ON SEMIGROUPS OF ENDOMORPHISMS OF GENERALIZED BOOLEAN RINGS

DOUGLAS B. SMITH, JR. and JIANG LUH

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1. Introduction

Magill in [4] first proved that two Boolean rings are isomorphic if and only if their respective endomorphism semigroups are isomorphic. His proof, however, relied on topological techniques. More recently Maxson has published a proof of the above using purely algebraic techniques [5]. In this paper, structure theorems are given which allow us to extend the above result to the p^k-rings of Foster [1]. As a special case, the result is shown to apply also to p-rings. An example is given to show that a further extension to J-rings is impossible.

Throughout this paper a *p-ring* will be a ring R with unity 1_R of characteristic p, where p is prime, and having the property that $x^p = x$ for all $x \in R$. We will consider two types of p^k-rings, the type always being identified by its author's name. Let p be a prime integer and k a positive integer. Then a *p^k-ring (McCoy)* R is a ring with unity 1_R of characteristic p such that $x^{p^k} = x$ for all $x \in R$. These were first introduced in [6]. The following more restrictive definition was introduced by Foster in [1]. Again let p be a prime integer and k a positive integer. A ring R is a *p^k-ring (Foster)* if the following hold:

(i) $1_R \in R$

(ii) $x^{pk} = x$ for all $x \in R$

(iii) R has at least one subring F which is isomorphic to the Galois field of p^k elements, $GF(p^k)$, and

(iv) $1_R \in F$.

Any subring F of a p^k-ring (Foster) satisfying (iii) and (iv) is called a *normal subfield* of R.

Note that since $1_R \in F$ and F is of characteristic p, R is of characteristic p, and hence a p^k-ring (Foster) is a p^k-ring (McCoy). The reverse is not true, as illustrated by the ring $GF(2) \oplus GF(2^2)$, which is a p^k-ring (McCoy) but not a p^k-ring (Foster). Both types of p^k-rings are p-rings when $k = 1$. We observe also

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that if R is a p^k-ring (Foster) and F is a normal subfield of R, then R is an algebra over F.

A *J-ring* is any ring R for which there exists an integer $n > 1$ such that $x^n = x$ for all $x \in R$.

Each type of ring we have defined is commutative (cf. [3] page 217), so the set of idempotents R' of such a ring R is easily seen to be a semigroup under multiplication. The set of ring endomorphisms of R, *End* R, is a semigroup under composition of functions. Thinking of a p^k-ring (Foster) as an algebra over some normal subfield F, the set of algebra endomorphisms of R over F, denoted by $End_F R$, is also a semigroup under composition of functions.

The mapping $e \to \phi_e$, where $\phi_e(r) = er$ for all $r \in R$, is easily seen to embed R' in End R for each of the rings discussed above. If R is a p^k-ring (Foster) and F a normal subfield of R, then the same mapping embeds R' in $\text{End}_F R$.

2. p^k-rings

We now present some structure theorems for the p^k-rings of McCoy and Foster. McCoy in [7] has shown that if R is a p-ring, then R is isomorphic to a subdirect sum of fields $GF(p)$, and that if R is a p^k-ring (McCoy), then R is isomorphic to a subdirect sum of fields of the form $GF(p^{k_i})$. If R is a p^k-ring (McCoy) and S a homomorphic image of R, then S is a p^k-ring (McCoy). Further, if S is subdirectly irreductible, then S is isomorphic to $GF(p^t)$, where $t \mid k$.

THEOREM 2.1. *Any nonzero homomorphic image of a p^k-ring (Foster) is a p^k-ring (Foster).*

PROOF. Suppose θ: $R \to S$ is an epimorphism, where R is a p^k-ring (Foster). If $x \in S$ then obviously $x^{p^k} = x$. If F is a normal subfield of R, then necessarily $\theta(F) \simeq F \simeq GF(p^k)$. $1_R \in F$ so $1_S = \theta(1_R) \in \theta(F) \subseteq S$ and S is a p^k-ring (Foster).

The following theorem forms the basis for the main result of this paper.

