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EXPLICIT OPTIMAL STRATEGY FOR THE VARDI CASINO WITH LIMITED PLAYING TIME

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Abstract

The Vardi casino with parameter $0 < c < 1$ consists of infinitely many tables indexed by their odds, each of which returns the same (negative) expected winnings $-c$ per dollar. A gambler seeks to maximize the probability of reaching a fixed fortune by gambling repeatedly with suitably chosen stakes and tables (odds). The optimal strategy is derived explicitly subject to the constraint that the gambler is allowed to play only a given finite number of times. Some properties of the optimal strategy are also discussed.

Keywords: Gambling theory; bold play; primitive casino; finite horizon

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1. Introduction

In the classical Dubins–Savage subfair primitive casino gambling problem with parameters r and w ($r > 0$, $0 < w < 1$, and $w(1+r) - 1 < 0$), the gambler can stake any amount in his possession, winning r times the stake with probability w and losing the stake with probability $1 - w$. The gambler seeks to maximize the probability of reaching a fixed fortune (to be normalized to unity) by gambling repeatedly with suitably chosen stakes. Dubins and Savage [4] proved that the maximum probability of reaching fortune 1 (the goal) is attained by the bold strategy: 'staking on each play as much as possible without risk of overshooting the goal', i.e. staking $\min\{f, (1-f)/r\}$ if the current fortune is $0 < f < 1$. When the gambler is allowed to play at most n times ($1 \le n < \infty$), Dvoretsky showed that the bold strategy remains optimal provided that either $r = 1$ and $w \le \frac{1}{2}$ or $r \le 1$ and $w = \frac{1}{2}$, and also constructed examples demonstrating that the bold strategy is not optimal in general for subfair primitive casinos with limited playing time; see [4, pp. 110–111]. It has recently been established in [8] that, for each fixed $3 \le n < \infty$, the condition of $r \le 1$ and $w \le \frac{1}{2}$ is necessary and sufficient for the bold strategy to be optimal at all initial fortunes $f \in (0, 1)$ when the gambler is allowed to play at most n times. For other recent developments on various extensions of the Dubins–Savage problem see [1], [2], [3], and [6].

In contrast to the Dubins–Savage casino with a fixed table (r, w), Vardi introduced a casino (with a given parameter $0 < c < 1$) consisting of infinitely (uncountably) many tables indexed by (r, w) with $w(1+r) - 1 = -c$, i.e.

$$w = w(r) = \frac{1-c}{1+r}, \qquad 0 < r < \infty. \tag{1}$$

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In other words, in the Vardi casino, the gambler with current fortune f is allowed to choose, on each play, both the stake s ($0 \le s \le f$) and the odds $r > 0$, so that his fortune becomes $f + rs$ if he wins (with probability $w(r) = (1-c)/(1+r)$) and $f - s$ otherwise (with probability $1 - w(r)$). Shepp [7] has shown that when the gambler plays in the Vardi casino with initial fortune $0 < f < 1$, the supremum over all betting strategies of the probability to reach the goal 1 is $1-(1-f)^{1-c}$. While this optimal winning probability is not attainable by any strategy, Shepp has demonstrated that it is the limit (as $\alpha \downarrow 0$) of the winning probability of the strategy S_α, i.e. staking $s = f$ (current fortune) on the table with odds $r = (1-f)/s = (1-f)/f$ if $f \le \alpha$ and staking $s = \alpha(1-f)/(1-\alpha)$ on the table with odds $r = (1-f)/s = (1-\alpha)/\alpha$ if $\alpha \le f < 1$. Note that the stake s and the odds r are related by $r = (1-f)/s$, so that the goal can be reached in one successful play. Also note that under the strategy S_α, the gambler always bets on the table with odds $(1-\alpha)/\alpha$ as long as his fortune is not less than α.

Shepp's work reveals the following two interesting facts about the Vardi casino.

(i) There is a simple explicit expression for the optimal winning probability.

(ii) The optimal strategy is only a limiting strategy and so does not exist.

To further understand the structures of the problem, we consider, in this note, betting strategies subject to the constraint of limited playing time. While finite-horizon problems usually admit no closed-form solutions, in Section 2 the optimal strategy and corresponding winning probability when the gambler is allowed to play only a given finite number of times is explicitly derived. In Section 3 some properties of the optimal strategy are discussed. In particular, it is shown that under the optimal strategy, the gambler should stay with the same table throughout the betting process, which depends on the initial fortune f and the number, n, of plays allowed. Indeed, interestingly and surprisingly, the optimal strategy is exactly Shepp's strategy S_α with $\alpha = 1-(1-f)^{1/n}$, which involves n independent and identically distributed (i.i.d.) Bernoulli trials such that the goal is reached if and only if one of the trials results in a success.

