, and $\pi\colon Y \to Y/G$ be the quotient morphism. Then we have*

* The author was informed this by Prof. T. Oda.

$$H^\nu(Y, \mathscr{F})^G \simeq H^\nu(Y/G, (\pi_*\mathscr{F})^G) .$$

§2. Hilbert modular forms

3. Let K be a totally real algebraic number field of degree n (>1), and O_K be the ring of integers. $SL_2(O_K)$ acts on the product H^n of n copies of the upper half plane $H = \{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$ by the modular substitution;

$$z = (z_1, \cdots, z_n) \longrightarrow Mz = \left(\frac{\alpha^{(1)}z_1 + \beta^{(1)}}{\gamma^{(1)}z_1 + \delta^{(1)}}, \cdots, \frac{\alpha^{(n)}z_n + \beta^{(n)}}{\gamma^{(n)}z_n + \delta^{(n)}}\right)$$

$$\text{for } M = \begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix} \in SL_2(O_K)$$

where $\alpha^{(1)}, \cdots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$ in some fixed order. Let Γ be a subgroup of $SL(O_K)$ of finite index. A holomorphic function f on H^n is called a *Hilbert modular form* for Γ of weight k if it satisfies

$$(3) \qquad f(Mz) = \prod_{i=1}^{n} (\gamma^{(i)}z_i + \delta^{(i)})^k f(z) \qquad \text{for any } M = \begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix} \in \Gamma .$$

The symmetric group $\mathfrak{S}_n$ of n letters acts on H^n as permutations of the coordinates

$$z = (z_1, \cdots, z_n) \longrightarrow \sigma z = (z_{\sigma(1)}, \cdots, z_{\sigma(n)}) \qquad \sigma \in \mathfrak{S}_n .$$

The automorphism group $\operatorname{Aut}(K/\boldsymbol{Q})$ can be regarded as a subgroup of $\mathfrak{S}_n$ because it acts on n-tuples $(\alpha^{(1)}, \cdots, \alpha^{(n)})$ as permutations, i.e., for $\sigma \in \operatorname{Aut}(K/\boldsymbol{Q})$ $((\sigma\alpha)^{(1)}, \cdots, (\sigma\alpha)^{(n)})$ is nothing else but the permutation of $(\alpha^{(1)}, \cdots, \alpha^{(n)})$. Let us fix some subgroup S of $\operatorname{Aut}(K/\boldsymbol{Q}) \subset \mathfrak{S}_n$, and let $\hat{\Gamma}$ be the composite of S and Γ as groups acting on H^n. In what follows, we shall always suppose $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$, in other words,

$$(4) \qquad \sigma\Gamma\sigma^{-1} = \Gamma \qquad \text{for any } \sigma \in S .$$

A holomorphic function f on H^n is called a (symmetric) Hilbert modular form for $\hat{\Gamma}$ if it satisfies both (3) and the identity $f(\sigma z) = f(z)$ for $\sigma \in S$. We shall denote by $A(\hat{\Gamma}) = \oplus A(\hat{\Gamma})_k$, the graded $\boldsymbol{C}$-algebra of Hilbert modular forms for $\hat{\Gamma}$, $A(\hat{\Gamma})_k$ being the vector space of Hilbert modular forms of weight k, and denote by $A(\hat{\Gamma})^{(r)}$, the subring $\oplus_{k\equiv 0(r)} A(\hat{\Gamma})_k$.

4. Let h denote the number of the cusps for $\hat{\Gamma}$. $X = H^n/\hat{\Gamma}$ is compactified by adding h points, and we get a normal projective variety X^*, which is isomorphic to Proj $(A(\hat{\Gamma}))$. We shall denote by X_0, the regular open subset of X, hence of X^*.