THEOREM 2.2. *If R is a p^k-ring (Foster) and F a normal subfield of R, then each element $r \in R$ can be uniquely expressed in the form*

$$r = \sum_i \alpha_i x_i,$$

where the α_i are the nonzero elements of F and the x_i are idempotent elements of R such that $x_m x_n = 0$ if $m \neq n$ and $\sum_i x_i = 1_R$.

The proof of this theorem, in a somewhat more general setting, may be found in [2].

As a result of this structure theorem we have the following theorem.

THEOREM 2.3. *If R is a subdirect sum of finitely many $p_i^{k_i}$-rings (Foster) then R is isomorphic to a direct sum of some of these same rings.*

PROOF. Let R be a subdirect sum of rings $M_i (i = 1, 2, \cdots, n)$, where M_i is a

$p_i^{k_i}$-ring (Foster) containing a normal subfield $F_i \simeq GF(p_i^{k_i})$. We prove the theorem by induction on n. Clearly the theorem is true for $n = 1$. Suppose now that the theorem holds for all rings that are subdirect sums of $k - 1 \geqq 1$ rings, and suppose that R is a subdirect sum of $p_i^{k_i}$-rings (Foster) M_i $(i = 1, 2, \cdots, k)$. Let $\mu\colon R \to \sum_{i=1}^{k} \oplus M_i$ be a monomorphism and $\pi_j\colon \sum_{i=1}^{k} \oplus M_i \to M_j$ be the projection epimorphism such that $\pi_j \mu$ is an epimorphism for each $j = 1, 2, \cdots, k$. Define $T_i = \{\mu(x) \mid x \in R \text{ and } \pi_j \mu(x) = 0 \text{ for all } j \neq i\}$ for each $i = 1, 2, \cdots, k$. We consider two cases.

CASE 1. For each $i, T_i \neq \{0\}$. Then for each i there exists a nonzero $a_i \in M_i$ such that $(0, \cdots, 0, a_i, 0, \cdots, 0) \in \mu(R)$, where a_i is the ith component. Now M_i is a $p_i^{k_i}$-ring (Foster), so by 2.2, $a_i = \sum_m \alpha_m x_m$, where the α_m are the nonzero elements of F_i and the x_m the appropriate idempotent elements in M_i. Since for each m, $\alpha_m^{-1} x_m \in M_i$, there exists an $r \in R$ such that $\pi_i \mu(r) = \alpha_m^{-1} x_m$, and consequently there is an element in $\sum_{i=1}^{k} \oplus M_i$, say $(b_1^{(m)}, b_2^{(m)}, \cdots, b_i^{(m)}, \cdots, b_k^{(m)}) = \mu(r)$, where $b_i^{(m)} = \alpha_m^{-1} x_m$. Thus $(0, \cdots, 0, x_m, 0, \cdots, 0) = (0, \cdots, 0, a_i, 0, \cdots, 0)$ $(b_1^{(m)}, b_2^{(m)}, \cdots, b_i^{(m)}, \cdots b_k^{(m)}) \in \mu(R)$, where x_m is the ith component. This is true for each m, so the sum of all such elements is in $\mu(R)$. But $\sum_m x_m = 1_R$, so $(0, \cdots, 0, 1_R, 0, \cdots, 0)$, where 1_R is the ith component is in $\mu(R)$. Since i was arbitrary we have $\mu(R) = \sum_{i=1}^{k} \oplus M_i$, and R is isomorphic to a direct sum of the M_i.

CASE 2. $T_i = \{0\}$ for some i. Without loss of generality, assume $T_k = \{0\}$. We define a map ϕ of $\mu(R)$ into the direct sum $\sum_{i=1}^{k} \oplus M_i$ by $\phi(x_1, x_2, \cdots, x_{k-1}, x_k) = (x_1, x_2, \cdots, x_{k-1})$. Since $T_k = \{0\}$, ϕ is a monomorphism. Hence $\phi\mu$ is a monomorphism of R into $\sum_{i=1}^{k-1} \oplus M_i$ and $\pi_j \phi\mu$ is an epimorphism for $j = 1, 2, \cdots, k - 1$. R is thus a subdirect sum of $M_1, \cdots, M_{k-1}$, so by the inductive assumption, R is a direct sum of some of the $M_1, \cdots, M_{k-1}$.

