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ASYMPTOTICS OF IMPLIED VOLATILITY FAR FROM MATURITY

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Abstract

This note explores the behaviour of the implied volatility of a European call option far from maturity. Asymptotic formulae are derived with precise control over the error terms. The connection between the asymptotic implied volatility and the cumulant generating function of the logarithm of the underlying stock price is discussed in detail and illustrated by examples.

Keywords: Implied volatility; large deviations; saddle-point approximation

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1. Introduction

Recall that the implied volatility of a European call option with strike K and time to maturity τ is defined as the unique nonnegative solution Σ (if it exists) to the equation

$$\frac{C}{S} = \mathrm{BS}\biggl(\log\biggl(\frac{K}{S}\biggr) - r\tau, \tau\Sigma^2\biggr), \tag{1.1}$$

where C is the current price of the option, S is the price of the underlying stock (assumed to pay no dividends), r is the yield on a zero-coupon bond with the same maturity as the option, and the Black–Scholes call price function $\mathrm{BS}\colon \mathbb{R}\times[0,\infty)\to[0,1)$ is defined by

$$\mathrm{BS}(k,v) = \begin{cases} \Phi\biggl(-\dfrac{k}{\sqrt{v}} + \dfrac{\sqrt{v}}{2}\biggr) - \mathrm{e}^k\Phi\biggl(-\dfrac{k}{\sqrt{v}} - \dfrac{\sqrt{v}}{2}\biggr) & \text{if } v>0,\\ (1-\mathrm{e}^k)^+ & \text{if } v=0, \end{cases}$$

where $\Phi(x) = \int_{-\infty}^{x}(1/\sqrt{2\pi})\exp[-y^2/2]\,\mathrm{d}y$ is the standard normal distribution function and $a^+ = \max\{a,0\}$ as usual. Just as the yield of a zero-coupon bond is a dimensionless quantification of the value of the bond, the implied volatility is often used to quote an option's price. Since $v\mapsto \mathrm{BS}(k,v)$ is strictly increasing onto the interval $[(1-\mathrm{e}^k)^+,1)$, (1.1) has a solution (and, hence, the implied volatility is well defined) if and only if $(S-K\mathrm{e}^{-r\tau})^+\le C<S$.

This note provides a detailed asymptotic analysis of the implied volatility of options very far from maturity. We work within a standard no-arbitrage framework for modelling the prices of a given underlying stock and the options written on it. The motivation of this analysis is to better understand the constraints on the possible shapes of the implied volatility surface imposed by the no-arbitrage condition. Furthermore, the simple asymptotic formula can be used for model calibration, especially in models where the moment generating function is known explicitly.

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Many of the techniques used here are familiar from the theory of large deviations, including saddle-point-type approximations. However, the treatment of the borderline and irregular cases seems to be new. These results extend and refine the asymptotics found by Jacquier [7] and Lewis [10, Chapter 6].

In Section 2 we set up the notation and the standing mathematical assumptions used throughout the paper. In Section 3 we obtain an asymptotic formula for the implied volatility of an option very far from maturity, with uniform control on the error. In Section 4 we relate the long implied volatility to asymptotics of the cumulant generating function of the logarithm of the underlying stock price. Finally, in Section 5 we compute some explicit examples.

2. The mathematical set-up and notation

We consider a market model where the stock price $(S_t)_{t\geq 0}$ is modelled as a nonnegative stochastic process defined on a probability space, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. (Since most of the following results do not depend on whether the time parameter t is discrete or continuous, no distinction is made unless otherwise indicated.) Without loss of generality, we normalize by taking $S_0 = 1$, and to simplify the presentation, we assume that the prevailing risk-free interest rate is identically zero.

We make the following assumption.

Assumption 2.1. *There exists a locally equivalent measure* P *such that S is a local martingale with* $\mathrm{P}(S_t > 0) > 0$ *for all $t \geq 0$.*

Unless otherwise indicated, all probabilistic statements should be interpreted with respect to P, and the notation E will denote expectation with respect to P. It is well known that this market model admits no arbitrage.

We now introduce a family of European call options to this market. We assume that the price $C(t, T, K)$ of a call with strike K and maturity T is given by

$$C(t, T, K) = S_t - \mathrm{E}[S_T \wedge K \mid \mathcal{F}_t].$$

In this way, the call price processes $(C(t, T, K))_{t\in[0,T]}$ are local martingales, and the market augmented by these options is free of arbitrage. Furthermore, since

$$(S_t - K)^+ \leq C(t, T, K) < S_t,$$

implied volatilities are well defined.

Remark 2.1. We could have equivalently priced put options by the formula

$$P(t, T, K) = \mathrm{E}[(K - S_T)^+ \mid \mathcal{F}_t],$$

and then priced the call options by put-call parity

$$C(t, T, K) = S_t - K + P(t, T, K).$$

It is important to note that the call prices in this framework are not necessarily martingales. Indeed, note that since S is only assumed to be a nonnegative local martingale, and, hence, a

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supermartingale, we generally have the inequality

$$\begin{aligned} \mathrm{E}[C(T,T,K)\mid \mathcal{F}_t] &= \mathrm{E}[(S_T-K)^+\mid \mathcal{F}_t] \\ &= \mathrm{E}[S_T - S_T\wedge K \mid \mathcal{F}_t] \\ &\le S_t - \mathrm{E}[S_T\wedge K\mid \mathcal{F}_t] \\ &= C(t,T,K), \end{aligned}$$

with equality only if S is a true martingale. We have chosen to price the calls in this, perhaps unorthodox, way because there are models (for instance, the inverse of a three-dimensional Bessel process) in which the inequality

$$\mathrm{E}[(S_T-K)^+] < (S_0-K)^+$$

holds for large enough T. In particular, if S is a strictly local martingale then the implied volatility of a sufficiently long dated option may not be well defined if we were to price the calls by expectation. See [1] for further discussion of the technicalities that arise when S is a strictly local martingale.

We now come to the object of our study.

Definition 2.1. The (time-0) implied total variance for log-moneyness $k = \log(K/S_0)$ and time to maturity $\tau \ge 0$ is the unique nonnegative solution $V(k,\tau)$ of the equation

$$\mathrm{BS}(k, V(k,\tau)) = \frac{C(0,\tau,\mathrm{e}^k)}{S_0} = 1 - \mathrm{E}[S_\tau \wedge \mathrm{e}^k].$$

We also define the implied volatility by the formula

$$\Sigma(k,\tau) = \sqrt{\frac{V(k,\tau)}{\tau}},$$

but we will find it more convenient to express most of the results in terms of the implied total variance.

Of course, if S is given by the Black–Scholes model, that is, a geometric Brownian motion of the form $S_t = \exp[-\sigma_0^2 t/2 + \sigma W_t]$ for a Wiener process W and a constant $\sigma_0 > 0$, then $\mathrm{E}[(S_\tau - \mathrm{e}^k)^+] = \mathrm{BS}(k, \tau\,\sigma_0^2)$ and $\Sigma(k,\tau) = \sigma_0$ for all $k \in \mathbb{R}$ and $\tau > 0$.

We make one further assumption.

Assumption 2.2. *As $t \uparrow \infty$, $S_t \to 0$ almost surely.*

Recall that a nonnegative supermartingale must converge almost surely to some nonnegative random variable S_∞. The assumption that $S_\infty = 0$ is not motivated by no-arbitrage considerations, but can be justified on the following economic grounds. It is easy to see that Assumption 2.2 is equivalent to the reasonable property (exhibited by many models, including Black–Scholes) that $C(0,T,K) \to S_0$ for all $K > 0$ as $T \uparrow \infty$. Alternatively, Assumption 2.2 holds if and only if $V(k,\tau) \to \infty$ as $\tau \uparrow \infty$. See [12] for details.

Remark 2.2. Note that if $\mathrm{P}(S_\infty > 0) > 0$ then $\sup_{T>0} C(0,T,K) < S_0$. In particular, for each $k \in \mathbb{R}$, the implied total variance $V(k,\tau)$ is bounded by a finite constant, and, hence, the asymptotic formulae given below do not apply.

3. Asymptotic formula

The main result is the following.

Theorem 3.1. *The implied total variance at long maturities is given asymptotically by*

$$V(k,\tau) = -8\log \mathrm{E}[S_\tau \wedge \mathrm{e}^k] - 4\log[-\log \mathrm{E}[S_\tau \wedge \mathrm{e}^k]] + 4k - 4\log\pi + \varepsilon(k,\tau), \tag{3.1}$$

where, for all $\kappa > 0$, we have the limit

$$\sup_{-\kappa\le k\le\kappa} |\varepsilon(k,\tau)| + \sup_{-\kappa\le k_1<k_2\le\kappa} \frac{|\varepsilon(k_2,\tau)-\varepsilon(k_1,\tau)|}{k_2-k_1} \to 0 \quad \text{as } \tau\uparrow\infty.$$

The proof is contained in a series of lemmas.