Let $\mathscr{L}(i)$ denote the coherent sheaf on X^* corresponding to modular forms of weight $i \in \boldsymbol{Z}$, and let $\mathscr{L} = \mathscr{L}(1)$. Obviously we have $\mathscr{L}(i) \otimes \mathscr{L}(j) \subset \mathscr{L}(i + j)$ for $i, j \geq 0$. Let $\pi: \tilde{X} \to X^*$ be a desingularization. The canonical coherent sheaf K_{X^*} on X^* is given by $K_{X^*} = \pi_* K_{\tilde{X}}$, $K_{\tilde{X}}$ being the canonical invertible sheaf on $\tilde{X}$. K_{X^*} is determined up to designularizations (Grauert-Riemenschneider [7]). We shall need also the dualizing sheaf ω_{X^*} which gives rise to the functorial isomorphism $\operatorname{Hom}(\mathscr{F}, \omega_{X^*}) \simeq H^n(X^*, \mathscr{F})^\vee$ for coherent sheaves $\mathscr{F}$. Again by [7], ω_{X^*} equals $i_* K_{X_0}$ where i denotes the inclusion of X_0 to X^*. If X^* is Cohen-Macaulay, then there are natural isomorphisms $H^\nu(X^*, \mathscr{F}) \simeq H^{n-\nu}(X^*, \mathscr{F}^\vee \otimes \omega_{X^*})^\vee$ for any locally free sheaf $\mathscr{F}$ and for its dual $\mathscr{F}^\vee$. We have the canonical inclusion $K_{X^*} \subset \omega_{X^*}$ (loc. cit). Moreover by Freitag [5] Satz 1 we have an equality $K_{X^*}|_X = \omega_{X^*}|_X$.

If S is a subgroup of the alternating group, then ω_{X^*}, $\mathscr{L}(2)$ are isomorphic. Let us show this. Let X^0 be the open subset of X which is the complement of the fixed points set. Obviously $X^0 \subset X_0$. For an open subset U of X^0 if f is a section of $\Gamma(U, \mathscr{L}(2))$, then $f dz_1 \wedge \cdots \wedge dz_n$ gives a section of $\Gamma(U, K_{X_0})$ and vice versa. So K_{X_0} and $\mathscr{L}(2)$ are isomorphic on X^0. Since the codimension of X^0 in X is larger than or equal to two as one can easily see, K_{X_0}, $\mathscr{L}(2)$ are isomorphic on X_0 by the extendability of holomorphic functions. So by Lemma 1 we get $\omega_{X^*}|_X \simeq \mathscr{L}(2)|_X$. Let ∞ be any cusp, and let U be an open neighborhood at ∞. Then a section f of $\Gamma(U - \{\infty\}, \mathscr{L}(j))$, $j \in \boldsymbol{Z}$, admits the Fourier expansion

$$f(z) = c_0 + \sum_\lambda c_\lambda \exp(2\pi\sqrt{-1}(\lambda^{(1)} z_1 + \cdots + \lambda^{(n)} z_n))$$

where λ varies over some lattice of totally positive numbers in K. So f is holomorphic at ∞, and hence we get $i_*(\mathscr{L}(j)|_{U-\{\infty\}}) = \mathscr{L}(j)$, i being the inclusion $U - \{\infty\} \to U$. Thus

$$\mathscr{L}(2) \simeq \omega_{X^*} .$$

Suppose that S is a group not contained in the alternating group. Let S' be the normal subgroup of S of index two which is a subgroup of the algernating group. Let $\psi: (H^n/\hat{\Gamma}')^* \to X^* = (H^n/\hat{\Gamma})^*$ be the canonical projection where $\hat{\Gamma}' = S' \cdot \Gamma$. If $\mathscr{L}'(2)$ is the coherent sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight two, then we define a coherent sheaf $\mathscr{L}(2)_-$ on X^* by

(5) $\qquad \Gamma(U, \mathscr{L}(2)_-) = \{f \in \Gamma(\psi^{-1}(U), \mathscr{L}'(2)) \mid f(\sigma z) = \operatorname{sgn}(\sigma) f(z), \sigma \in S\}$,

U being an open subset of X^*. Any section of $\mathscr{L}(2)_-$ vanishes along the fixed points set under the action of the group S/S'. Then $\mathscr{L}(2)_-|_{X_0}$ is isomorphic to K_{X_0} by the similar argument as above and by [5] the proof of Hilfssatz 4, and moreover

$$\mathscr{L}(2)_- \simeq \omega_{X^*} .$$

5. Let $X^* = (H^n/\hat{\Gamma})^*$, $\mathscr{L}(i)$, $\mathscr{L} = \mathscr{L}(1)$ be as above.

LEMMA 3. *Assume that* $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$ *acts freely on* H^n. *Then*
i) $\mathscr{L}(i)$ *is invertible, and* $\mathscr{L}^i = \mathscr{L}(i)$,
ii) $H^\nu(X^*, \mathscr{L}(i)) = 0$ *for* $i \geq 2$, $\nu > 0$, $(i, \nu) \neq (2, n)$.