COROLLARY 2.4. (*Foster*) *If R is a finite p^k-ring* (*Foster*), *then R is isomorphic to a direct sum of finitely many copies of $GF(p^k)$.*

PROOF. This is an immediate consequence of Theorem 2.3 and that of the note which precedes Theorem 2.1.

3. Endomorphisms of p^k-rings

Throughout this section let p be a fixed prime integer, k a fixed positive integer, R and S p^k-rings (Foster) with normal subfields F and G respectively, and R' and S' the semigroups of idempotents of R and S, respectively. We will show that if $\text{End}_F R \simeq \text{End}_G S$ as semigroups, then $R' \simeq S'$ as semigroups.

We will identify R' and S' with their isomorphic images in $\text{End}_F R$ and $\text{End}_G S$, respectively. The elements of R' will be denoted by ϕ_r, where $r = r^2 \in R$, and those of S' by ψ_S, where $s = s^2 \in S$. Specifically the zero and unit el.m nts of R' will be ϕ_0 and ϕ_1, while those of S' will be ψ_0 and ψ_1.

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In some of the proofs that follow, we will refer, for example, to $\phi_e + \phi_r$, where $e = e^2$, $r = r^2 \in R$, although addition is not defined in End R. We can legitimately do this if we consider ϕ_e and ϕ_r as elements of the ring End$(R, +)$, where we are considering all endomorphisms of the abelian group $(R, +)$.

Let π: $\mathrm{End}_F R \to \mathrm{End}_G S$ be a semigroup isomorphism.

LEMMA 3.1. *$\pi(\phi_0) = \psi_0$ and $\pi(\phi_1) = \psi_1$.*

LEMMA 3.2. *If $\psi_s \in S'$, $\phi = \pi^{-1}(\psi_s)$, and $\phi_e \in R'$, then $\phi\phi_e = \phi_e\phi$.*

PROOF. Note that $\phi_1 - \phi_e = \phi_{1-e} \in R' \subseteq \mathrm{End}_F R$, so $\phi_e\phi(\phi_1 - \phi_e) \in \mathrm{End}_F R$ We show now that $\phi_e\phi(\phi_1 - \phi_e) = \phi_0$.

$$\begin{aligned}[\pi(\phi_e\phi(\phi_1 - \phi_e))](1_S) &= [\pi(\phi_e)\psi_s\pi(\phi_1 - \phi_e)](1_S) = \pi(\phi_e)\{s \cdot [\pi(\phi_1 - \phi_e)](1_S)\}\\ &= [\pi(\phi_e)(s)][\pi(\phi_e - \phi_e)(1_S)] = [\pi(\phi_e)(s)][\psi_0(1_S)] = 0.\end{aligned}$$

Thus $\pi(\phi_e\phi(\phi_1 - \phi_e)) = \psi_0$ and hence $\phi_e\phi(\phi_1 - \phi_e) = \phi_0$, so $\phi_e\phi = \phi_e\phi\phi_e$. Similarly $\phi\phi_e = \phi_e\phi\phi_e$. Thus, $\phi\phi_e = \phi_e\phi$.

LEMMA 3.3. *If $\psi_s \in S'$* and *$\phi = \pi^{-1}(\psi_s)$* then *$\phi(ee') = e\phi(e')$* for all *$e = e^2$, $e' = (e')^2 \in R$.*

PROOF. $\phi(ee') = \phi\phi_e(e') = \phi_e\phi(e') = e\phi(e')$ by 3.2 since $\phi_e \in R'$.