Lemma 3.1. *We have*

$$\log(1-\mathrm{BS}(k,v)) = -\frac{v}{8} - \frac{1}{2}\log v + \frac{k}{2} + \frac{1}{2}\log\biggl(\frac{8}{\pi}\biggr) + \varepsilon_1(k,v),$$

where $\lim_{v\uparrow\infty}\sup_{k\in[-\kappa,\kappa]}|\varepsilon_1(k,v)| = 0$ *for all* $\kappa > 0$.

Proof. We have the calculation

$$\begin{aligned}1-\mathrm{BS}(k,v) &= \Phi[-d_1(k,v)] + \mathrm{e}^k\Phi[d_2(k,v)] \\ &= \phi[d_1(k,v)]\{U[d_1(k,v)] + U[-d_2(k,v)]\},\end{aligned}$$

where, as usual, $\phi(x) = \exp[-x^2/2]/\sqrt{2\pi}$ is the standard normal density,

$$d_1(k,v) = -\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2} \quad\text{and}\quad d_2(k,v) = -\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2},$$

and $U(x) = \Phi(-x)/\phi(x)$ is Mills' ratio.

Since $|xU(x)-1| \le 1/x^2$ for $x > 0$, we have

$$\sqrt{v}U[d_1(k,v)] \to 2 \quad\text{and}\quad \sqrt{v}U[-d_2(k,v)] \to 2$$

as $v\uparrow\infty$, uniformly for $k\in[-\kappa,\kappa]$. The result now follows.

Lemma 3.2. *Let the inverse of the Black–Scholes function* $\mathrm{IBS}\colon \mathbb{R}\times[0,1)\to\mathbb{R}_+$ *be defined implicitly by*

$$\mathrm{BS}(k,\mathrm{IBS}(k,c)) = c.$$

Then

$$\mathrm{IBS}(k,c) = -8\log(1-c) - 4\log[-\log[1-c]] + 4k - 4\log\pi + \varepsilon_2(k,c),$$

where $\lim_{c\uparrow 1}\sup_{k\in[-\kappa,\kappa]}|\varepsilon_2(k,c)| = 0$ *for all* $\kappa > 0$.

Proof. We first have the crude estimate

$$\mathrm{BS}(k,N) \le \mathrm{BS}\biggl(-\frac{N}{4},N\biggr) = \Phi\biggl(\frac{\sqrt{N}}{4}\biggr) - \mathrm{e}^{-N/4}\Phi\biggl(-\frac{3\sqrt{N}}{4}\biggr) \le 1-\mathrm{e}^{-N}$$

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for all $k \geq -N/4$ and large enough N. Therefore, $\mathrm{IBS}(k,c) \geq -\log[1-c]$ whenever $k \geq \log[1-c]/4$ and $1-c$ is small enough. In particular, $\inf_{k\in[-\kappa,\kappa]} \mathrm{IBS}(k,c) \to \infty$ as $c \uparrow 1$.

Now, Lemma 3.1 tells us that

$$\mathrm{IBS}(k,c) = -8\log[1-c] - 4\log\left[\frac{\mathrm{IBS}(k,c)}{8}\right] + 4k - 4\log\pi + 8\varepsilon_1(k, \mathrm{IBS}(k,c)).$$

Dividing both sides by $\mathrm{IBS}(k,c)$, and using the limits $\log v/v \to 0$ and $\varepsilon_1(k,v) \to 0$ uniformly, we have

$$\frac{-8\log(1-c)}{\mathrm{IBS}(k,c)} \to 1$$

uniformly in $k \in [-\kappa,\kappa]$. Since

$$\varepsilon_2(k,c) = 8\varepsilon_1(k, \mathrm{IBS}(k,c)) + 4\log\left(\frac{\mathrm{IBS}(k,c)}{-8\log(1-c)}\right),$$

the proof is complete.

Proof of Theorem 3.1. Lemma 3.2 yields

$$\varepsilon(k,\tau) = \varepsilon_2(k, 1-\mathrm{E}[S_\tau \wedge \mathrm{e}^k]).$$

But the inequality $\mathrm{E}[S_\tau \wedge \mathrm{e}^k] \leq \mathrm{E}[S_\tau \wedge \mathrm{e}^\kappa]$, the assumption $S_t \to 0$ almost surely, and the dominated convergence theorem, together imply that $\varepsilon(k,\tau) \to 0$ uniformly in $k \in [-\kappa,\kappa]$.

It remains to show that $\sup_{-\kappa\leq k_1<k_2<\kappa} |\varepsilon(k_2,\tau) - \varepsilon(k_1,\tau)|/(k_2-k_2) \to 0$. To this end, let $c(k,\tau) = 1 - \mathrm{E}[S_\tau \wedge \mathrm{e}^k]$ and note that, for each $\tau > 0$, the function $k \to c(k,\tau)$ has both left-hand and right-hand derivatives at each point, given by

$$D_- c(k,\tau) = -\mathrm{e}^k\,\mathrm{P}(S_\tau > \mathrm{e}^k) \quad \text{and} \quad D_+ c(k,\tau) = -\mathrm{e}^k\,\mathrm{P}(S_\tau \geq \mathrm{e}^k),$$

respectively. Therefore, computing the derivatives implicitly from the definition of implied total variance, we have the expression

$$D_- V(k,\tau) = 2\sqrt{V(k,\tau)}\left(\frac{\Phi\{d_2[k, V(k,\tau)]\} - \mathrm{P}(S_\tau \geq \mathrm{e}^k)}{\phi\{d_2[k, V(k,\tau)]\}}\right).$$

Of course, we have a similar expression for $D_+ V(k,\tau)$. On the other hand, differentiating (3.1) yields

$$D_- V(k,\tau) = 4 - 8\mathrm{e}^k \frac{\mathrm{P}(S_\tau \geq \mathrm{e}^k)}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]}\left(1 + \frac{1}{2\log(\mathrm{E}(S_\tau \wedge \mathrm{e}^k))}\right) + D_-\varepsilon(k,\tau).$$

The result follows from noting the following uniform limits:

$$\frac{\sqrt{v}\,\Phi[d_2(k,v)]}{\phi[d_2(v,k)]} \to 2$$

and

$$\frac{\mathrm{e}^k\,\Phi\{d_2[k, V(k,\tau)]\}}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]} \to \frac{1}{2},$$

and the inequality $\mathrm{P}(S_\tau \geq \mathrm{e}^k)/\,\mathrm{E}[S_\tau \wedge \mathrm{e}^k] \leq 1$.

A first corollary of Theorem 3.1 is the leading order term of the long implied total variance.

Corollary 3.1. *For all* $\kappa > 0$,

$$\sup_{k\in[-\kappa,\kappa]}\left|\frac{V(k,\tau)}{-8\log \mathrm{E}[S_\tau \wedge 1]} - 1\right| \to 0 \quad \text{as } \tau \uparrow \infty.$$

Proof. This follows quickly from Theorem 3.1 upon noting the inequality

$$\mathrm{e}^{-\kappa} \le \log\frac{S_\tau \wedge \mathrm{e}^k}{S_\tau \wedge 1} \le \mathrm{e}^{\kappa} \quad \text{for all } k \in [-\kappa,\kappa].$$

The above corollary shows that the implied volatility surface flattens at long maturities. The next corollary gives an asymptotic formula for the skew.

Corollary 3.2. *We have*

$$D_\pm V(k,\tau) = 4\left(\frac{\mathrm{E}[S_\tau \mathbf{1}_{\{S_\tau<\mathrm{e}^k\}} - \mathrm{e}^k \mathbf{1}_{\{S_\tau>\mathrm{e}^k\}}] \pm \mathrm{e}^k\,\mathrm{P}(S_\tau = \mathrm{e}^k)}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]}\right) + \varepsilon_\pm(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]}|\varepsilon_\pm(k,\tau)| \to 0$ *as* $\tau \uparrow \infty$ *for all* $\kappa > 0$. *In particular, we have the following bound:*

$$\limsup_{\tau\uparrow\infty} \max\{|D_- V(k,\tau)|, |D_+ V(k,\tau)|\} \le 4. \tag{3.2}$$

Remark 3.1. A consequence of the corollary is that the implied volatility flattens in the precise sense that

$$\sup_{-\kappa\le k_1<k_2<\kappa} |\Sigma(k_1,\tau) - \Sigma(k_2,\tau)| \to 0 \quad \text{as } \tau \uparrow \infty,$$

where $\Sigma(k,\tau) = \sqrt{V(k,\tau)/\tau}$. This is a model-independent result, and is not a consequence, for instance, of some notion of ergodicity. This flattening phenomenon has been noticed before: Gatheral [5] has shown that the gradient of the implied volatility, if it exists, decays pointwise like $1/\tau$. See also [9]. Equation (3.2) was established in [12], where the constant 4 was shown to be sharp.

4. The cumulant generating function

From Corollary 3.1 and the simple inequality $S \wedge 1 \le S^p$ which holds for all $0 \le p \le 1$ and $S \ge 0$, we obtain the bound

$$\liminf_{\tau\uparrow\infty} \frac{V(k,\tau)}{-8\inf_{0\le p\le 1}\log \mathrm{E}[S_\tau^p]} \ge 1.$$

In this section we explore conditions under which the above bound can be strengthened to equality. Indeed, we show how the behaviour of the long implied total variance is related to the cumulant generation function of the log stock price.