We close this section by noting that Grigorescu *et al.* [5] have recently investigated the Vardi casino with interest payments, which is considerably more complicated than without interest payments.

2. Explicit optimal strategy and winning probability

For $n = 1, 2, \ldots,$ let

$$\mathrm{P}_n(f) = \begin{cases} 1 - [c + (1-c)(1-f)^{1/n}]^n, & 0 \le f < 1, \\ 1, & f \ge 1. \end{cases} \tag{2}$$

Theorem 1. *If the gambler has fortune $0 < f < 1$ and is allowed to play at most n times, then the initial (unique) optimal play is to stake $s_n(f)$ on the table with odds $r_n(f)$, where*

$$s_n(f) = (1-f)^{(n-1)/n} - (1-f) \le f \quad \textit{and} \quad r_n(f) = \frac{1-f}{s_n(f)} > 0. \tag{3}$$

The optimal winning probability is $\mathrm{P}_n(f)$.

Proof. The theorem will be proved by induction on n. The case in which $n = 1$ is obvious. Suppose that the theorem holds for all $n \le m$ ($m \ge 1$). We now consider the case in which the gambler can play at most $m + 1$ times with initial fortune $0 < f < 1$. If the gambler initially

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stakes $0 \le s \le f$ on the table with odds $r > 0$ and then proceeds optimally thereafter, then he will *not* reach the goal with probability

$$\begin{aligned} Q(r,s) &= w(r)(1-\mathrm{P}_m(f+rs)) + (1-w(r))(1-\mathrm{P}_m(f-s)) \\ &= \frac{1-c}{1+r}(1-\mathrm{P}_m(f+rs)) + \left(1-\frac{1-c}{1+r}\right)(1-\mathrm{P}_m(f-s)), \end{aligned} \tag{4}$$

by the induction hypothesis, where $w(r)$ is given in (1). By (2) and (3), we have

$$\begin{aligned} \mathrm{P}_m(f + r_{m+1}(f)s_{m+1}(f)) &= \mathrm{P}_m(1) = 1, \\ 1 - \frac{1-c}{1+r_{m+1}(f)} &= c + (1-c)(1-f)^{1/(m+1)}, \\ 1 - \mathrm{P}_m(f - s_{m+1}(f)) &= (c + (1-c)\{1-f+s_{m+1}(f)\}^{1/m})^m \\ &= (c + (1-c)(1-f)^{1/(m+1)})^m, \end{aligned}$$

so, by (2) and (4), we have

$$Q(r_{m+1}(f), s_{m+1}(f)) = (c + (1-c)(1-f)^{1/(m+1)})^{m+1} = 1 - \mathrm{P}_{m+1}(f). \tag{5}$$

We will establish the following facts.

(a) For $r > 0$, we have $Q(r, 0) > 1 - \mathrm{P}_{m+1}(f)$.

(b) For each fixed $0 < s \le f$, $Q(r,s)$, as a function of $r > 0$, has a unique minimum at $r = (1-f)/s$; i.e. $Q(r,s) > Q((1-f)/s, s)$ for all $0 < r \ne (1-f)/s$.

(c) As a function of $s \in (0, f]$, $Q((1-f)/s, s)$ has a unique minimum at $s = s_{m+1}(f) = (1-f)^{m/(m+1)} - (1-f) \le f$.

The statements (a)–(c) together with (5) imply that the minimum of $Q(r,s)$ over $r > 0$ and $0 \le s \le f$ equals $1 - \mathrm{P}_{m+1}(f)$, which is uniquely attained at $r = r_{m+1}(f)$ and $s = s_{m+1}(f)$. This proves that the theorem holds for $n = m+1$.

Finally we prove statements (a)–(c). Let X be a random variable with $P(X = 1) = c = 1 - P(X = 1-f)$. Noting that

$$1 - \mathrm{P}_k(f) = [\mathrm{E}(X^{1/k})]^k =: \|X\|_{1/k},$$

we have, by Lyapounov's inequality,

$$Q(r, 0) = 1 - \mathrm{P}_m(f) = \|X\|_{1/m} > \|X\|_{1/(m+1)} = 1 - \mathrm{P}_{m+1}(f).$$

This completes the proof of statement (a).