LEMMA 3.4. *If $\psi_s \in S'$* and *$\phi = \pi^{-1}(\psi_s)$*, then *$\phi(rr') = \phi(r)r'$*, for all *r, $r' \in R$.*

PROOF. By 2.2 we may uniquely write r and r' as $r = \sum_i \alpha_i x_i$, $r' = \sum_k \beta_j x'_j$, where $\alpha_i, \beta_j \in F$ and $x_i = (x_i)^2, x'_j = (x'_j)^2 \in R$ are such that $x_m x_n = x'_m x'_n = 0$ if $m \neq n$ and $\sum_i x_i = \sum_j x_j = 1_R$.
Thus

$$\begin{aligned}\phi(rr') &= \phi\left(\sum_i \alpha_i x_i \sum_j \beta_j x'_j\right) = \phi\left(\sum_{i,j} \alpha_i\beta_j x_i x'_j\right)\\ &= \sum_{i,j} \phi(\alpha_i\beta_j)\phi(x_i x'_j) \text{ since } \phi \in \mathrm{End}_F R\\ &= \sum_{i,j} \phi(\alpha_i)\beta_j\,\phi(x_i)x'_j \text{ since } \phi \in \mathrm{End}_F R \text{ and by 3.3}\\ &= \sum_i \phi(\alpha_i x_i)\sum_j \beta_j x'_j = \phi(r)r'.\end{aligned}$$

LEMMA 3.5. *If $\psi_s \in S'$ and $\phi = \pi^{-1}(\psi_s)$, then $\phi \in R'$.*

PROOF. If $r \in R$ then $\phi(r) = \phi(1_R \cdot r) = \phi(1_R) \cdot r$ by 3.4. Thus if $e = \phi(1_R)$ then $e = e^2$ and $\phi = \phi_e \in R'$.

THEOREM 3.6. *If $\mathrm{End}_F R \simeq \mathrm{End}_G S$ then $R' \simeq S'$.*

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PROOF. By 3.5, $\pi^{-1}(S') \subseteq R'$ so $S' \subseteq \pi(R')$. By a similar argument we can show that $\pi(R') \subseteq S'$, giving $S' \subseteq \pi(R') \subseteq S'$, so $\pi(R') = S'$. Since π preserves multiplication and is one-one, the theorem is proved.

4. The main theorem

Let p be a fixed prime integer, k a fixed positive integer, and R and S p^k-rings (Foster) with normal subfields F and G, respectively. Let R' and S' be the semi-groups of idempotents of R and S, respectively, and let $\pi: R' \to S'$ be a semi-group isomorphism. Since $F \simeq GF(p^k) \simeq G$, let $\sigma: F \to G$ be a field isomorphism. We will use the next two lemmas freely, without specific reference to them.

LEMMA 4.1. *$\pi(0) = 0$ and $\pi(1_R) = 1_S$.*

PROOF. The proof is basically the same as that of 3.1.

LEMMA 4.2. *If $x \in R'$ then $\pi(1_R - x) = 1_S - \pi(x)$.*

PROOF. Trivially $1_R - x \in R'$ if $x \in R'$. Suppose $\pi(1_R - x) = 1_S - s$ for some $s \in S$. Then since $\pi(1_R - x) \in S'$, $s = 1_S - \pi(1_R - x) \in S'$. Hence $s = \pi(y)$ for some $y \in R'$, i.e.,

$$\pi(1_R - x) = \pi(1_R) - \pi(y), \tag{1}$$

so that by multiplying by $\pi(x)$ we have $0 = \pi(x) - \pi(xy)$. Since π is one-one, $x = xy$. Multiplying (1) by $\pi(y)$ gives $y = xy$, so $x = y$.

LEMMA 4.3. *Suppose that $\alpha \in F$, $x \in R'$, and $\alpha x \in R'$. Then $\pi(\alpha x) = \sigma(\alpha)\pi(x)$.*

PROOF. If $x = 0$ the conclusion is obvious. Suppose $x \neq 0$. Then since $\alpha x, x \in R'$, $\alpha x = (\alpha x)^2 = \alpha^2 x$, so

$$(\alpha^2 - \alpha)x = 0. \tag{2}$$

Now since $\alpha^2 - \alpha \in F$, $\alpha^2 - \alpha = 0$, else we could multiply (2) by $(\alpha^2 - \alpha)^{-1}$ and obtain $x = 0$. But $\alpha(\alpha - 1) = 0$ implies $\alpha = 0$ or $\alpha = 1$ because F is a field. Since σ is a field isomorphism, $\sigma(0) = 0$ and $\sigma(1) = 1$, the conclusion following immediately.