Definition 4.1. The moment generating function $M_\tau : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$M_\tau(p) = \mathrm{E}[S_\tau^p \mathbf{1}_{\{S_\tau>0\}}] \quad \text{for all } \tau \ge 0.$$

The cumulant generating function $\Lambda_\tau : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Lambda_\tau(p) = \log M_\tau(p) \quad \text{for all } \tau \ge 0.$$

Note that $M_\tau(0) = \mathrm{P}(S_\tau > 0)$ and $M_\tau(1) = \mathrm{E}[S_\tau]$, so by Assumption 2.1 and Hölder's inequality, $M_\tau(p)$ takes values in $(0, 1]$ for all $p \in [0, 1]$. In particular, M_τ is finite valued on the open interval, and, hence, can be extended analytically to the vertical strip $\{p + \mathrm{i}q : 0 < p < 1,\ q \in \mathbb{R}\}$ in the complex plane. Furthermore, there exists a neighbourhood $\{p+\mathrm{i}q : 0 < p < 1,\ |q| < \varepsilon\}$ on which this extension of M_τ is nonzero and so Λ_τ can be extended to the principal branch of the logarithm of M_τ.

Roughly speaking, the asymptotics of the implied total variance depends on the location of the minimum of the convex function Λ_τ. Essentially, three types of behaviour are possible, which we call the regular, borderline, and irregular cases below. Note that if the stock price S is strictly positive and a true martingale, then the regular case covers most of the interesting examples. However, there are many popular models in which either S hits zero in finite time with positive probability, or is a strictly local martingale, or both, and the borderline and irregular cases become important.

4.1. The regular case

In this subsection we now make the following assumption.

Assumption 4.1. *There exist a* $0 < p^* < 1$ *and a positive increasing function* C *with* $C(\tau) \uparrow \infty$ *such that*

$$\Lambda_\tau\biggl(p^* + \mathrm{i}\frac{\theta}{C(\tau)}\biggr) - \Lambda_\tau(p^*) \to -\frac{\theta^2}{2}$$

as $\tau \uparrow \infty$ *for all real* θ.

Remark 4.1. The above assumption can be verified in practice as follows. Suppose that there exists a strictly convex function $\bar{\Lambda}$ on $(0, 1)$ which can be extended to an analytic function on $\{0 < p < 1,\ |q| < \varepsilon\}$ for some $\varepsilon > 0$ such that

$$\frac{1}{\tau}\Lambda_\tau(p + \mathrm{i}q) \to \bar{\Lambda}(p + \mathrm{i}q)$$

as $\tau \uparrow \infty$ for all $0 < p < 1$ and $|q| < \varepsilon$. Furthermore, suppose that $p^* \in (0, 1)$ is the unique minimizer of $\bar{\Lambda}$, so that in particular, we have $\bar{\Lambda}'(p^*) = 0$.

Letting $\bar{\Lambda}''(p^*) = a^2 > 0$, and

$$\frac{\Lambda_\tau(p^* + \mathrm{i}q)}{\tau} - \bar{\Lambda}(p^* + \mathrm{i}q) = \beta_\tau(q),$$

we have

$$\Lambda_\tau\biggl(p^* + \mathrm{i}\frac{\theta}{a\sqrt{\tau}}\biggr) - \Lambda_\tau(p^*) = \tau\biggl[\bar{\Lambda}\biggl(p^* + \mathrm{i}\frac{\theta}{a\sqrt{\tau}}\biggr) - \bar{\Lambda}(p^*)\biggr] + \theta\frac{\sqrt{\tau}}{a}\beta'_\tau\biggl(\frac{\hat{\theta}}{a\sqrt{\tau}}\biggr)$$

for some $|\hat{\theta}| \le |\theta|$ by the mean value theorem. Assuming that $\sqrt{\tau}\beta'_\tau(\theta/\sqrt{\tau}) \to 0$ uniformly in θ on compacts, Assumption 4.1 is satisfied with $C(\tau) = a\sqrt{\tau}$ since the first bracketed term above converges to $-\theta^2/2$ by Taylor's theorem.

In this section we will let

$$\phi_\tau(\theta) = \frac{1}{\sqrt{2\pi}}\frac{M_\tau(p^* + \mathrm{i}\theta/C(\tau))}{M_\tau(p^*)}.$$

By assumption, $\phi_\tau(\theta) \to \phi(\theta) = \exp[-\theta^2/2]/\sqrt{2\pi}$, the standard normal density, for each $\theta \in \mathbb{R}$.

Before stating and proving the main result of this section, we need a further assumption. We have assumed that $S_t \to 0$ almost surely, which implies that $\Lambda_\tau(p) \to -\infty$ for each $p \in (0, 1)$, since the process $(S_t^p)_{t\geq 0}$ is uniformly integrable. In this section we make the following stronger technical assumption.

Assumption 4.2. *As* $\tau \uparrow \infty$,

$$\frac{C(\tau)}{\Lambda_\tau(p^*)} \to 0.$$

Remark 4.2. In most examples $C(\tau) \sim a\sqrt{\tau}$ and $\Lambda_\tau(p^*) \sim b\tau$ for constants $a, b > 0$, so that Assumption 4.2 is satisfied.

Our first result gives the leading order of long implied total variance with no other assumption on the cumulant generating function. The proof of this result is similar to the proof of the classical Cramér large deviation principle; see Chapter 5.11 of [6], for instance. The following result appears (without the uniform control of the error term) in [13] as an application of the Cramér theorem in the case where $\log S$ has independent stationary increments. This result also appears in Chapter 6 of [10] for stochastic volatility models under a heuristic asymptotic factorization assumption on the function M_τ.

Theorem 4.1. *As* $\tau \uparrow \infty$,

$$\sup_{k\in[-\kappa,\kappa]} \left| \frac{V(k,\tau)}{-8\Lambda_\tau(p^*)} - 1 \right| \to 0.$$

Proof. By the inequality $S_\tau \wedge 1 \leq S_\tau^p$, which holds for all $0 < p < 1$, we have the upper bound

$$\log \mathrm{E}[S_\tau \wedge 1] \leq \Lambda(p^*).$$

Now, for each $\tau \geq 0$, let X_τ be a random variable such that $\mathrm{E}[\exp[\mathrm{i}\theta X_\tau]] = M_\tau(p^* + \mathrm{i}\theta/C(\tau))/M_\tau(p^*)$. Note that by Assumptions 4.1 the distribution of X_τ converges to that of a standard normal random variable. For each $b > 0$, we have

$$\begin{aligned}
\log \mathrm{E}[S_\tau \wedge 1] &= \Lambda_\tau(p^*) + \log \mathrm{E}[\exp[-p^* C(\tau) X_\tau] \exp[C(\tau)X_\tau] \wedge 1] \\
&\geq \Lambda_\tau(p^*) + \log \mathrm{E}[\exp[-p^* C(\tau) X_\tau](\exp[C(\tau)X_\tau] \wedge 1)\, \mathbf{1}_{\{X_\tau < -b\Lambda_\tau(p^*)/C(\tau)\}}] \\
&\geq \Lambda_\tau(p^*)(1 + p^* b) + \log \mathrm{E}[(\exp[C(\tau)X_\tau] \wedge 1)\, \mathbf{1}_{\{X_\tau < -b\Lambda_\tau(p^*)/C(\tau)\}}].
\end{aligned}$$

Since $(\exp[C(\tau)X_\tau] \wedge 1)\, \mathbf{1}_{\{X_\tau < -b\Lambda_\tau(p^*)/C(\tau)\}}$ is bounded and converges in distribution to a Bernoulli random variable with parameter $\frac{1}{2}$ by Assumption 4.2, we have

$$\limsup_{\tau\uparrow\infty} \left| \frac{\log \mathrm{E}[S_\tau \wedge 1]}{\Lambda_\tau(p^*)} - 1 \right| \leq bp^*.$$

Now let $b \downarrow 0$, and apply Corollary 3.1.

If we supplement this pointwise convergence, $\phi_\tau \to \phi$, with some sort of uniform integrability condition, we can get good estimates of the long implied volatility.

Theorem 4.2. *If*

$$\int_{-\infty}^{\infty} \frac{|\phi_\tau(\theta)|}{1 + \theta^2/C(\tau)^2}\, \mathrm{d}\theta \to 1$$

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then

$$V(k,\tau) = -8\Lambda_\tau(p^*) + 4k(2p^*-1) + 8\log\biggl[\frac{C(\tau)p^*(1-p^*)}{\sqrt{-\Lambda_\tau(p^*)/2}}\biggr] + \delta(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]} |\delta(k,\tau)| \to 0$ *as* $\tau \uparrow \infty$ *for each* $\kappa > 0$.