Proof. Let $\pi: \tilde{X} \to X^*$ be the desingularization. Then the cohomology group $H^\nu(\tilde{X}, K_{\tilde{X}})$ $(\simeq H^{n-\nu}(\tilde{X}, \mathcal{O}_{\tilde{X}})^\vee)$ vanishes for $0 < \nu < n$ (cf. [6]). Since all the higher direct image sheaves $R^\nu\pi_* K_{\tilde{X}}$ $(\nu \geq 1)$ vanish by [7], also $H^\nu(X^*, K_{X^*})$, $0 < \nu < n$, vanish by using the Leray spectral sequence.

At first let us suppose $\hat{\Gamma} = \Gamma$. Then i) is obvious, and this implies that $\mathscr{L}$ is ample. We have an exact sequence

$$0 \longrightarrow K_{X^*} \longrightarrow \mathscr{L}^2 \longrightarrow \mathscr{L}^2/K_{X^*} \longrightarrow 0 ,$$

where $\mathscr{L}^2/K_{X^*}$ is supported only at cusps by the observation of Section 2.4. Tensoring $\mathscr{L}^i$ and taking the long exact sequence

$$\longrightarrow H^\nu(X^*, \mathscr{L}^i \otimes K_{X^*}) \longrightarrow H^\nu(X^*, \mathscr{L}^{2+i}) \longrightarrow H^\nu(X^*, \mathscr{L}^i \otimes (\mathscr{L}^2/K_{X^*})) \longrightarrow ,$$

we get the desired result since $H^\nu(X^*, \mathscr{L}^i \otimes (\mathscr{L}^2/K_{X^*}))$, $\nu > 0$, vanishes, and since also $H^\nu(X^*, \mathscr{L}^i \otimes K_{X^*})$, $i \geq 0$, $\nu > 0$, $(i, v) \neq (0, n)$, vanishes by the generalized Kodaira vanishing theorem (cf. [7] Satz 2.1) and by the above observation.

Let us consider the general case. Let us put $Y = H^n/\Gamma$, and let $\mathscr{M}$ be the invertible sheaf on Y^* corresponding to modular forms of weight one. We have shown above that i), ii) hold for $Y^*, \mathscr{M}$. To prove i) for $X^*, \mathscr{L}$, it is enough to show that for any point x of X^*, there is an neighborhood V at x such that $\Gamma(V, \mathscr{L})$ has a section not vanishing at x as a function. If x is not a ramification point of the canonical projection $p: Y^* \to Y^*$, then nothing is a problem. If x is such a point, then we can take a point $y \in Y$ with $x = p(y)$, and its neighborhood W

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such that $\Gamma(W, \mathscr{M})$ has a section f not vanishing at y. Then $g = \sum \sigma f$, σ running over the stabilizer subgroup at y of $S \simeq \hat{\Gamma}/\Gamma$, is a desired element, indeed if we take a sufficiently small neighborhood V at x, then $g|_V$ is a section of $\Gamma(V, \mathscr{L})$ whose value $g(x)$ at x is not zero. This shows i). ii) is a direct consequence of Lemma 2, noticing $\mathscr{L} = (p_*\mathscr{M})^S$. q.e.d.

For any $\hat{\Gamma}$, there is a normal subgroup $\hat{\Gamma}'$ of finite index such that $\hat{\Gamma}' \cap SL_2(O_K)$ acts freely on H^n. Then $\hat{\Gamma}/\hat{\Gamma}'$ acts on $(H^n/\hat{\Gamma}')^*$ as a finite automorphism group, and the quotient morphism $p: (H^n/\hat{\Gamma}')^* \to (H^n/\hat{\Gamma})^* = X^*$ is induced. If $\mathscr{L}'(i)$ is the invertible sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight i, then $\mathscr{L}(i)$ equals to $(p_*\mathscr{L}'(i))^{\hat{\Gamma}/\hat{\Gamma}'}$. So by Lemma 3 and Lemma 2, we have the following;

Proposition 1.

$$H^\nu(X^*, \mathscr{L}(i)) = 0 \quad \textit{for} \quad i \geq 2, \quad \nu > 0, \quad (i, \nu) \neq (2, n) .$$

6. Our main theorem is as follows;

Theorem 1. *Let n, Γ, $\hat{\Gamma}$, $A(\hat{\Gamma})$ be as in Section 2.3, and let Γ be under the condition (4) Then $A(\hat{\Gamma})^{(r)}$ is Cohen-Macaulay for any (equivalently some) $r \geq 2$ if and only if*

$$(6) \qquad \begin{aligned} &n \leq 2\,, \quad \textit{or} \\ &n = 3\,, \quad \hat{\Gamma}_\infty/\Gamma_\infty \simeq \mathbf{Z}/3\mathbf{Z} \quad \textit{at each cusp} \quad \infty\,, \quad \textit{or} \\ &n = 4\,, \quad \hat{\Gamma}_\infty/\Gamma_\infty \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \quad \textit{at each cusp} \quad \infty\,, \end{aligned}$$

where $\hat{\Gamma}_\infty$ (resp. Γ_∞) is denoting the stabilizer subgroup of $\hat{\Gamma}$ (resp. Γ) at ∞.