LEMMA 4.4. *Let $x_1, x_2, \cdots, x_n \in R'$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$. If*

$$\sum_{i=1}^{n} \alpha_i(x_1 x_i) \in R' \text{ then } \pi\left[\sum_{i=1}^{n} \alpha_i(x_1 x_i)\right] = \sum_{i=1}^{n} \sigma(\alpha_i)\pi(x_1 x_i).$$

PROOF. We proceed by induction. By 4.3 the conclusion holds for $n = 1$. Suppose the lemma is true for $n = k$. Then

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$$\begin{aligned}\pi\left[\sum_{i=1}^{k+1}\alpha_i(x_1x_i)\right] &= \pi\left[\sum_{i=1}^{k+1}\alpha_i(x_1x_i)\right][\pi(x_1x_{k+1})+\pi(1_R)-\pi(x_1x_{k+1})]\\ &= \pi\left[\left(\sum_{i=1}^{k+1}\alpha_ix_1x_i\right)(x_1x_{k+1})\right]+\pi\left[\left(\sum_{i=1}^{k+1}\alpha_ix_1x_i\right)(1_R-x_1x_{k+1})\right]\\ &= \pi\left[\sum_{i=1}^{k+1}\alpha_ix_1x_ix_{k+1}\right]+\pi\left[\sum_{i=1}^{k+1}\alpha_ix_1x_i-\sum_{i=1}^{k+1}\alpha_ix_1x_ix_{k+1}\right]\\ &= \pi\left[(\alpha_1+\alpha_{k+1})x_1x_{k+1}+\sum_{i=2}^{k}\alpha_ix_1x_ix_{k+1}\right]\\ &\quad+\pi\left[\sum_{i=1}^{k}\alpha_ix_1x_i(1_R-x_{k+1})\right].\end{aligned}$$

Since each of the quantities enclosed by brackets is in R' and in a form which allows us to use our inductive assumption, we do to obtain

$$\begin{aligned}&\sigma(\alpha_1+\alpha_{k+1})\pi(x_1x_{k+1})+\sum_{i=2}^{k}\sigma(\alpha_i)\pi(x_1x_ix_{k+1})+\sum_{i=1}^{k}\sigma(\alpha_i)\pi(x_1x_i)(1_S-\pi(x_{k+1}))\\ &=\sum_{i=1}^{k+1}\sigma(\alpha_i)\pi(x_1x_i)\end{aligned}$$

after cancellation, using 4.2 and the additivity of σ.

LEMMA 4.5. *If* $x_1, x_2, \cdots, x_n \in R'$, $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$, and $\sum_{i=1}\alpha_ix_i \in R'$, then $\pi[\sum_{i=1}^{n}\alpha_ix_i] = \sum_{i=1}^{n}\sigma(\alpha_i)\pi(x_i)$.

PROOF. Again we proceed by induction. The lemma is true for $n = 1$ by 4.3. We now suppose the lemma to be true for $n = k$. Then following a technique similar to the proof of 4.4 we have

$$\begin{aligned}\pi\left[\sum_{i=1}^{k+1}\alpha_ix_i\right] &= \left[\pi\left(\sum_{i=1}^{k+1}\alpha_ix_i\right)\right][\pi(x_1)+\pi(1_R)-\pi(x_1)] = \pi\left[\sum_{i=1}^{k+1}\alpha_ix_ix_1\right]\\ &\quad+\pi\left[\sum_{i=1}^{k+1}\alpha_ix_i(1_R-x_1)\right]\\ &= \sum_{i=1}^{k+1}\sigma(\alpha_i)\pi(x_ix_1)+\pi\left[\sum_{i=2}^{k+1}\alpha_ix_i(1_R-x_1)\right] \quad \text{by 4.4 and cancellations}\\ &= \sum_{i=1}^{k+1}\sigma(\alpha_i)\pi(x_ix_1)+\sum_{i=2}^{k+1}\sigma(\alpha_i)\pi[x_i(1_R-x_1)] \quad \text{by the inductive hypothesis}\\ &= \sum_{i=1}^{k+1}\sigma(\alpha_i)\pi(x_i) \text{ after cancellations.}\end{aligned}$$