Remark 4.3. Based on the theorem above, we might guess that the implied total variance is always approximately affine in the log-moneyness for long maturities. We will see in Section 5 that this conjecture is false. The symmetric binomial model is affine in the regular case with $p^* = \frac{1}{2}$, but the long implied total variance is not affine. Of course, for this example, the integrability condition in Theorem 4.2 fails to hold.

We begin with a lemma.

Lemma 4.1. *The following identity is valid for all* $p \in (0,1)$, $k \in \mathbb{R}$, *and* $\tau \geq 0$*:*

$$\mathrm{E}[S_\tau \wedge \mathrm{e}^k] = \frac{\mathrm{e}^{k(1-p)}}{2\pi}\int_{-\infty}^{\infty} \frac{M_\tau(p+\mathrm{i}y)\mathrm{e}^{-\mathrm{i}ky}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y.$$

Proof. It is straightforward to show by contour integration that the formula

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}zy}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y = 2\pi\begin{cases}\mathrm{e}^{-pz} & \text{if } z \geq 0,\\ \mathrm{e}^{(1-p)z} & \text{if } z < 0\end{cases}$$

holds for all real z. Furthermore, since

$$\int_{-\infty}^{\infty} \mathrm{E}\biggl|\frac{\exp[(p+\mathrm{i}y)\log S_\tau - \mathrm{i}ky]\,\mathbf{1}_{\{S_\tau>0\}}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\biggr|\,\mathrm{d}y \leq M_\tau(p)\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{p(1-p)+y^2} < \infty,$$

Fubini's theorem implies that

$$\begin{aligned}
&\frac{\mathrm{e}^{k(1-p)}}{2\pi}\int_{-\infty}^{\infty}\frac{M_\tau(p+\mathrm{i}y)\,\mathrm{e}^{-\mathrm{i}ky}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y\\
&\quad= \frac{1}{2\pi}\mathrm{E}\biggl[\int_{-\infty}^{\infty}\frac{\exp[p\log S_\tau + k(1-p) + \mathrm{i}y(\log S_\tau - k)]\,\mathbf{1}_{\{S_\tau>0\}}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y\biggr]\\
&\quad= \mathrm{E}[S_\tau \wedge \mathrm{e}^k].
\end{aligned}$$

Remark 4.4. Note that, for all real z, contour integration yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}zy}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y = 2\pi\begin{cases}-\mathrm{e}^{-pz}(\mathrm{e}^z-1)^+ & \text{if } p > 1,\\ \mathrm{e}^{-pz}(1-\mathrm{e}^z)^+ & \text{if } p < 0.\end{cases}$$

In particular, the above proof cannot be valid if $p^* > 1$ or $p^* < 0$. We will see shortly that, in fact, genuinely different behaviour arises in these cases.

Proof of Theorem 4.2. By Lemma 4.1 and the change of variables $\theta = yC(\tau)$, we have

$$\mathrm{E}[S_\tau \wedge \mathrm{e}^k] = \frac{\exp[k(1-p^*) + \Lambda_\tau(p^*)]}{p^*(1-p^*)C(\tau)\sqrt{2\pi}}\int_{-\infty}^{\infty} f_\tau(\theta,k)\,\mathrm{d}\theta,$$

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where

$$f_\tau(\theta,k)=\frac{\phi_\tau(\theta)\mathrm{e}^{-\mathrm{i}k\theta/C(\tau)}}{[1+\mathrm{i}\theta/(p^*C(\tau))][1-\mathrm{i}\theta/((1-p^*)C(\tau))]}.$$

Define auxiliary functions g_τ and h_τ by

$$g_\tau(\theta)=\frac{\phi_\tau(\theta)}{1+\theta^2/C(\tau)^2}$$

and

$$h_\tau(\theta)=\frac{\mathrm{e}^{\mathrm{i}k\theta/C(\tau)}f_\tau(\theta,k)}{g_\tau(\theta)}=\frac{1+\theta^2/C(\tau)^2}{[1+\mathrm{i}\theta/(p^*C(\tau))][1-\mathrm{i}\theta/((1-p^*)C(\tau))]}.$$

First note that $g_\tau\to\phi$ and $h_\tau\to 1$ pointwise. We have the computation

$$\begin{aligned}|f_\tau(\theta,k)-\phi(\theta)|&=|g_\tau(\theta)\mathrm{e}^{-\mathrm{i}k\theta/C(\tau)}h_\tau(\theta)-\phi(\theta)|\\&\le|g_\tau(\theta)-\phi(\theta)||h_\tau(\theta)|+\phi(\theta)|\mathrm{e}^{-\mathrm{i}k\theta/C(\tau)}-1|+\phi(\theta)|h_\tau(\theta)-1|\\&\le|g_\tau(\theta)-\phi(\theta)|+\phi(\theta)\bigg(\frac{\kappa|\theta|}{C(\tau)}\wedge 2\bigg)+\phi(\theta)|h_\tau(\theta)-1|\end{aligned}$$

for all $k\in[-\kappa,\kappa]$, where we have used the inequalities $|h_\tau(\theta)|\le 1$ and $|\mathrm{e}^{\mathrm{i}x}-1|\le|x|\wedge 2$. All three of the terms above vanish pointwise as $\tau\uparrow\infty$, and the second and third terms are dominated by integrable functions. Furthermore, since

$$\int_{-\infty}^{\infty}|g_\tau(\theta)|\,\mathrm{d}\theta\to 1=\int_{-\infty}^{\infty}\phi(\theta)\,\mathrm{d}\theta$$

by hypothesis, we have the convergence

$$\int_{-\infty}^{\infty}|g_\tau(\theta)-\phi(\theta)|\,\mathrm{d}\theta\to 0$$

by Scheffé's theorem.

If we write $\int_{-\infty}^{\infty}f_\tau(\theta,k)\,\mathrm{d}\theta=1+\delta_1(k,\tau)$, the above computation shows that

$$\sup_{k\in[-\kappa,\kappa]}|\delta_1(k,\tau)|\le\int_{-\infty}^{\infty}\sup_{k\in[-\kappa,\kappa]}|f_\tau(\theta,k)-\phi(\theta)|\,\mathrm{d}\theta\to 0$$

by the dominated convergence theorem.

Now, applying Theorem 3.1 we have

$$\begin{aligned}\delta(k,\tau)&=\varepsilon(k,\tau)-8\log[1+\delta_1(k,\tau)]\\&\quad-4\log\bigg[1+\frac{k(1-p^*)-\log[p^*(1-p^*)C(\tau)\sqrt{2\pi}/(1+\delta_1(k,t))]}{\Lambda_\tau(p^*)}\bigg],\end{aligned}$$

and the conclusion follows since $\log[C(\tau)]/\Lambda_\tau(p^*)\to 0$ by Assumption 4.2.

Remark 4.5. The asymptotic formula appearing in Theorem 4.2 is essentially Equation (3.8) in Chapter 6 of Lewis's book [10] under a suggestive, if not completely rigorous, assumption of the form $M_\tau(p)\approx\mathrm{e}^{-\lambda(p)\tau}u(p)$. Under a similar assumption, Jacquier [7] showed that Lewis's formula as stated cannot be correct since the constant term is missing. The main contribution of this paper is both the explicit sufficient condition on the function Λ_τ under which the formula holds and the uniform control on the error term.

We can now study the skew. If the distribution of S_τ is continuous, the smile is differentiable, and intuitively, $DV(k,\tau) \to 4(2p^* - 1)$ by Theorem 4.2. The following theorem makes this intuition precise.

Theorem 4.3. *Suppose that the distribution of S_τ is continuous for all $\tau \geq 0$. If*

$$\int_{-\infty}^{\infty} \frac{|\phi_\tau(\theta)|}{1 + |\theta|/C(\tau)}\,\mathrm{d}\theta \to 1$$

then

$$DV(k,\tau) = 4(2p^* - 1) + \delta'(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]} |\delta'(k,\tau)| \to 0$ *as* $\tau \uparrow \infty$ *for each* $\kappa > 0$.

Remark 4.6. Note that if the asymptotic formula $DV(k,\tau) \to 4(2p - 1)$ holds for some p then, by Corollary 3.2, we must have $0 \leq p \leq 1$.

The proof of this result closely follows the proof of Theorem 4.2. Hence, we only outline the argument. Again, we begin with a lemma.

Lemma 4.2. *The following identity is valid for all $p \in (0,1)$, $k \in \mathbb{R}$, and $\tau \geq 0$:*

$$\begin{aligned}\mathrm{E}[S_\tau\, &\mathbf{1}_{\{S_\tau<\mathrm{e}^k\}} - \mathrm{e}^k\, \mathbf{1}_{\{S_\tau>\mathrm{e}^k\}}] \\ &= (2p-1)\,\mathrm{E}[S_\tau \wedge \mathrm{e}^k] - \frac{\mathrm{e}^{k(1-p)}}{\pi \mathrm{i}} \lim_{N\to\infty} \int_{-N}^{N} \frac{yM_\tau(p+\mathrm{i}y)\mathrm{e}^{-\mathrm{i}ky}}{(p+\mathrm{i}y)(1-p-\mathrm{i}y)}\,\mathrm{d}y.\end{aligned}$$

Proof. Again, contour integration and Fubini's theorem proves the lemma.