By Thomas and Vasquez [20], Theorem 2, we also get the following corollary;

Corollary. *Let $n = 2$, and let Γ be $SL_2(O_K)$ or its torsion free subgroup. Then $A(\Gamma)^{(2)}$ is Gorenstein.*

To prove Theorem 1 the following is a key proposition.

Proposition C (Freitag [6]). *The condition (6) and the following two conditions (a), (b) are all equivalent to each other.*

(a) *X^* is Cohen-Macaulay,*

(b) *$H^\nu(X^*, \mathscr{O}_{X^*}) = 0$ for $0 < \nu < n$.*

By the fact we saw in Section 1.1, $A(\hat{\Gamma})^{(r)}$ cannot be Cohen-Macaulay for any r unless (6) is the case. In [20] it is shown that $A(\Gamma)^{(2)}$ is never

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Gorenstein under some condition on Γ with $n=3$. But it is not even Cohen-Macaulay.

7. *Proof of Theorem* 1. By Propositions A and C it is enough to show merely 'if' part. We shall give the proofs of two kinds, however one is available only for $n=2$. At first we assume $n=2$. Then X^* is a normal surface and hence it is Cohen-Macaulay. We may assume $\Gamma(=\hat{\Gamma}\cap SL_2(O_K))$ acts freely on H^2 by replacing $\hat{\Gamma}$ by a normal subgroup $\hat{\Gamma}'$ of finite index if necessary. Indeed if $A(\hat{\Gamma}')^{(r)}$ is Cohen-Macaulay, then so is $A(\hat{\Gamma})^{(r)}$ by Proposition B because $A(\hat{\Gamma})^{(r)}$ is the invariant subring of $A(\hat{\Gamma}')^{(r)}$ under the action of $\hat{\Gamma}/\hat{\Gamma}'$. So we may assume that $\mathscr{L}$ is an ample invertible sheaf by Lemma 3. Since X^* is Cohen-Macaulay, we have an isomorphism between the cohomology groups

$$H^\nu(X^*, \mathscr{L}^{-i}) \simeq H^{n-\nu}(X^*, \mathscr{L}^i\otimes\omega_{X^*})^\vee$$

by Serre's duality theorem. As we have seen in Section 2.4, K_{X^*} is a subsheaf of ω_{X^*} and ω_{X^*}/K_{X^*} is supported only at cusps. Now the similar argument as in the proof of Lemma 3 will derive the vanishing of the cohomology groups $H^{n-\nu}(X^*, \mathscr{L}^i\otimes\omega_{X^*})$ for $n-\nu>0$, $i>0$, $(i, n-\nu)\neq(0, n)$ (use ω_{X^*} instead of $\mathscr{L}^2$). So $H^\nu(X^*, \mathscr{L}^{-1})$ vanishes for $0<\nu<n$, $i>0$. Together with Proposition 1 and Proposition C (b) we get

$$H^\nu(X^*, \mathscr{L}^i)=0$$
$$\text{for}\quad 0<\nu<n\,,\quad i\equiv 0 \bmod r\,,$$

where r is any integer greater than one. By Corollary to Proposition A this implies that $A(\hat{\Gamma})^{(r)}$ is Cohen-Macaulay, and our assertion is proved when $n=2$.

In the case $n=3, 4$ the above argument does not work since $(H^n/\hat{\Gamma}')^*$ may not be Cohen-Macaulay even if so is $(H^n/\hat{\Gamma})^*$, where $\hat{\Gamma}'$ is a subgroup of $\hat{\Gamma}$. Let us take a normal subgroup Γ' of $\hat{\Gamma}$ which acts freely on H^n. Then by the virtue of [1] we have a smooth toroidal compactification $\overline{X}'$ of $X'=H^n/\Gamma'$, on which, we may assume, the finite quotient group $\hat{\Gamma}/\Gamma'$ acts in the natural way (cf. [1], [22]). Let us put $\overline{X}=\overline{X}'/(\hat{\Gamma}/\Gamma')$ that has only quotient singularities. $\overline{X}$ (resp. $\overline{X}'$) has $X=H^n/\hat{\Gamma}$ (resp. X') as its Zariski open subset. Let π (resp. π') be the morphism of the blowing up $\overline{X}\to X^*$ (resp. $\overline{X}'\to X'^*$), and let ψ (resp. $\bar{\psi}$) be the quotient map of X'^* to X^* (resp. $\overline{X}'$ to $\overline{X}$). We have a commutative diagram