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THEOREM 4.6. *If R' and S' are isomorphic as semigroups, then R and S are isomorphic as rings.*

PROOF. We define a function $\pi^*: R \to S$ as follows: If $r \in R$ has as its unique representation $r = \sum_i \alpha_i x_i$ guaranteed by 2.2, let

$$\pi^*(r) = \sum_i \sigma(\alpha_i)\pi(x_i).$$

Note that the image of r is indeed a legitimate representation of an element of S — in particular $\sum_i \pi(x_i) = 1_S$ by 4.1 and 4.5. By the uniqueness of the representation of r, π^* is a one-one function and obviously onto.

To show that π^* is additive, let $r = \sum_i \alpha_i x_i$, $r' = \sum_i \alpha_i x_i'$, and $r + r' = \sum_i \alpha_i x_i''$ be the unique representations. Then

$$\sum_i \alpha_i x_i'' = \sum_i \alpha_i x_i + \sum_i \alpha_i x_i'.$$

Multiplying by $\alpha_k^{-1} x_k''$ we have

$$x_k'' = \sum_i \alpha_k^{-1}\alpha_i x_i x_k'' + \sum_i \alpha_k^{-1}\alpha_i x_i' x_k'' \in R'.$$

Thus by 4.5

$$\pi(x_k'') = \sum_i \sigma(\alpha_k^{-1})\sigma(\alpha_i)\pi(x_i)\pi(x_k'') + \sum_i \sigma(\alpha_k^{-1})\sigma(\alpha_i)\pi(x_i')\pi(x_k'')$$

and since σ is a field isomorphism,

$$\sigma(\alpha_k)\pi(x_k'') = \pi(x_k'')\left[\sum_i \sigma(\alpha_i)\pi(x_i) + \sum_i \sigma(\alpha_i)\pi(x_i')\right].$$

Summing over all k and using the fact that $\sum_k \pi(x_k'') = 1_S$, we have

$$\pi^*(r + r') = \sum_k \sigma(\alpha_k)\pi(x_k'') = \sum_i \sigma(\alpha_i)\pi(x_i) + \sum_i \sigma(\alpha_i)\pi(x_i')$$
$$= \pi^*(r) + \pi^*(r').$$

A similar technique shows π^* to be multiplicative, and thus an isomorphism.

COROLLARY 4.7. *If $End_F R \simeq End_G S$ then $R \simeq S$.*

PROOF. This follows immediately from 3.6 and 4.6.

Note that each p-ring R is a p^k-ring in the sense of Foster, the normal subfield F being isomorphic to $GF(p)$. Further R is an algebra over F and $\text{End}_F R = \text{End}\, R$. With this in mind we have

COROLLARY 4.8. *Let p be a fixed prime integer. If R and S are p-rings such that End $R \simeq$ End S, then $R \simeq S$.*

5. Remarks

It is not known whether the Corollary 4.8 can be extended to the p^k-rings of Foster, wherein the entire semigroups of ring endomorphisms are used, to the

p^k-rings of McCoy, or to direct sums of p^{k_i}-rings in both senses. It does not extend to direct sums of p-rings, where p takes on at least two distinct values, or to J-rings as illustrated by the following example.

Let $R = GF(2) \oplus GF(2) \oplus GF(3)$ and $S = GF(3) \oplus GF(3) \oplus GF(2)$. Each of these rings has the property that $x^6 = x$ for each x in the ring and End $R \simeq$ End S, but R is not isomorphic to S.

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USAF Academy, Colorado 80840
U. S. A.

and

North Carolina State University at Raleigh 27607
U. S. A.