Proof of Theorem 4.3. By Lemma 4.2 and the proof of Theorem 4.2, we need only show that

$$\int_{-\infty}^{\infty} \frac{\phi_\tau(\theta)\mathrm{e}^{-\mathrm{i}k\theta/C(\tau)}\theta/C(\tau)}{[p^* + \mathrm{i}\theta/C(\tau)][(1-p^*) - \mathrm{i}\theta/C(\tau)]}\,\mathrm{d}\theta \to 0$$

uniformly in k. The given condition is sufficient for this convergence.

4.2. The borderline cases

In this section we tackle the cases where the saddle-point method outlined above still yields asymptotics. However, since these cases sit on the borderline between the regular and irregular cases, the asymptotic formulae are different. The following assumption will be in force throughout this section.

Assumption 4.3. *There exist a $p^* \in \{0,1\}$ and an increasing function C with the following properties: there exists a neighborhood of p^* on which Λ_τ is finite valued for all $\tau \geq 0$, and, hence, the mapping $y \mapsto \Lambda_\tau(p^* + yi)$ is well defined and smooth. The function is such that $C(\tau) \to \infty$ and*

$$\Lambda_\tau\biggl(p^* + \mathrm{i}\frac{\theta}{C(\tau)}\biggr) - \Lambda_\tau(p^*) \to -\frac{\theta^2}{2} \quad \textit{for all } \theta \in \mathbb{R}.$$

Curiously, for the borderline cases (and with the irregular cases treated below), no extra uniform integrability condition is required to obtain the full asymptotics.

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Theorem 4.4. *If* $p^* = 1$ *then*

$$V(k,\tau) = -8\Lambda_\tau(1) - 4\log[-\Lambda_\tau(1)] + 4k - 4\log\left[\frac{\pi}{4}\right] + \delta(k,\tau),$$

and if $p^* = 0$ *then*

$$V(k,\tau) = -8\Lambda_\tau(0) - 4\log[-\Lambda_\tau(0)] - 4k - 4\log\left[\frac{\pi}{4}\right] + \delta(k,\tau),$$

where in both cases $\sup_{k\in[-\kappa,\kappa]} |\delta(k,\tau)| \to 0$ *for all* $\kappa > 0$.

Proof. Let X_τ be a random variable with characteristic function

$$\mathrm{E}[\exp[\mathrm{i}\theta X_\tau]] = \frac{M_\tau(p^* + \mathrm{i}\theta/C(\tau))}{M_\tau(p^*)}.$$

If $p^* = 1$ then

$$\mathrm{E}[S_\tau \wedge \mathrm{e}^k] = \mathrm{E}[S_\tau]\,\mathrm{E}[1 \wedge \exp[k - C(\tau)X_\tau]].$$

But $1 \wedge \exp[k - C(\tau)X_\tau]$ is bounded and converges in distribution to a Bernoulli random variable with mean $\frac{1}{2}$. Furthermore, since

$$\mathrm{E}[1 \wedge \exp[-\kappa - C(\tau)X_\tau]] \le \mathrm{E}[1 \wedge \exp[k - C(\tau)X_\tau]] \le \mathrm{E}[1 \wedge \exp[\kappa - C(\tau)X_\tau]],$$

we have

$$\frac{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]}{\mathrm{E}[S_\tau]} \to \frac{1}{2}$$

uniformly for $k \in [-\kappa,\kappa]$, and the conclusion follows from Theorem 3.1.

Similarly, if $p^* = 0$ then

$$\mathrm{E}[S_\tau \wedge \mathrm{e}^k] = \mathrm{P}(S_\tau > 0)\,\mathrm{E}[\exp[C(\tau)X_\tau] \wedge \mathrm{e}^k],$$

and the conclusion follows as before.

We can apply Corollary 3.2 to obtain the asymptotic skew, again with no further integrability assumption.

Theorem 4.5. *Assume that the distribution of* S_τ *is continuous. If* $p^* = 1$ *then*

$$DV(k,\tau) = 4 + \delta'(k,\tau),$$

and if $p^* = 0$ *then*

$$DV(k,\tau) = -4 + \delta'(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]} |\delta'(k,\tau)| \to 0$ *for all* $\kappa > 0$.

Proof. We must estimate the quantity

$$\frac{\mathrm{E}[S_\tau \mathbf{1}_{\{S_\tau<\mathrm{e}^k\}} - \mathrm{e}^k \mathbf{1}_{\{S_\tau>\mathrm{e}^k\}}]}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]} = 1 - 2\frac{\mathrm{e}^k\,\mathrm{P}(S_\tau > \mathrm{e}^k)}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]} = 2\frac{\mathrm{E}[S_\tau \mathbf{1}_{\{S_\tau<\mathrm{e}^k\}}]}{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]} - 1.$$

As in the proof above, let X_τ have characteristic function

$$\mathrm{E}(\exp[\mathrm{i}\theta X_\tau]) = \frac{M_\tau(p^* + \mathrm{i}\theta/C(\tau))}{M_\tau(p^*)}.$$

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If $p^* = 1$ then

$$\frac{\mathrm{E}[S_\tau \mathbf{1}_{\{S_\tau < \mathrm{e}^k\}}]}{\mathrm{E}[S_\tau]} = \mathrm{P}(C(\tau)X_\tau < k) \to \frac{1}{2},$$

and if $p^* = 0$ then

$$\frac{\mathrm{P}(S_\tau > \mathrm{e}^k)}{\mathrm{P}(S_\tau > 0)} = \mathrm{P}(C(\tau)X_\tau > k) \to \frac{1}{2}.$$

In both cases the convergence is uniform in $k \in [-\kappa, \kappa]$, proving the claim.

4.3. The irregular cases

Now we deal with the irregular cases. As before, we split them into subcases.

Assumption 4.4. *There exists a p^* such that either*

1. $p^* > 1$ *and* $\Lambda_\tau(p^*) - \Lambda_\tau(1) \to -\infty$, *or*
2. $p^* < 0$ *and* $\Lambda_\tau(p^*) - \Lambda_\tau(0) \to -\infty$.

Theorem 4.6. *If $p^* > 1$ then*

$$V(\tau, k) = -8\Lambda_\tau(1) - 4\log[-\Lambda_\tau(1)] + 4k - 4\log\pi + \delta(k,\tau),$$

and if $p^* < 0$ then

$$V(\tau, k) = -8\Lambda_\tau(0) - 4\log[-\Lambda_\tau(0)] - 4k - 4\log\pi + \delta(k,\tau).$$

In both cases, $\sup_{k\in[-\kappa,\kappa]} |\delta(k,\tau)| \to 0$ *for all* $\kappa > 0$.

Proof. If $p^* > 1$, we have the inequality

$$\begin{aligned} \mathrm{E}[S_\tau] &\geq \mathrm{E}[S_\tau \wedge \mathrm{e}^k] \\ &\geq \mathrm{E}[S_\tau \wedge \mathrm{e}^{-\kappa}] \\ &= \mathrm{E}[S_\tau - (S_\tau - \mathrm{e}^{-\kappa})^+] \\ &\geq \mathrm{E}[S_\tau] - \mathrm{e}^{(p^*-1)\kappa}\,\mathrm{E}[S_\tau^{p^*}] \end{aligned}$$

for all $k \in [-\kappa, \kappa]$, where we have used the simple inequality $(a-b)^+ \leq a^p b^{1-p}$ which holds for all $a, b > 0$ and $p > 1$. Hence, if $p^* > 1$, we have the bound

$$\left|\frac{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]}{\mathrm{E}[S_\tau]} - 1\right| \leq \exp[(p^*-1)\kappa + \Lambda_\tau(p^*) - \Lambda_\tau(1)] \to 0.$$

Similarly, if $p^* < 0$, we have the inequality

$$\begin{aligned} \mathrm{e}^k\,\mathrm{P}(S_\tau > 0) &\geq \mathrm{E}[S_\tau \wedge \mathrm{e}^k] \\ &= \mathrm{E}[\mathrm{e}^k \mathbf{1}_{\{S\tau > 0\}} - (\mathrm{e}^k - S_\tau)^+ \mathbf{1}_{\{S\tau > 0\}}] \\ &\geq \mathrm{E}[\mathrm{e}^k \mathbf{1}_{\{S\tau > 0\}} - (\mathrm{e}^\kappa - S_\tau)^+ \mathbf{1}_{\{S\tau > 0\}}] \\ &\geq \mathrm{e}^k\,\mathrm{P}(S_\tau > 0) - \mathrm{e}^{(1-p^*)\kappa}\,\mathrm{E}[S_\tau^{p^*} \mathbf{1}_{\{S_\tau > 0\}}] \end{aligned}$$

for all $k \in [-\kappa, \kappa]$, where we have used the inequality $(b-a)^+ \le a^p b^{1-p}$ which holds for all $a, b > 0$ and $p < 0$. Thus, if $p^* < 0$, we have the corresponding bound

$$\left| \frac{\mathrm{E}[S_\tau \wedge \mathrm{e}^k]}{\mathrm{P}(S_\tau > 0)} - \mathrm{e}^k \right| \le \exp[(1-p^*)\kappa + \Lambda_\tau(p^*) - \Lambda_\tau(0)] \to 0.$$

The result now follows from Theorem 3.1.