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$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\bar{\psi}} & \bar{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X'^* & \xrightarrow[\psi]{} & X^* \end{array} .$$

We shall show that the morphism π enjoys

$$R^\nu \pi_* \mathcal{O}_{\bar{X}} = 0 , \qquad 0 < \nu < n - 1 ,$$

by using our assumption of X^* being Cohen-Macaulay. Let $\tilde{\pi}: \tilde{X} \to \bar{X}$ be the desingularization. Since $\bar{X}$ has only rational singularities, the higher direct image sheaves $R^\nu \tilde{\pi}_* \mathcal{O}_{\tilde{X}}$, $\nu > 0$, vanish. $\pi \circ \tilde{\pi}: \tilde{X} \to X^*$ is the desingularization of X^*, and by the same reason as above $R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}}|_X$ vanish for $\nu > 0$. Since $(R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_X)_\infty = 0$ for $0 < \nu < n - 1$ if the local ring at a cusp ∞ is Cohen-Macaulay as Freitag showed [6] 4.5, $R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}}$ vanish for $0 < \nu < n - 1$. Considering the Leray spectral sequence $E_2^{p,q} = R^p \pi_*(R^q \tilde{\pi}_* \mathcal{O}_{\tilde{X}}) \Rightarrow R^{p+q}(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}}$, we get the vanishing of $R^\nu \pi_* \mathcal{O}_{\bar{X}}$ for $0 < \nu < n - 1$.

If $\mathcal{L}'(i)$ is the invertible sheaf on X'^* corresponding to modular forms of weight i, then $(\bar{\psi}_* \pi'^* \mathcal{L}'(i))^{\hat{\Gamma}/\Gamma'}$ is equal to $\pi^* \mathcal{L}(i)$ by using the facts that (i) π, π' are birational, (ii) X^*, X'^* are normal, and (iii) $\psi_* \mathcal{L}'(i)^{\hat{\Gamma}/\Gamma'} = \mathcal{L}(i)$. $H^\nu(\bar{X}', \pi'^* \mathcal{L}'(i))$ vanishes for $\nu < n$, $i < 0$ by the generalized Kodaira vanishing theorem [7]. So applying Lemma 2, we have $H^\nu(\bar{X}, \pi^* \mathcal{L}(i)) = 0$ for $\nu < n$, $i < 0$. Since $\mathcal{L}(i)$ is locally free near at each cusp, the projection formula $R^\nu \pi_* \mathcal{L}(i) = \mathcal{L}(i) \otimes R^\nu \pi_* \mathcal{O}_{\bar{X}}$ holds and hence there exists the Laray spectral sequence $E_2^{p,q} = H^p(X^*, \mathcal{L}(i) \otimes R^q \pi_* \mathcal{O}_{\bar{X}}) \Rightarrow H^{p+q}(\bar{X}, \pi^* \mathcal{L}(i))$. It follows from this that $H^\nu(X^*, \mathcal{L}(i)) = 0$ for $\nu < n$, $i < 0$. Now the same argument as above shows our assertion. q.e.d.

In the above proof, we have shown under the condition (6) that $H^\nu(X^*, \mathcal{L}(i))$ vanishes for $0 < \nu < n$ if $i \neq 1$. As a consequence of this we get the following;

PROPOSITION 2. *$A(\hat{\Gamma})$ is Cohen-Macaulay if and only if*

$$H^\nu(X^*, \mathcal{L}) = 0 \qquad \textit{for } 0 < \nu < n$$

together with the condition (6).