Note that

$$\mathrm{P}_m(x) = \begin{cases} 1 - \sum_{k=0}^{m} \binom{m}{k} c^{m-k}(1-c)^k(1-x)^{k/m}, & 0 \le x < 1, \\ 1, & x = 1, \end{cases}$$

is convex in $0 \le x \le 1$. For fixed $0 < s \le f$, we have, by (4),

$$\begin{aligned} Q(r,s) &= 1 - \mathrm{P}_m(f-s) - s(1-c)\left(\frac{\mathrm{P}_m(f+rs) - \mathrm{P}_m(f-s)}{(f+rs)-(f-s)}\right) \\ &= 1 - \mathrm{P}_m(f-s) - s(1-c)G(f+rs), \end{aligned} \tag{6}$$

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where

$$G(x) := \frac{\mathrm{P}_m(x) - \mathrm{P}_m(f-s)}{x-(f-s)}, \qquad x > f.$$

Observing that $G(x)$ is the slope of the line segment connecting the points $(x, \mathrm{P}_m(x))$ and $(f-s, \mathrm{P}_m(f-s))$, we have, by the convexity of $\mathrm{P}_m(x)$, that $G(x)$ is increasing in $x \in (f, 1)$, so

$$\begin{aligned}\sup_{f<x<1} G(x) = \lim_{x\uparrow 1} G(x) &= \frac{1-c^m-\mathrm{P}_m(f-s)}{1-(f-s)} \\ &< \frac{1-\mathrm{P}_m(f-s)}{1-(f-s)} \\ &= G(1).\end{aligned}$$

Furthermore, $G(x)$ is strictly decreasing in $x \in [1, \infty)$, since $\mathrm{P}_m(x) = 1$ for $x \geq 1$. It follows that $G(x) < G(1)$ for all $f < x \neq 1$, which, by (6), implies that $Q(r, s) > Q((1-f)/s, s)$ for all $0 < r \neq (1-f)/s$. This completes the proof of statement (b).

Letting

$$x := (1-f+s)^{1/m}, \tag{7}$$

we have

$$\begin{aligned}Q((1-f)/s, s) &= \left(1 - \frac{1-c}{1+(1-f)/s}\right)(1-\mathrm{P}_m(f-s)) \\ &= \left(c + \frac{(1-c)(1-f)}{x^m}\right)(c+(1-c)(1-f+s)^{1/m})^m \\ &= \left(c + \frac{(1-c)(1-f)}{x^m}\right)(c+(1-c)x)^m \\ &=: H(x).\end{aligned}$$

Since $H(x)$ is of the form $\sum_{k=-m}^{m} a_k x^k$ with $a_k > 0$ for all k, $H(x)$ is strictly convex in $x > 0$. Noting that $H'(x) = 0$ at $x = (1-f)^{1/(m+1)}$, $Q((1-f)/s, s)$ has a unique minimum at $s = s^*$ satisfying (by (7))

$$(1-f+s^*)^{1/m} = (1-f)^{1/(m+1)},$$

i.e.

$$s^* = s_{m+1}(f) = (1-f)^{m/(m+1)} - (1-f) \leq f.$$

This completes the proof of statement (c).

3. Properties of the optimal strategy

We are now in a position to discuss some properties of the optimal strategy and corresponding winning probability.

Property 1. The optimal stake

$$s_n(f) = (1-f)^{(n-1)/n} - (1-f)$$

depends on the current fortune and the number of plays allowed. It is decreasing in n and concave in f with the maximum value attained at $f = 1 - ((n-1)/n)^n$. As $n \to \infty$,

$$s_n(f) = n^{-1}(1-f)\log(1-f)^{-1} + O(n^{-2}).$$

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Furthermore, $f - s_n(f) = 1 - (1-f)^{(n-1)/n}$, the remaining fortune after a losing bet, is increasing in f and in n.

Property 2. The optimal odds

$$r_n(f) = \frac{1-f}{s_n(f)} = \frac{1}{(1-f)^{-1/n} - 1}$$

is decreasing in f and increasing in n. As f increases from 0 to 1, $r_n(f)$ decreases from ∞ to 0. As $n \to \infty$,

$$r_n(f) = n(\log(1-f)^{-1})^{-1} + O(1).$$

Property 3. The optimal winning probability $\mathrm{P}_n(f)$ given in (2) is convex and increasing in $0 \le f \le 1$ with a discontinuity at $f = 1$, i.e.

$$\lim_{f \uparrow 1} \mathrm{P}_n(f) = 1 - c^n < 1 = \mathrm{P}_n(1).$$

As $n \to \infty$, $\mathrm{P}_n(f)$ increases to $1 - (1-f)^{1-c}$, the optimal winning probability for the Vardi casino with unlimited playing time. More precisely, we have, as $n \to \infty$,

$$\mathrm{P}_n(f) = 1 - (1-f)^{1-c} - 2^{-1}n^{-1}c(1-c)(1-f)^{1-c}(\log(1-f))^2 + O(n^{-2}).$$