In fact, we also have convergence of the skews.

Theorem 4.7. *Assume that S_τ has a continuous distribution for all $\tau \ge 0$. If $p^* > 1$ then*

$$DV(k,\tau) = 4 + \delta'(k,\tau),$$

and if $p^* < 0$ then

$$DV(k,\tau) = -4 + \delta'(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]} |\delta'(k,\tau)| \to 0$ for all $\kappa > 0$.

Proof. Note that if $p^* > 1$ then, by Chebychev's inequality,

$$0 \le \mathrm{e}^k\, \mathrm{P}(S_\tau > \mathrm{e}^k) \le \mathrm{e}^{k(1-p^*)}\, \mathrm{E}[S_\tau^{p^*}] \le \exp[\kappa(p^*-1) + \Lambda_\tau(p^*)]$$

for all $k \in [-\kappa, \kappa]$. Similarly, if $p^* < 0$, we have

$$0 \le \mathrm{E}[S_\tau \mathbf{1}_{\{S_\tau < \mathrm{e}^k\}}] \le \exp[\kappa(p^*-1) + \Lambda_\tau(p^*)].$$

The theorem now follows from Corollary 3.2 and the estimates in the proof of Theorem 4.6.

5. Examples

In this section we consider some examples to illustrate Theorem 4.2.

5.1. Models with stationary, independent increments

The easiest case to analyse is when $S_t = \exp[X_t]$, where X has stationary, independent increments such that S is a martingale. If time is continuous, we take X to be a Lévy process, but our discussion is also valid for discrete time. When X has stationary, independent increments, the cumulant generating function has the nice form

$$\Lambda_\tau(p) = \tau\, \Lambda_1(p),$$

and $\Lambda_1(0) = \Lambda_1(1) = 0$. The function Λ_1 has a unique global minimum at some point $p^* \in (0,1)$ at which $\Lambda_1'(p^*) = 0$.

Letting $a^2 = \Lambda_1''(p^*) > 0$ we have, by Taylor's theorem,

$$\tau\left[\Lambda_1\left(p^* + \mathrm{i}\frac{\theta}{a\sqrt{\tau}}\right) - \Lambda_1(p^*)\right] \to -\frac{\theta^2}{2},$$

and Assumption 4.1 is satisfied with $C(\tau) = a\sqrt{\tau}$. Clearly, Assumption 4.2 is also satisfied. In particular, the leading order behaviour of the implied volatility is given by

$$\sup_{k\in[-\kappa,\kappa]} \left| \Sigma(k,\tau)^2 + 8 \min_{p\in\mathbb{R}} \log \mathrm{E}[S_1^p] \right| \to 0.$$

Example 5.1. (*The Black–Scholes model.*) This example is simply a reality-check. Let $S_t = \exp[-\sigma_0^2 t/2 + \sigma_0 W_t]$ for a Brownian motion W. The cumulant generating function is

$$\Lambda_1(p) = \tfrac{1}{2}\sigma_0^2 p(p-1),$$

which is minimized at $p^* = \frac{1}{2}$, and we may take $C(\tau) = \sigma_0\sqrt{\tau}$ in Assumption 4.1. Now $\phi_\tau = \phi$ for all $\tau > 0$, and in particular, $\sup_{\tau>0}\phi_\tau$ is integrable. Hence, both Theorems 4.2 and 4.3 apply, and we have

$$V(k,\tau) = \sigma_0^2\tau + \delta(\tau,k)$$

and

$$DV(k,\tau) = \delta'(\tau,k).$$

Of course, for this example, $\delta(\tau,k) = 0$ identically.

5.1.1. *A sufficient condition.* Let the stock price be given by $S_t = \exp[X_t]$, where X has independent, stationary increments. As usual, let $M_1(p) = \mathrm{E}[\exp[pX_1]]$, $\Lambda_1(p) = \log M_1(p)$, and $p^* = \operatorname{argmin}\Lambda_1$. We now exhibit a sufficient condition for Theorem 4.2 to hold.

Theorem 5.1. *Suppose that, for some $b > 0$, the inequality*

$$|M_1(p^* + \mathrm{i}q)| \le \exp[-b\,(q^2 \wedge 1)]M_1(p^*)$$

holds for all $q \in \mathbb{R}$. Then the asymptotic implied total variance is given by

$$V(k,\tau) = -8\Lambda_1(p^*)\tau + 4k(2p^*-1) + 4\log\left[\frac{2\Lambda_1''(p^*)[p^*(1-p^*)]^2}{-\Lambda_1(p^*)}\right] + \delta(k,\tau),$$

where $\sup_{k\in[-\kappa,\kappa]}|\delta(k,\tau)| \to 0$ for all $\kappa > 0$.

Proof. Let $a = \sqrt{\Lambda_1(p^*)}$. Since

$$\begin{aligned}\int_{|\theta|>a\sqrt{\tau}} \frac{|\phi_\tau(\theta)|}{1+\theta^2/\tau}\,\mathrm{d}\theta &\le \frac{1}{\sqrt{2\pi}}\int_{|\theta|>a\sqrt{\tau}} \frac{\mathrm{e}^{-b\tau}}{1+\theta^2/\tau}\,\mathrm{d}\theta \\ &= \sqrt{\frac{2\tau}{\pi}}\mathrm{e}^{-b\tau}\tan^{-1}(1/a) \\ &\to 0,\end{aligned}$$

and $\mathbf{1}_{\{|\theta|\le a\sqrt{\tau}\}}\,|\phi_t(\theta)| < \exp[-b\theta^2/a^2]/\sqrt{2\pi}$ is integrable and converges pointwise to $\phi(\theta)$, we have

$$\int_{-\infty}^{\infty} \frac{|\phi_\tau(\theta)|}{1+\theta^2/\tau}\,\mathrm{d}\theta \to 1$$

by the dominated convergence theorem. Hence, Theorem 4.2 applies.

Remark 5.1. For models satisfying the hypothesis of Theorem 5.1, the long implied total variance is approximately affine in both the log-moneyness k and the time to maturity τ:

$$V(k,\tau) \approx A\tau + Bk + C.$$

In Chapter 5 of [4], it is observed, in the context of a fast-mean reverting stochastic volatility model, that such affine structure could be exploited for model calibration. Indeed, we need only regress observed values of $V(k,\tau)$ against (k,τ) for small k and large τ to obtain estimates of A, B, and C. For a model with three parameters, such as the variance gamma model studied below, the values of A, B, and C can be inverted to yield the model parameters.

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Example 5.2. (*Subordinated Brownian motion and the variance gamma process.*) For an example of a Lévy process which satisfies the sufficient condition given by Theorem 5.1, consider the following construction. Let Y be a subordinator with characteristic function

$$\mathrm{E}[\exp[\mathrm{i}qY_t]] = \exp\biggl[\mathrm{i}qta + \int_{(0,\infty)} t(\mathrm{e}^{\mathrm{i}qx} - 1)\mu(\mathrm{d}x)\biggr],$$

where $a \geq 0$ and μ is a measure such that

$$\int_{(0,\infty)} (x \wedge 1)\mu(\mathrm{d}x) < \infty.$$

Now let W be an independent Brownian motion and define a new Lévy process X by

$$X_t = \sigma W(Y_t) + \Theta Y_t + mt$$

for real constants σ and Θ such that m defined by

$$m = -a\biggl(\Theta + \frac{\sigma^2}{2}\biggr) - \int_{(0,\infty)} \biggl(\exp\biggl[\biggl(\Theta + \frac{\sigma^2}{2}\biggr)x\biggr] - 1\biggr)\mu(\mathrm{d}x)$$

is finite. Then, by construction, the process $S_t = \exp[X_t]$ defines a martingale.

The cumulant generating function of X_1 is given by

$$\Lambda_1(p) = a\sigma^2\frac{p(p-1)}{2} + \int_{(0,\infty)} \biggl(\exp\biggl[\biggl(p\Theta + \frac{p^2\sigma^2}{2}\biggr)x\biggr] - \exp\biggl[\biggl(\Theta + \frac{\sigma^2}{2}\biggr)x\biggr]\biggr)\mu(\mathrm{d}x),$$

and, hence,

$$\begin{aligned}
&\mathrm{Re}(\Lambda_1(p + \mathrm{i}q)) - \Lambda_1(p)\\
&\quad = -a\sigma^2\frac{q^2}{2} + \int_{(0,\infty)} \exp\biggl[\biggl(p\Theta + \frac{p^2\sigma^2}{2}\biggr)x\biggr]\\
&\qquad\qquad \times \biggl(\cos[(\Theta + \sigma^2 p)qx]\exp\biggl[-\frac{q^2\sigma^2 x}{2}\biggr] - 1\biggr)\mu(\mathrm{d}x)\\
&\quad \leq -a\sigma^2\frac{q^2}{2} - q^2 \wedge 1\int_{(0,\infty)} \exp\biggl[\biggl(p\Theta + \frac{p^2\sigma^2}{2}\biggr)x\biggr]\biggl(\frac{\sigma^2 x}{2 + \sigma^2 x/2}\biggr)\mu(\mathrm{d}x),
\end{aligned}$$

where we have used the inequalities $\cos\alpha < 1$ and $\mathrm{e}^{-\alpha\beta} - 1 \leq -(\alpha \wedge 1)\beta/(1 + \beta)$ for all $\alpha, \beta > 0$. Therefore, Theorem 5.1 applies, as claimed.