8. In what follows, we always assume $n = 2$, and that Γ acts freely on H^2. Let a be the index $a = [SL_2(O_K); \Gamma]$, and let χ be the arithmetic

genus $\sum_{\nu=0}^{2}(-1)^{\nu}\dim H^{\nu}(\tilde{X}, \mathcal{O}_{\tilde{X}})$ where $\tilde{X}$ is the nonsingular model of $X^* = (H^2/\Gamma)^*$. χ is equal to $1 +$ dim. of the space of cusp forms of weight two. By Shimizu [17] (see also Hirzebruch [10] § 2 Theorem, Freitag [6] 7.2 Satz) we have a Hilbert polynomial $P(k)$ of $A(\Gamma)$:

$$P(k) = \tfrac{1}{2}\cdot\zeta_K(-1)\cdot ak(k-2) + \chi + h\,, \tag{7}$$

where ζ_K is the zeta function of K, and h is the number of cusps. $P(k)$ gives the dimension of $A(\Gamma)_k$ for $k \geq 3$, and $P(2)$ equals $\dim A(\Gamma)_2 + 1$. $P(k)$ must be equal to the Euler-Poincaré characteristic $\chi(\mathscr{L}^k) = \sum_{\nu=0}^{2}(-1)^{\nu}\dim H^{\nu}(X^*, \mathscr{L}^k)$, which is known to be a polynomial of k (cf. [15]]). Hence we have

$$\begin{aligned} &-\tfrac{1}{2}\zeta_K(-1)\cdot a + \chi + h \\ &\qquad = \dim H^0(X^*, \mathscr{L}) - \dim H^1(X^*, \mathscr{L}) + \dim H^2(X^*, \mathscr{L})\,. \end{aligned}$$

Since $\mathscr{L}^2$ is now Serre's dualizing sheaf (§ 2.4), $H^2(X^*, \mathscr{L})$ is just dual to $H^0(X^*, \mathscr{L})$. So we obtain

$$\dim A(\Gamma)_1 = \tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)a + \chi + h) + \tfrac{1}{2}\dim H^1(X^*, \mathscr{L})\,.$$

Especially the inequality

$$\dim A(\Gamma)_1 \geq \tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)a + \chi + h)$$

always holds, and $A(\Gamma)$ is Cohen-Macaulay if and only if the equality (1) holds by Proposition 2.

(1) is a nice equality in the following sence. If (1) is the case, then we can compute the generating function $Q(t) = \sum \dim A(\Gamma)_k t^k$ together with (7) and with $\dim A(\Gamma)_2 = P(2) - 1$ as

$$\begin{aligned} Q(t) = \frac{1}{(1-t)^3}\{1 + t^5 + (t + t^4)\{\tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)a + \chi + h) - 3\} \\ + (t^2 + t^3)\{\tfrac{1}{2}(\tfrac{3}{2}\zeta_K(-1)a - \chi - h) + 2\}\}\,. \end{aligned}$$

It is easy to see $Q(t)$ satisfies $-t^2Q(t^{-1}) = Q(t)$. By Stanley [18] this implies that $A(\Gamma)$ is Gorenstein.

9. Let X^*, Γ be as above. Let $\hat{X}^*$ denote $(H^2/\hat{\Gamma})^*$ where $\hat{\Gamma} = \mathfrak{S}_2\cdot\Gamma$, and let $\hat{\mathscr{L}}$ be the invertible sheaf on $\hat{X}$ corresponding to symmetric Hilbert modular forms of weight one. Let us suppose (4). Then if $p\colon X^* \to \hat{X}^*$ is the canonical projection, we have the direct decomposition

$$p_*\mathscr{L} = \hat{\mathscr{L}} \oplus \mathscr{L}_-$$

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where $\mathscr{L}_-$ is the coherent sheaf given in the similar way as (5). Since $\hat{\mathscr{L}} \otimes \mathscr{L}_- = \mathscr{L}(2)_-$, it gives Serre's dualizing sheaf on $\hat{X}^*$ (§ 2.4). Thus we have

$$H^1(\hat{X}^*, \mathscr{L}_-) = \mathrm{Ext}^1(\mathscr{O}_{\hat{X}^*}, \mathscr{L}_-) \simeq \mathrm{Ext}^1(\hat{\mathscr{L}}, \hat{\mathscr{L}} \otimes \mathscr{L}_-) \simeq H^1(\hat{X}^*, \hat{\mathscr{L}})^\vee$$

and hence

$$\begin{aligned} H^1(X^*, \mathscr{L}) &\simeq H^1(\hat{X}^*, p_*\mathscr{L}) \simeq H^1(\hat{X}^*, \hat{\mathscr{L}}) \oplus H^1(\hat{X}^*, \mathscr{L}_-) \\ &\simeq H^1(\hat{X}^*, \hat{\mathscr{L}}) \oplus H^1(\hat{X}^*, \hat{\mathscr{L}})^\vee , \end{aligned}$$

and hence $\dim H^1(X^*, \mathscr{L}) = 2 \dim H^1(\hat{X}^*, \hat{\mathscr{L}})$. Thus $A(\Gamma)$ and $A(\hat{\Gamma})$ are Cohen-Macaulay or not alike by Proposition 2. Summing up the above, we shall state it as the proposition.