Property 4. Suppose that the gambler has fortune $0 < f < 1$ and is allowed to play at most n times. Under the optimal strategy, the initial play is to stake $s_n(f)$ on the table with odds $r_n(f)$. The goal is reached in case of a win (with probability $(1-c)/(1+r_n(f))$). Otherwise, the gambler's fortune reduces to $f - s_n(f)$. Then the next play is to stake

$$s_{n-1}(f - s_n(f)) = s_{n-1}(1 - (1-f)^{(n-1)/n}) = (1-f)^{(n-2)/n} - (1-f)^{(n-1)/n}$$

on the table with odds

$$\begin{aligned} r_{n-1}(f - s_n(f)) &= \frac{1 - (f - s_n(f))}{s_{n-1}(f - s_n(f))} \\ &= \frac{(1-f)^{(n-1)/n}}{(1-f)^{(n-2)/n} - (1-f)^{(n-1)/n}} \\ &= \frac{1}{(1-f)^{-1/n} - 1} \\ &= r_n(f). \end{aligned}$$

By induction, it is readily shown that if each of the first $k-1$ plays results in a loss ($k = 1, 2, \dots, n$), the gambler's fortune reduces to $1 - (1-f)^{(n-k+1)/n}$. Then the kth play is to stake

$$s_{n-k+1}(1 - (1-f)^{(n-k+1)/n}) = (1-f)^{(n-k)/n} - (1-f)^{(n-k+1)/n}$$

on the table with odds

$$r_{n-k+1}(1 - (1-f)^{(n-k+1)/n}) = \frac{1}{(1-f)^{-1/n} - 1} = r_n(f).$$

In other words, in order to maximize the probability of reaching the goal in n plays, the gambler divides the total fortune f into n parts, i.e.

$$(1-f)^{(n-k)/n} - (1-f)^{(n-k+1)/n}, \qquad k = 1, 2, \dots, n,$$

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and then stakes them sequentially on the (fixed) table with odds $1/((1-f)^{-1/n}-1)$ until he wins a bet (and reaches the goal). This optimal strategy is exactly Shepp's strategy S_α with $\alpha = 1-(1-f)^{1/n}$. (To see this, note that under the optimal strategy, the gambler's remaining fortune is never below $1-(1-f)^{1/n}$ unless he loses all the n bets (and goes broke), and note that under the strategy S_α with $\alpha = 1-(1-f)^{1/n}$, the gambler always bets on the table with odds $r = (1-\alpha)/\alpha = 1/((1-f)^{-1/n}-1)$ as long as his current (remaining) fortune is not less than $\alpha = 1-(1-f)^{1/n}$.) So the optimal strategy involves n i.i.d. Bernoulli trials with common success probability

$$w_n(f) := w\left(\frac{1}{(1-f)^{-1/n}-1}\right) = 1-(c+(1-c)(1-f)^{1/n}),$$

such that the gambler reaches the goal if at least one of the n trials results in a success, which occurs with probability $1-(1-w_n(f))^n = \mathrm{P}_n(f)$. Incidentally, since the common success probability $w_n(f)$ satisfies $nw_n(f) \to -(1-c)\log(1-f)$ as $n \to \infty$, the number of successes in n trials has, as $n \to \infty$, a Poisson distribution with mean $-(1-c)\log(1-f)$, from which it follows that

$$\lim_{n\to\infty} \mathrm{P}_n(f) = 1-\exp((1-c)\log(1-f)) = 1-(1-f)^{1-c}.$$

Property 5. A table with odds r corresponds to a two-valued random variable X with distribution $P(X=r) = w(r) = 1-P(X=-1)$ and mean $\mathrm{E}(X) = -c$ such that the gambler's fortune becomes $f+sX$ when staking $0 \le s \le f$ on the table. In the more general problem, where the gambler is allowed to choose any (not necessarily two-valued) random variable X with $\mathrm{E}(X) = -c$ and $P(X \ge -1) = 1$, Shepp [7] argued that the optimal winning probability $1-(1-f)^{1-c}$ for the (infinite-horizon) Vardi casino cannot be increased for the following reason. The set of distributions of random variables X with $\mathrm{E}(X) = -c$ and $P(X \ge -1) = 1$ is a convex set whose extreme points are the distributions of two-valued random variables. So, any random variable with mean $-c$ is a mixture of two-valued random variables. Also note that staking s on a table X with $\mathrm{E}(X) = -c$ and $P(X \in \{a,b\}) = 1$ $(-1 < a < 0, a < b)$ is equivalent to staking $(-a)s$ on a table $X/(-a)$ whose mean equals $(-c)/(-a) < -c$. The same reasoning along with a standard induction argument shows that the strategy described in Property 4 (i.e. Shepp's strategy S_α with $\alpha = 1-(1-f)^{1/n}$) remains (uniquely) optimal even when the gambler is allowed to choose, in each of the n plays, any random variable X with $\mathrm{E}(X) = -c$ and $P(X \ge -1) = 1$.

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