One realization of the above construction, popularised by Madan *et al.* [11], is when $\mu(\mathrm{d}x) = (1/\nu x)\mathrm{e}^{-x/\nu}\,\mathrm{d}x$ for some constant $\nu > 0$ and $a = 0$, so that Y is a gamma process, X is a variance gamma process, and

$$\Lambda_1(p) = \frac{p\log[1 - (\Theta + \sigma^2/2)\nu] - \log[1 - (p\Theta + p^2\sigma^2/2)\nu]}{\nu}.$$

In this case, the minimizer $p^* \in (0, 1)$ of Λ_1 can be found by solving the equation $\Lambda_1'(p) = 0$, which is equivalent to the quadratic equation

$$\frac{\sigma^2}{2}p^2 + \biggl(\Theta - \frac{\sigma^2}{\log[1 - (\Theta + \sigma^2/2)\nu]}\biggr)p - \biggl(\frac{1}{\nu} + \frac{\Theta}{\log[1 - (\Theta + \sigma^2/2)\nu]}\biggr) = 0.$$

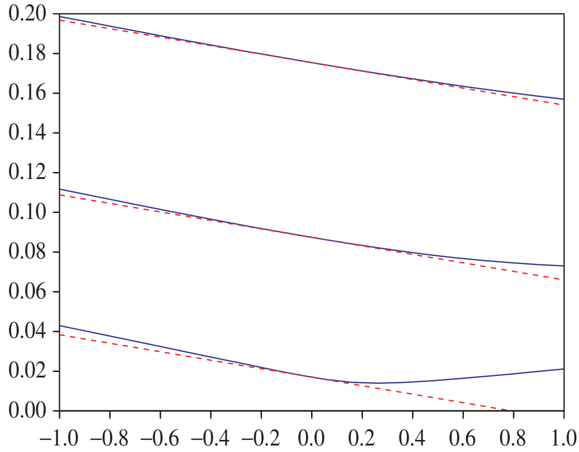

FIGURE 1: A comparison of the results of numerical integration (*solid lines*) with the affine approximation (*dashed lines*) of the variance gamma model with parameters $\sigma = 0.1213$, $\nu = 0.1686$, and $\Theta = -0.1436$ for times to maturity $\tau = 1, 5$, and 10 years. The horizontal axis is log-moneyness $k = \log[K/S_0]$ and the vertical axis is implied total variance $V(k, \tau)$.

Note that, for the variance gamma model, the call option prices can be expressed as

$$
\begin{aligned}
&\mathrm{E}[(S_\tau - \mathrm{e}^k)^+] \\
&\quad = \mathrm{E}\bigg[\exp\bigg[m\tau + \bigg(\Theta + \frac{\sigma^2}{2}\bigg)Y_\tau\bigg]\bigg(\exp[-\sigma^2 Y_\tau + \sigma W(Y_\tau)] \\
&\qquad\qquad\qquad\qquad - \exp\bigg[k - m\tau - \bigg(\Theta + \frac{\sigma^2}{2}\bigg)Y_\tau\bigg]\bigg)^+\bigg] \\
&\quad = \mathrm{E}\bigg[\exp\bigg[m\tau + \bigg(\Theta + \frac{\sigma^2}{2}\bigg)Y_\tau\bigg]\mathrm{BS}\bigg(k - m\tau - \bigg(\Theta + \frac{\sigma^2}{2}\bigg)Y_\tau, \sigma^2 Y_\tau\bigg)\bigg] \\
&\quad = \int_0^\infty \exp\bigg[m\tau + \bigg(\Theta + \frac{\sigma^2}{2}\bigg)\nu u\bigg]\mathrm{BS}\bigg(k - m\tau - \bigg(\Theta + \frac{\sigma^2}{2}\bigg)\nu u, \sigma^2 \nu u\bigg)\frac{u^{t/\nu - 1}\mathrm{e}^{-u}}{\Gamma(t/\nu)}\,\mathrm{d}u,
\end{aligned}
$$

and, hence, can be calculated by numerical integration.

In Figure 1 we compare the affine approximation as given by Theorem 5.1 with the 'true' smile given by numerical integration with the parameter values $\sigma = 0.1213$, $\nu = 0.1686$, and $\Theta = -0.1436$ as found by Madan *et al.* [11] to fit S&P 500 European option prices between 1992 and 1994. The fit is surprisingly good even when the time to maturity is five years.

5.1.2. *A counterexample.* We now consider the extremely simple binomial model where we can do the calculations very explicitly. It turns out that the long implied total variance is not approximately affine in log-moneyness k as suggested by Theorem 5.1. Of course, the sufficient condition on the cumulant generating function fails to hold in this example.

Example 5.3. (*Binomial model.*) Suppose that $S_{\tau+1} = \xi_{\tau+1} S_\tau$, where ξ is a sequence of independent random variables such that

$$
\mathrm{P}(\xi_\tau = \mathrm{e}^b) = \frac{1}{\mathrm{e}^b + 1} = 1 - \mathrm{P}(\xi_\tau = \mathrm{e}^{-b})
$$

for a constant $b > 0$. In this case, the moment generating function of $\log S_1$ is given by

$$M_1(p) = \frac{e^{bp} + e^{b(1-p)}}{e^b + 1} = \frac{\cosh[b(p-1/2)]}{\cosh(b/2)}.$$

The minimizing exponent is $p^* = \frac{1}{2}$ by symmetry. A naive application of Theorem 5.1 would predict the following formula:

$$V(k,\tau) = 8\tau \log\cosh\left(\frac{b}{2}\right) + 4\log\left(\frac{b^2}{8\log\cosh(b/2)}\right) + \delta(k,\tau).$$

However, the above formula is *not correct*! Indeed, note that $M_1(\frac{1}{2} + iy) = \cos(by)M_1(\frac{1}{2})$, so the sufficient condition in Theorem 5.1 does not hold. In this simple example we can actually compute the long implied total variance explicitly:

$$V(k,\tau) = 8\tau\log\cosh\left(\frac{b}{2}\right) - 8\log\cosh g(k,\tau) + 4\log\left(\frac{(\sinh b/2)^2}{2\log\cosh(b/2)}\right) + \delta(k,\tau),$$

where $g(\cdot,\tau)$ is the $2b$-periodic function whose restriction to the interval $(-b, b]$ is given by

$$g(k,\tau) = \begin{cases} \dfrac{|k|}{2} & \text{if } \tau \text{ is odd,} \\[2ex] \dfrac{b-|k|}{2} & \text{if } \tau \text{ is even.} \end{cases}$$

The above asymptotic formula may be regarded as something of a curiosity, as it would be hard to argue that the binomial model provides a good fit to stock price data. However, it serves as a warning that the integrability condition cannot be dropped from the statement of Theorem 4.2.

The above asymptotic formula is a consequence of the following proposition.

Proposition 5.1. *For each* $\tau \in \mathbb{N}$, *let*

$$F_\tau(y) = \int_{-\infty}^{\infty} \frac{(\cos x)^\tau \sqrt{\tau} e^{ixy}}{a^2 + x^2}\, dx.$$

Then $F_{2m+1} \to H$ *and* $F_{2m} \to H(\cdot + 1)$ *uniformly on compacts, where* H *is the* 2*-periodic function whose restriction to* $(-1, 1]$ *is given by*

$$H(y) = \sqrt{2\pi}\frac{\cosh(ay)}{a\sinh a}.$$

Proof. We have by the absolute integrability of the integrand for each fixed $\tau \in \mathbb{N}$ and $y \in \mathbb{R}$ the calculation

$$\begin{aligned} F_\tau(y) &= \sum_{n\in\mathbb{Z}} \int_{(n-1/2)\pi}^{(n+1/2)\pi} \frac{\sqrt{\tau}(\cos x)^\tau e^{ixy}}{a^2 + x^2}\, dx \\ &= \sum_{n\in\mathbb{Z}} \int_{-\pi/2}^{\pi/2} (-1)^{n\tau} \frac{\sqrt{\tau}(\cos z)^\tau (-1)^{\tau n} e^{izy + iyn\pi}}{a^2 + (z+n\pi)^2}\, dz \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{\tau}(\cos z)^\tau e^{izy} G(z,y,\tau)\, dz, \end{aligned}$$

where

$$G(z,y,\tau) = \sum_{n\in\mathbb{Z}} \frac{(-1)^{n\tau} e^{iny\pi}}{a^2 + (z+n\pi)^2}.$$

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For all τ, the series defining $G(\cdot,\cdot,\tau)$ converges absolutely and uniformly on $[-\pi/2,\pi/2]\times[-\kappa,\kappa]$, and, hence, defines a continuous function.