PROPOSITION 3. *Let K be a real quadratic field, and Γ be a subgroup of $SL_2(O_K)$ of finite index acting freely on H^2. Then the following are equivalent*;

(a) *$A(\Gamma)$ is Gorenstein.*

(b) *$A(\Gamma)$ is Cohen-Macaulay.*

(c) *The equality* (1) $\dim A(\Gamma)_1 = \frac{1}{2}(-\frac{1}{2}\zeta_K(-1)a + \chi + h)$ *holds.*

Assuming (4) *for* $S = \mathfrak{S}_2$,

(d) *$A(\hat{\Gamma})$ is Cohen-Macaulay.*

The known examples of full rings $A(\Gamma)$ for above Γ are quite a few yet. At any rate such examples in Hirzebruch [11], which are

$$K = Q(\sqrt{5}), \qquad \Gamma = \Gamma(\sqrt{5}) = \{M \in SL_2(O_K) \mid M \equiv 1_2 \bmod \sqrt{5}\} ,$$

and

$$K = Q(\sqrt{2}) , \qquad \Gamma = \Gamma(2) \cdot \left\langle \begin{pmatrix} 1+\sqrt{2} & \\ & 1-\sqrt{2} \end{pmatrix} \right\rangle ,$$
$$\Gamma(2) = \{M \in SL_2(O_K) \mid M \equiv 1_2 \bmod 2\} ,$$

are satisfying the conditions in Proposition 3. It may not be unreasonable to expect it in more general case.

10. Let Γ be as above. By the method of [20], we can show that $A(\Gamma)$ may possibly be a complete intersection ring only in a finite number of cases as following. The index $a = [SL_2(O_K) \colon \Gamma]$ is divisible by 6 because $SL_2(O_K)$ has torsion points of order 2, 3, on the other hand Γ not (cf. Hirzebruch [10] § 1.7). Let us put $a = 6f$. Then $A(\Gamma)$ may possibly be a complete intersection ring only if $(2\zeta_K(-1)f, h + \chi)$ is one of

$$(32/3, 32),\ (8, 26),\ (16/3, 20),\ (8/3, 14),\ (4, 16),\ (2, 11)\ ,$$
$$(4/3, 8),\ (10/3, 15),\ (4/3, 10),\ (2/3, 7),\ (2/3, 9)\ .$$

Considering the vlaues of the zeta functions at -1, this cannot happen if the discriminant of K is larger than 105. We skip the proof, which will be almost the same as in [20].

§ 3. Single modular forms

11. Let H_n be the Siegel space of degree n, i.e., $\{Z \in M_n(\boldsymbol{C}) \mid {}^tZ = Z, \operatorname{Im} Z > 0\}$. The symplectic group $Sp(\boldsymbol{R})$ acts on H_n by the usual modular substitution

$$Z \longmapsto MZ = (AZ + B)(CZ + D)^{-1} \qquad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp_n(\boldsymbol{R})\ .$$

We shall denote by $\Gamma_n(\ell)$, the principal congruence subgroup of level ℓ; $\{M \in Sp_n(\boldsymbol{Z}) \mid M \equiv 1_{2n} \bmod \ell\}$.

Let f be a holomorphic function on H_n. f is called a *Siegel modular form* of weight k for a congruence subgroup Γ, if it satisfies

$$f(MZ) = |CZ + D|^k f(Z) \qquad \text{for } M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma\ .$$

When $n = 1$, we need an additional condition that f is holomorphic also at cusps, which is automatic if $n > 1$. We denote by $A(\Gamma) = \oplus_{k \geq 0} A(\Gamma)_k$ (resp. $S(\Gamma) = \oplus_{k \geq 0} S(\Gamma)_k$), the graded ring of modular forms (resp. the graded ideal of cusp forms).

Let $X = H_n/\Gamma$, and let X^* be its Satake compactification, which is a normal projective variety isomorphic to $\operatorname{Proj}(A(\Gamma))$.

12. Let Γ be neat, and let $\mathscr{L}$ be an invertible sheaf on X^* corresponding to modular forms of weight one. The regular open subset of X^* coincides with X, and $\mathscr{L}^{n+1}_{|X}$ is isomorphic to the canonical invertible sheaf K_X on X. Then by [7], the dualizing sheaf ω_{X^*} on X^* is given by $i_* \mathscr{L}^{n+1}_{|X}$, i being the inclusion map of X into X^*, where ω_{X^*} gives rise to the functorial isomorphism $\operatorname{Hom}(\mathscr{F}, \omega_{X^*}) \simeq H^{n(n+1)/2}(X^*, \mathscr{F})^\vee$ for coherent sheaves $\mathscr{F}$ on X^*. Here we note

$$\omega_{X^*} \simeq \mathscr{L}^{n+1}$$

by Koecher's principle. So if X^* is Cohen-Macaulay, then $H^\nu(X^*, \mathscr{L}^k)$ is isomorphic to the dual of $H^{n(n+1)/2-\nu}(X^*, \mathscr{L}^{n+1-k})$ and hence $P(k) =$

$(-1)^{n(n+1)/2}P(n+1-k)$, $P(k)$ denoting the Hilbert polynomial of the graded ring $A(\Gamma)$ or equivalently $\chi(\mathscr{L}^k)$.

On the other hand it is shown in [2] Vol. 2–16 that

$$P(k) = \dim A(\Gamma)_k = \dim S(\Gamma)_k + \sum_{\Gamma' \subset Sp_{n-1}(\boldsymbol{R})} \dim S(\Gamma')_k + \cdots + \sum_{\Gamma' \subset Sp_1(\boldsymbol{R})} \dim S(\Gamma')_k + \#\,(\text{0-dimensional cusps}).$$

for $k \gg 0$ where Γ' varies over the set of all the subgroups attached to cusps of X^*. (The above is shown in [2] for $k \gg 0$, $k \equiv 0 \bmod 2$. However both sides must be numerical polynomials of k for $k \gg 0$, so we get the above formula.)

13. Let us consider the case $n = 2$. $A(\Gamma_2(2))$ was shown to be Cohen-Macaulay in Igusa [14]. So for any arithmetic group Γ containing $\Gamma_2(2)$ as a normal subgroup, $A(\Gamma)$ is Cohen-Macaulay by Proposition B. However the Cohen-Macaulayness fails for $\Gamma_2(\ell)$ $\ell \geq 6$. We shall show it.

Let X^* be the Satake compactification of $H^2/\Gamma_2(\ell)$ for some $\ell \geq 3$, and let $P(k)$ be the Hilbert polynomial for $A(\Gamma_2(\ell))$. Then if X^* is Cohen-Macaulay, we would have $P(3/2) = 0$ since $P(k) = -P(3-k)$ by the observation in Section 3.12. By Yamazaki [23] we can actually calculate $P(k)$ and hence $P(3/2)$;

$$P(3/2) = 2^{-4}3^{-1}\ell^4 \prod_{p \mid \ell} (1-p^4)\{(\ell^3 - 6\ell^2) \prod_{p \mid \ell} (1-p^{-2}) + 2^3 3\}\,.$$

This is not zero if $\ell \geq 6$, so in this case X cannot be Cohen-Macaulay and hence $A(\Gamma_2(l))$ $\ell \geq 6$ are not Cohen-Macaulay algebras.

The similar argument works also for $\Gamma = \Gamma_3(\ell)$ $\ell \geq 3$ by using the formula by Tsushima [21]. Indeed if $(H_3/\Gamma_3(\ell))^*$ were Cohen-Macaulay, then the Hilbert polynomial $P(k)$ of the graded ring $A(\Gamma_3(\ell))$ would satisfy $P(k-2) - P(2-k) = 0$ by the observation in Section 3.12. However actually we have

$$P(k-2) - P(2-k) = 2^{-7}3^{-3}5^{-1}\ell^{16} \prod_{p \mid \ell} (1-p^{-2})(1-p^{-4})(1-p^{-6})k^3 + O(k^2)\,.$$

So $(H_3/\Gamma_3(\ell))^*$ is not Cohen-Macaulay. We obtain the following;

PROPOSITION 4. *Let $\Gamma = \Gamma_n(\ell)$ with $n = 2$, $\ell \geq 6$ or $n = 3$, $\ell \geq 3$. Then the Satake compactification of $H_n/\Gamma_n(\ell)$ is not a Cohen-Macaulay variety. Especially if $A(\Gamma)$ denotes the ring of Siegel modular forms for Γ, then $A(\Gamma)^{(r)}$ is not Cohen-Macaulay for any integer r.*

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2856-235, Sashiogi,
Omiya-shi,
Saitama, 330
Japan