Make the substitution $z=\theta/\sqrt{\tau}$ in the last integral above. First note that the integrand is uniformly bounded by an integrable function of θ, since $G(\cdot,\cdot,\cdot)$ is bounded on $[-\pi/2,\pi/2]\times[-\kappa,\kappa]\times\mathbb{N}$ and the inequality $\cos z\le 1-z^2/\pi$ on $[-\pi/2,\pi/2]$ implies that

$$\left|\cos\left(\frac{\theta}{\sqrt{\tau}}\right)^{\tau}\mathbf{1}_{[-\pi\sqrt{\tau}/2,\pi\sqrt{\tau}/2]}(\theta)\right|\le\exp\left[-\frac{\theta^2}{\pi}\right].$$

Now letting τ take only odd values, we have the pointwise in $\theta\in\mathbb{R}$ uniform in $y\in[-\kappa,\kappa]$ convergence

$$\left[\cos\left(\frac{\theta}{\sqrt{\tau}}\right)\right]^{2m+1}\mathrm{e}^{\mathrm{i}y\theta/\sqrt{\tau}}G\left(\frac{\theta}{\sqrt{\tau}},y,\tau\right)\mathbf{1}_{[-\pi\sqrt{\tau}/2,\pi\sqrt{\tau}/2]}(\theta)\to\exp[-\theta^2/2]G(0,y,1),$$

and, hence, the dominated convergence theorem implies that $F_{2m+1}(y)\to\sqrt{2\pi}G(0,y,1)$ uniformly. Similarly, $F_{2m}(y)\to\sqrt{2\pi}G(0,y,0)$ uniformly.

It is now a simple matter to check that the series

$$G(0,y,1)=\sum_{n\in\mathbb{Z}}\frac{(-1)^n\mathrm{e}^{\mathrm{i}yn\pi}}{a^2+n^2\pi^2}$$

is the Fourier series for the function $(1/\sqrt{2\pi})H$. Since H is continuous and of bounded variation, its Fourier series converges everywhere. The case of even τ is similar.

To show that the long implied volatility in the binomial model is given by the announced formula, we need only work through the proof of Theorem 4.2 and substitute the integral in the above proposition with $a=b/2$ and $y=k/b$ in the appropriate place. In Figure 2 we compare the true implied total variance for the binomial model with $b=0.15$ with the approximation.

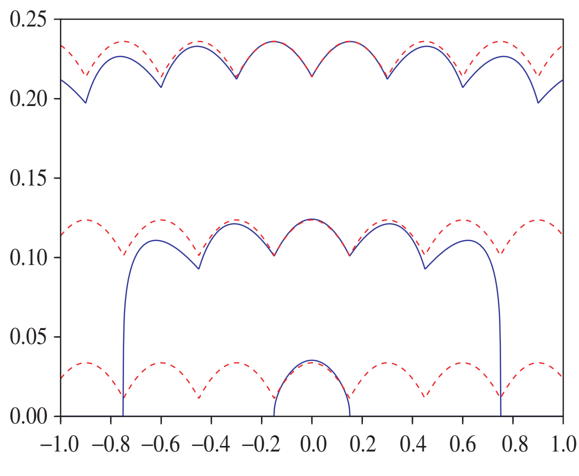

FIGURE 2: A comparison of the true implied total variance (*solid line*) with the approximation (*dashed line*) for the binomial model with $b=0.15$ for times to maturity $\tau=1,5$, and 10 years. The horizontal axis is log-moneyness $k=\log[K/S_0]$ and the vertical axis is implied total variance $V(k,\tau)$.

5.2. Affine models

In Markovian stochastic volatility models, the moment generating function of the stock price can be found generally by solving a partial differential equation. Affine models, however, have the attractive feature that the moment generating function can be found by solving a coupled family of ordinary differential equations, and in many cases are known in closed form. See [2] for a complete account of such models. We include the canonical example of an affine stochastic volatility model, the Heston model, to illustrate the technique. We only sketch the outline of the story. For full details for this example, see the recent preprint [3].

Example 5.4. (*Heston model.*) Consider the following coupled stochastic differential equation:

$$\mathrm{d}S_t = S_t\sqrt{V_t}\,\mathrm{d}W_t^{(S)}, \qquad \mathrm{d}V_t = \lambda(\theta - V_t)\,\mathrm{d}t + \zeta\sqrt{V_t}\,\mathrm{d}W_t^{(V)}$$

for correlated Wiener processes $W^{(X)}$ and $W^{(V)}$ with $\langle W^{(X)}, W^{(V)}\rangle_t = rt$, and positive constants λ, θ, and ζ.

Let

$$H(t, v, p) = \mathrm{e}^{A(t,p)v+B(t,p)},$$

where the function A solves the Riccati equation,

$$\frac{\partial}{\partial t}A(t, p) = \frac{1}{2}p(p-1) + (pr\zeta - \lambda)A(t, p) + \frac{1}{2}\zeta^2 A(t, p)^2, \qquad A(0, p) = 0,$$

and B is given by

$$B(t, p) = \lambda\theta\int_0^t A(s, p)\,\mathrm{d}s.$$

It is easy to see by Itô's formula that the process $M_t = S_t^p H(t, V_t, p)$ defines a positive local martingale. We suppose that M is a true martingale for each p in some set $\Gamma \subseteq \mathbb{R}$, so that

$$\Lambda_\tau(p) = A(\tau, p)V_0 + B(\tau, p) \quad \text{for all } p \in \Gamma.$$

Following Keller-Ressel's paper [8], we note that when $(pr\zeta - \lambda)^2 > \zeta^2 p(p-1)$, the Riccati equation has two fixed points, $A^-(p) < A^+(p)$, with $A^-(p)$ stable and $A^+(p)$ unstable. Hence, when $p \in (0, 1)$ or $pr\zeta < \lambda$, the unstable fixed point $A^+(p)$ is positive, so that $A(\tau, p)$ converges to $A^-(p)$, and in particular,

$$\bar{\Lambda}(p) = \lim_{\tau\uparrow\infty}\frac{\Lambda_\tau(p)}{\tau} = \lambda\theta A^-(p).$$

The minimizer p^* of this function $\bar{\Lambda}$ gives information about the long implied volatility, and was found in Chapter 6 of [10] to be

$$p^* = \frac{1}{2(1-r^2)\zeta}\big(\zeta - 2r\lambda + r\sqrt{\zeta^2 - 4\lambda\zeta r + 4\lambda^2}\big).$$

Note there are parameter values for which p^* defined by the above formula is such that $p^* > 1$ and $p^* r\zeta > \lambda$. (The referee has observed that the parameters $\zeta = 1$, $r = \frac{3}{4}$, and $\lambda = \frac{1}{4}$ have this property.) In these cases, it is not at all clear whether p^* satisfies Assumption 4.1 since $\Lambda_\tau(p)/\tau$ may not converge as $\tau \uparrow \infty$.

5.3. Irregular cases

We conclude with examples which illustrate what happens in the irregular case. Remember, the irregular case can only arise when the stock price is either a strictly local martingale, or hits zero in finite time with positive probability.

Example 5.5. The first example is extremely simple and is related to an example appearing in [12] verifying that the bound in Corollary 3.2 is sharp.

Let T be a random time where the distribution of $1/T$ is uniform on [0, 1], and let

$$S_t = \mathbf{1}_{\{0\le t<1\}} + t\,\mathbf{1}_{\{1\le t<T\}}.$$

Then S is a martingale with respect to its natural filtration. Now, it is easy to see that $\Lambda_\tau(p) = (p-1)\log t$ for $t \ge 1$. Letting $p^* = -1 < 0$, say, we have

$$\Lambda_\tau(p^*) - \Lambda_\tau(0) = -\log t \to -\infty,$$

so Theorem 4.6 applies. The full asymptotics are then

$$V(k,\tau) = 8\log\tau - 4\log\log\tau - 4k - 4\log\pi + \delta(\tau,k).$$

Example 5.6. (*CEV models.*) The CEV models, i.e. the models with constant elasticity of variance, are given by the stochastic differential equation

$$\mathrm{d}S_t = S_t^\alpha\,\mathrm{d}W_t.$$

It is well known that if $\alpha > 1$ then S is a strictly positive, strict local martingale. To illustrate the phenomenon, we consider the case $\alpha = 2$, corresponding to the inverse of a three-dimensional Bessel process. In this case we have

$$\mathrm{E}[S_t] = 2\Phi\left(\frac{1}{\sqrt{t}}\right) - 1,$$

so that $\Lambda_\tau(1)/\log\tau \to -\frac{1}{2}$. A routine calculation shows that $\Lambda_\tau(2)/\log\tau \to -1$, and, hence, the long implied total variance can be read off, using Theorem 4.6:

$$V(k,\tau) = 4\log\tau - 4\log\log\tau + 4k + \delta(k,\tau).$$

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